Last time we saw:

\[ H_0(\text{Fin}\text{-}\text{Set}_0) \cong k[Z] \]

\[ [n] \mapsto z^n \]

and the functor

\[ \text{Fin}\text{-}\text{Set}_0 \xrightarrow{\text{+1}} \text{Fin}\text{-}\text{Set}_0 \]

\[ S \mapsto S + 1 \]

gives the creation operator, \( a^* \) (covariant)

\[ (+1)^*: k[Z] \rightarrow k[Z] \]

\[ z^n \mapsto z^{n+1} \]

and the annihilation operator, \( a \) (contravariant)

\[ (+1)!: k[Z] \rightarrow k[Z] \]

\[ z^n \mapsto nz^{n-1} \]

Can we groupify the commutation relation

\[ aa^* - a^*a = 1 \]
Both $a$ and $a^*$ come from spans of groupoids. Recall:

given a span of groupoids:

\[
\begin{array}{c}
\text{S} \\
\downarrow p \\
\downarrow q \\
X \quad Y
\end{array}
\]

we get an operator

\[
\begin{array}{c}
\text{FinSet}^o \\
\downarrow 1 \\
\text{FinSet}^o
\end{array} \quad +1 \quad A^* \\
\begin{array}{c}
\text{FinSet}^o \\
\downarrow 1 \\
\text{FinSet}^o
\end{array}
\]

To get the creation operator this way, use:

This span really is the groupoidified version of the creation operator, $A^*$. To get the annihilation operator, use:

\[
\begin{array}{c}
\text{FinSet}^o \\
\downarrow +1 \\
\text{FinSet}^o
\end{array} \quad 1 \quad A \\
\begin{array}{c}
\text{FinSet}^o \\
\downarrow 1 \\
\text{FinSet}^o
\end{array}
\]
We've groupified $a, a^*$ to get $A \times A^*$; now can we show:

$$AA^* = A^*A + 1$$

First question: how do we compose spans of groupoids? Second: how do we add them?

Watch the video to see JB perform a marker experiment in quantum mechanics!

Given composable spans

\[
\begin{array}{c}
S \\
\downarrow \swarrow \\
X & \rightarrow & Y \\
\downarrow \searrow \\
T & \rightarrow & Z
\end{array}
\]

how do we compose them? We take the "weak pullback"
Given sets $S$ and functions:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow g & & \downarrow \\
\end{array}
\]

the pullback is

\[
S \times_T T = \{(s, t) \in S \times T : f(s) = g(t)\}
\]

Given groupoids $S$ and functors:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow g & & \downarrow \\
\end{array}
\]

we let the weak pullback be the groupoid

\[
S \times_T T = \{(s, t) \in S \times T \text{ equipped with } \xrightarrow{\xi} \text{ such that } f(s) \xrightarrow{\alpha} g(t)\}
\]

A morphism in $S \times_T T$ looks like:

\[
\begin{array}{ccc}
f(s) & \xrightarrow{\alpha} & g(t) \\
\downarrow h & & \downarrow k \\
\end{array}
\]

\[
\begin{array}{ccc}
f(s) & \xrightarrow{\xi'} & g(t') \\
\end{array}
\]

a commutative square in $Y$. 

Example:

\[
\begin{array}{c}
[\text{2-colored finite sets}] \\
\downarrow f \\
\text{FinSet}_0 \\
\downarrow g \\
[\text{3-colored finite sets}]
\end{array}
\]

What's the weak pullback?

An object is a 2-colored set \( X \): 3-colored set \( Y \) equipped with \( f(x) \mapsto g(y) \), i.e. a set with 2-coloring AND 3-coloring, i.e. a 6-colored set.

So the weak pullback is:

\[
[\text{6-colored sets}], \text{ aka } \text{FinSet}_6
\]

Example:

\[
\begin{array}{c}
\text{FinSet}_6 \\
\downarrow 1 \\
\text{FinSet}_0 \\
\downarrow +1 \\
\text{FinSet}_6
\end{array}
\]
An object in the weak pullback is a pair of finite sets $s, t$ equipped with an isomorphism

$$s \xrightarrow{s} t + 1.$$ 

So an answer is $\text{[pointed finite sets]} \simeq \text{FinSet}_0$.

$$(t+1, 1) \leftrightarrow t$$

$$(0, *) \rightarrow s - \{s\}$$

And equivalently,

$$t \in \text{FinSet}_0$$

$$+1 \quad \rightarrow \quad 1$$

$$\text{FinSet}_0 \quad \rightarrow \quad \text{FinSet}_0$$

$$1 \quad \rightarrow \quad +1$$

$$\text{FinSet}_0 \quad \rightarrow \quad \text{FinSet}_0$$
Let's compose $A : A^*$.

![Diagram]

An object in the weak pullback is:

$3 \times E \times \text{FinSet}_0 \cong 3+1 \rightarrow 6+1$. 
There are two choices: either $\alpha$ preserves the point or not.

In the first case, we really have just a finite set. ($s, sy$)

In the second case, we have a finite set $w$. 2 different marked points!

So: the weak pullback is

$$\left[ \text{finite set } s \text{ equipped with 2 points } x, y \in S \right]$$