Geometric Representation Theory Homework

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Homework 1

Show that

$$\psi_0(x) = e^{-x^2/2}$$

is an eigenvector of the Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2)$$

with eigenvalue $\frac{1}{2}$.

Solution

It's easy to compute

$$\begin{aligned}
H\psi_0(x) &= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) e^{-x^2/2} \\
&= \frac{1}{2}\left(-\frac{d}{dx}\left(-xe^{-x^2/2}\right) + x^2 e^{-x^2/2}\right) \\
&= \frac{1}{2}\left(e^{x^2/2} - x^2 e^{-x^2/2} + x^2 e^{-x^2/2}\right) \\
&= \frac{1}{2}\psi_0(x)
\end{aligned}$$

where we've made the usual substitutions $p = -i\dfrac{d}{dx}$ and $q = x$, So $\psi_0(x)$ is indeed an eigenvector with eigenvalue $\frac{1}{2}$.

Homework 2

Define the creation and annihilation operators by

$$a^* = \frac{1}{\sqrt{2}}(p + iq)$$

and

$$a = \frac{1}{\sqrt{2}}(p - iq)$$

respectively, where p *and* q *are as above and obey*

$$[q, p] = qp - pq = i.$$

Show that

$$[a, a^*] = 1$$

$$H = a^*a + \frac{1}{2}$$

$$[H, a^*] = a^*$$

$$[H, a] = -a$$

Then show that if

$$H\psi = \lambda\psi$$

then

$$Ha^*\psi = (\lambda + 1)a^*\psi$$

and

$$Ha\psi = (\lambda - 1)a\psi$$

Solution

Using the so-called canonical commutation relation (CCR)

$$[q, p] = i$$

with the definitions of a and a^*, we get

$$\begin{aligned}
[a, a^*] &= \left[\frac{1}{\sqrt{2}}(p - iq), \frac{1}{\sqrt{2}}(p + iq)\right] \\
&= \frac{1}{2}([p, p] - i[q, p] + i[p, q] + i^2[q, q]) \\
&= \frac{1}{2}(0 + (-i)i + i(-i) + 0) \\
&= 1
\end{aligned}$$

where we used the bilinearity of [,].

We can also use the CCR to show $H = a^*a + \frac{1}{2}$, even though at first glance H does not appear to involve a commutator:

$$\begin{aligned}
a^*a &= \frac{1}{\sqrt{2}}(p + iq)\frac{1}{\sqrt{2}}(p - iq) \\
&= \frac{1}{2}(p^2 + iqp - ipq + q^2) \\
&= \frac{1}{2}(p^2 + i[q, p] + q^2) \\
&= \frac{1}{2}(p^2 + q^2 - 1) \\
&= H - \frac{1}{2}
\end{aligned}$$

The commutators of the creation and annihilation operators with H are an easy consequence of $[a, a^*] = 1$, their commutation relatiun with each other:

$$\begin{aligned}
[H.a^*] &= [a^*a + \frac{1}{2}, a^*] \\
&= [a^*a, a^*] + [\frac{1}{2}, a^*] \\
&= a^*aa^* - a^*a^*a \\
&= a^*([a, a^*]) \\
&= a^*
\end{aligned}$$

and similarly for $[H, a] = -a$.

Now it's very straightforward to see that eigenvectors of H

$$H\psi = \lambda\psi$$

are "raised" by a^*

$$Ha^*\psi = (\lambda + 1)a^*\psi$$

since

$$\begin{aligned} Ha^*\psi &= a^*H\psi + [H, a^*]\psi \\ &= a^*\lambda + a^*\psi \\ &= (\lambda + 1)a^*\psi \end{aligned}$$

and "lowered" by a, since

$$\begin{aligned} Ha\psi &= aH\psi + [H, a]\psi \\ &= a\lambda - a\psi \\ &= (\lambda - 1)a\psi \end{aligned}$$