Harmonic Oscillator

\[ p, q : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \]

\[ q \psi(x) = x \psi(x) \]

\[ p \psi(x) = \frac{i}{\hbar} \psi'(x) \]

\[ [p, q] = \frac{i}{\hbar} \quad a^* = \frac{p + iq}{\sqrt{2}} \]

\[ [a, a^*] = 1 \quad a = \frac{p - iq}{\sqrt{2}} \]

\[ H = \frac{1}{2} (p^2 + q^2) = a^* a + \frac{1}{2} \]

\[ \psi_0 \in L^2(\mathbb{R}) \text{ given by } \psi_0(x) = e^{-x^2/\hbar} \]

has

\[ a \psi_0 = 0 \]

i.e.

\[ (p - iq) \psi_0 = 0 \]

\[ \left( \frac{1}{i} \frac{d}{dx} - ix \right) \psi_0 = 0 \]

\[ \frac{d}{dx} e^{-x^2/\hbar} = -xe^{-x^2/\hbar} \]

Thus:

\[ H \psi_0 = (a^* a + \frac{1}{2}) \psi_0 = \frac{1}{2} \psi_0 \]
or let
\[ N = H - \frac{1}{2} = a^* a \]
\[ N \psi = 0 \]

Also:
\[ [N, a^*] = a^* \]
\[ [N, a] = -a \]

\[ N \psi = \lambda \psi \implies N a^* \psi = a^* N \psi + a^* \psi = (\lambda + 1) a^* \psi \]
\[ N \psi = \lambda \psi \implies N a \psi = (\lambda - 1) a \psi \]

so if
\[ \psi_n = (a^*)^n \psi_0 \]

then
\[ N \psi_n = n \psi_n \]

so \( N \) is called the **number operator**.

Then - \( \psi_n \) form an orthogonal basis of \( L^2(\mathbb{R}) \), (a Hilbert space basis)
Notice:

\[ a^* \Psi_n = \Psi_{n+1} \]

and

\[ a \Psi_n = n \Psi_{n-1} \quad (n \geq 0) \]

which we prove by induction:

\[ a \Psi_0 = 0 \]

\[ a \Psi_n = n \Psi_{n-1} \implies a \Psi_{n+1} = a a^* \Psi_n \]

\[ = a^* a \Psi_n + \Psi_n \]

\[ = a^* n \Psi_{n-1} + \Psi_n \]

\[ = (n+1) \Psi_n \]

Thus:

\[ N \Psi_n = a^* a \Psi_n \]

\[ = n \Psi_n \]

So we have an analogy:

\[ \mathbb{C} [z] \xrightarrow{\alpha} L^2(\mathbb{R}) \]

\[ z^n \xrightarrow{\alpha} \Psi_n \]

\[ m_z z^n = z^{n+1} \quad a^* \Psi_n = \Psi_{n+1} \]

\[ \frac{d}{dz} z^n = n z^{n-1} \quad a \Psi_n = n \Psi_{n-1} \]
α will not be an isomorphism, but it is 1-1 if it has a dense range.

HW: Find the (unique) inner product on $C[Z]$ s.t. $m_z \land \frac{d}{dz}$ are adjoints, i.e.: 

$$\langle zz^n, z^m \rangle = \langle z^n, \frac{d}{dz} z^m \rangle$$

for $n, m \geq 1$, and $\|z\| = 1$.

With this inner product on $C[Z]$, \( \alpha : C[Z] \to L^2(\mathbb{IR}) \)

is 1-1, inner product preserving, if it has a dense range. So the Hilbert space completion of $C[Z]$, Fock space, is equipped with a unitary operator to $L^2(\mathbb{IR})$.

Henceforth we'll drop the analysis, replace $C$ by any field $k$, and work with $k[Z]$ with operators $a^* = m_z$, $a = \frac{d}{dz}$, $N = a^* a$. 
We can easily generalize to \( k[z_1, \ldots, z_n] \) with operators
\[
\alpha^* = m_{z_i}, \quad N_i = \alpha_i^* \alpha_i.
\]
\[
\alpha_i = \frac{d}{dz_i}, \quad N = \sum_i N_i.
\]

Now let's groupoidify \( k[z] \), \( \alpha \), \( \alpha^* \).

Degroupoidification turns groupoids into vector spaces, spans of groupoids into linear operators.

If \( X \) is a groupoid, there's a set \( X \) of isomorphism classes of objects. We then form a vector space with \( X \) as its basis — this vector space is called \( H_0(X) \), the "zeroth homology".

If we have a functor
\[
f: X \rightarrow Y
\]
between groupoids, we get an operator
\[ f_* : H_0(X) \rightarrow H_0(Y) \]
\[ [x] \mapsto [f(x)] \]

And sometimes:

\[ f^! : H_0(Y) \rightarrow H_0(X) \]
\[ [y] \mapsto |\text{Aut}(y)| \sum_{[x] \in f^!(y)} \frac{[x]}{|\text{Aut}(x)|} \]

Here \( f^{-1}(y) \) is the groupoid whose objects are \( x \in X \) with \( f(x) = y \) and whose morphisms are all morphisms between those. \( \text{Aut}(x) \) is the set of all isomorphisms from \( x \) to \( x \). \( f^!(x) \) is the set of isomorphism classes in \( f^{-1}(x) \).

Example: \( \text{Fin Set}_0 \) has \( H_0(\text{Fin Set}_0) = k[Z] \)

Consider:

\[
\begin{array}{c}
\text{Fin Set}_0 \xrightarrow{+1} \text{Fin Set}_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
S \quad S \cup \mathbb{R}^3 \\
\alpha \quad \alpha + 1 \\
T \quad T \cup \mathbb{R}^3
\end{array}
\]
What's

\[ H_0(\text{Fin Set}_0) \xrightarrow{(+1)^*} H_0(\text{Fin Set}_0) \]

\[ \mathbb{Z}^n \xrightarrow{(+1)^*} \mathbb{Z}^{n+1} \]

It's the creation operator. What's

\[ H_0(\text{Fin Set}_0) \xrightarrow{(+1)!} H_0(\text{Fin Set}_0) \]

\[ \mathbb{Z}^n \xrightarrow{(+1)!} \mathbb{Z}^{n+1} \]

Say \( y \) is an \( n \)-element set, so \([y] = \mathbb{Z}^n\).

\([x] = \{ (n-1)\text{-element sets} \} = \mathbb{Z}^{n-1} \]

So, \( |\text{Aut}(x)| = (n-1)! \); \( |\text{Aut}(y)| = n! \)

\[ (+1)! \mathbb{Z}^n = n! \frac{\mathbb{Z}^{n-1}}{(n-1)!} = n \mathbb{Z}^{n-1} \]

It's the annihilation operator!