Geometric Representation Theory - Winter 2008
Homework 1
Christopher Walker

Exercise 1. *If* $\Psi_0(x) = e^{-x^2/2}$ *then* $H\Psi_0 = \frac{1}{2}\Psi_0$

Answer: Recall $H = \frac{1}{2}(p^2 + q^2)$ where $(p\Psi)(x) = \frac{1}{i}\Psi'(x)$ and $(q\Psi)(x) = x\Psi(x)$. So,

$$\begin{aligned}
H\Psi_0(x) &= \tfrac{1}{2}(p^2+q^2)\Psi_0(x) \\
&= \tfrac{1}{2}(p^2+q^2)(e^{-x^2/2}) \\
&= \tfrac{1}{2}(p^2(e^{-x^2/2}) + q^2(e^{-x^2/2})) \\
&= \tfrac{1}{2}(p(p(e^{-x^2/2})) + q((e^{-x^2/2}))) \\
&= \tfrac{1}{2}(-(e^{-x^2/2})'' + x^2(e^{-x^2/2})) \\
&= \tfrac{1}{2}((xe^{-x^2/2})' + x^2e^{-x^2/2}) \\
&= \tfrac{1}{2}(-x^2e^{-x^2/2} + e^{-x^2/2} + x^2e^{-x^2/2}) \\
&= \tfrac{1}{2}(e^{-x^2/2}) \\
&= \tfrac{1}{2}\Psi_0(x)
\end{aligned}$$

□

Exercise 2. *Define the creation and annihilation operators by*

$$a^* = \frac{1}{\sqrt{2}}(p+iq)$$

$$a = \frac{1}{\sqrt{2}}(p-iq)$$

Show that:

- $H = a^*a + \frac{1}{2}$
- $[a, a^*] = 1$
- $[H, a] = a^*$
- $[H, a] = -a$

Then use these to show that if $H\Psi = \lambda\Psi$ *then* $Ha^*\Psi = (\lambda+1)a^*\Psi$ *and* $Ha\Psi = (\lambda-1)a\Psi$.

Answer: Recall that $[p, q] = \frac{1}{i}$.

$$\begin{aligned}
a^*a + \tfrac{1}{2} &= \tfrac{1}{\sqrt{2}}(p+iq)\tfrac{1}{\sqrt{2}}(p-iq) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p+iq)(p-iq) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p^2+q^2-pqi+qpi) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p^2+q^2) + \tfrac{1}{2}(-pqi+qpi) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p^2+q^2) + \tfrac{1}{2}(-i(\tfrac{1}{i})) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p^2+q^2) - \tfrac{1}{2} + \tfrac{1}{2} \\
&= \tfrac{1}{2}(p^2+q^2) \\
&= H
\end{aligned}$$

$$\begin{aligned}
[a,a^*] &= [\tfrac{1}{\sqrt{2}}(p+iq), \tfrac{1}{\sqrt{2}}(p-iq)] \\
&= \tfrac{1}{2}[p-iq,p+iq] \\
&= \tfrac{1}{2}([p,p]+[p,iq]+[-iq,p]+[-iq,iq]) \\
&= \tfrac{1}{2}([p,p]+[p,iq]+[p,iq]+[q,q]) \\
&= \tfrac{1}{2}(i[p,q]+i[p,q]) \\
&= \tfrac{1}{2}(2i[p,q]) \\
&= \tfrac{1}{2}(2i\tfrac{1}{i}) \\
&= \tfrac{1}{2}(2) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
[H,a^*] &= [a^*a+\tfrac{1}{2},a^*] \\
&= [a^*a,a^*]+[\tfrac{1}{2},a^*] \\
&= [a^*a,a^*] \\
&= a^*[a,a^*]+[a^*,a^*]a \\
&= a^*[a,a^*] \\
&= a^*
\end{aligned}$$

$$\begin{aligned}
[H,a] &= [a^*a+\tfrac{1}{2},a] \\
&= [a^*a,a]+[\tfrac{1}{2},a] \\
&= [a^*a,a] \\
&= a^*[a,a]+[a^*,a]a \\
&= [a^*,a]a \\
&= -[a,a^*]a \\
&= -a
\end{aligned}$$

Now, assume $H\Psi = \lambda\Psi$. Using the brackets above, we will show that $Ha^*\Psi = (\lambda+1)a^*\Psi$ and $Ha\Psi = (\lambda-1)a\Psi$. we begin with the creation operator a^*.

$$\begin{aligned}
Ha^*\Psi &= (a^*H+a^*)\Psi \\
&= a^*H\Psi+a^*\Psi \\
&= a^*\lambda\Psi+a^*\Psi \\
&= (\lambda+1)a^*\Psi
\end{aligned}$$

Similarly for the annihilation operator a we get:

$$\begin{aligned}
Ha\Psi &= (aH-a)\Psi \\
&= aH\Psi+a\Psi \\
&= a\lambda\Psi+a\Psi \\
&= (\lambda-1)a\Psi
\end{aligned}$$

□