

Last time we saw:

$$H_0(\text{FinSet}_0) \cong k[z]$$

$$[n] \longmapsto z^n$$

the functor

$$\text{FinSet}_0 \xrightarrow{+1} \text{FinSet}_0$$

$$S \longmapsto S + 1$$

gives the creation operator, a^* (covariant)

$$(+1)_*: k[z] \longrightarrow k[z]$$

$$z^n \longmapsto z^{n+1}$$

the annihilation operator, a (contravariant)

$$(+1)^!: k[z] \longrightarrow k[z]$$

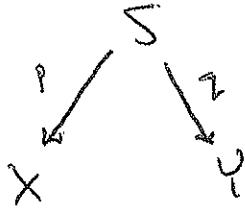
$$z^n \longmapsto n z^{n-1}.$$

Can we groupoidify the commutation relation

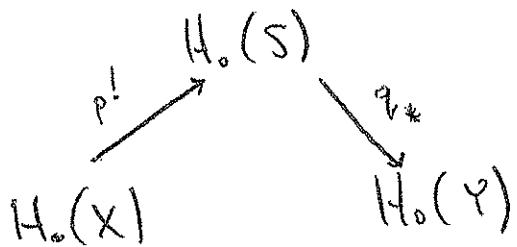
$$aa^* - a^*a = 1?$$

(2)

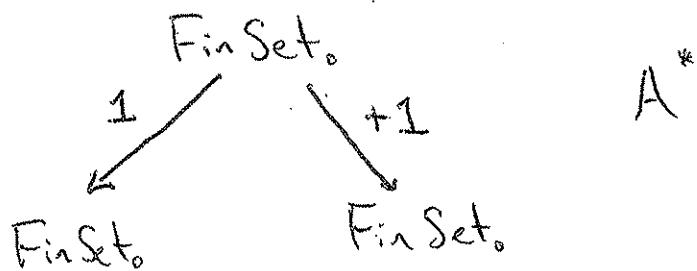
Both a & a^* come from spans of groupoids. Recall:
given a span of groupoids:



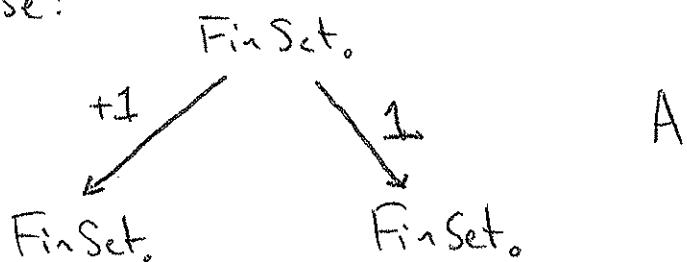
we get an operator



To get the creation operator this way, use:



This span really is the groupoidified version of the creation operator, A^* . To get the annihilation operator, use:



③

We've groupified a, a^* to get $A \in A^*$; now can we show:

$$AA^* \cong A^*A + 1$$

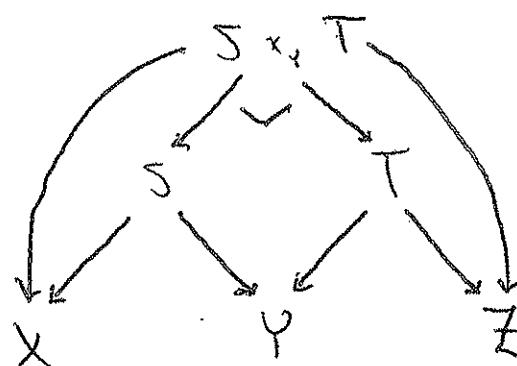
First question: how do we compose spans of groupoids? Second: how do we add them?

Watch the video to see JB perform a marker experiment in quantum mechanics!

Given composable spans

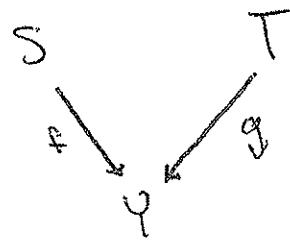


how do we compose them? We take the "weak pullback"



(4)

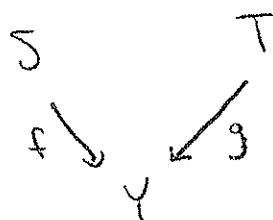
Given sets \mathcal{S} , \mathcal{T} ; functions:



the pullback is

$$\mathcal{S} \times_{\varphi} \mathcal{T} = \{(s, t) \in \mathcal{S} \times \mathcal{T} : f(s) = g(t)\}$$

Given groupoids \mathcal{S} , \mathcal{T} ; functors:



we let the weak pullback be the groupoid

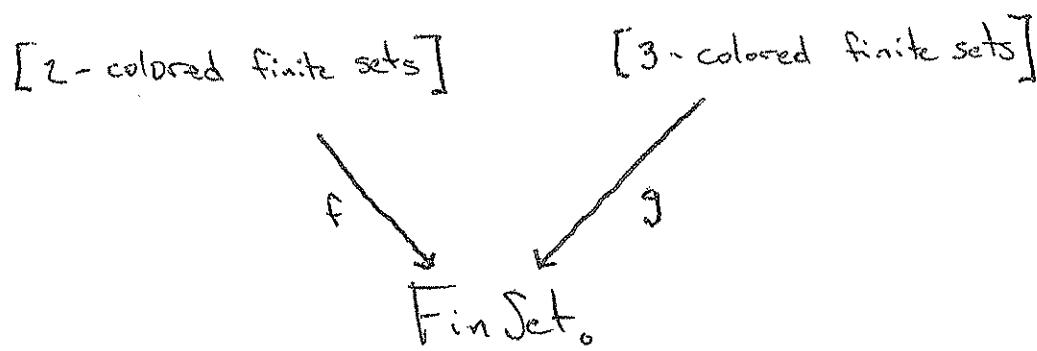
$$\mathcal{S} \times_{\varphi} \mathcal{T} = \left[(s, t) \in \mathcal{S} \times \mathcal{T} \text{ equipped with } f(s) \xrightarrow{\sim} g(t) \right]$$

A morphism in $\mathcal{S} \times_{\varphi} \mathcal{T}$ looks like:

$$\begin{array}{ccc} f(s) & \xrightarrow{\cong} & g(t) \\ h \downarrow & & \downarrow k \\ f(s') & \xrightarrow[\sim]{\alpha'} & g(t') \end{array}$$

a commutative square in φ .

Example:



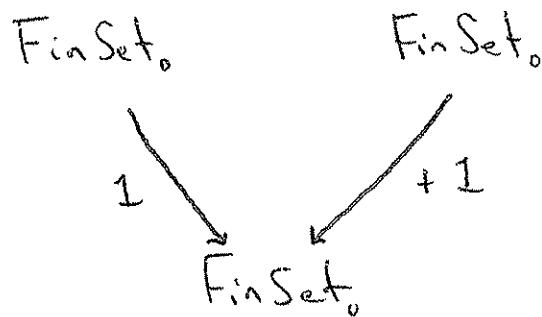
What's the weak pullback?

An object is a 2-colored set X in 3-colored set y equipped with $f(x) \xrightarrow{\alpha} g(y)$, i.e. a set with 2-coloring AND 3-coloring, i.e. a 6-colored set.

So the weak pullback is:

[6-colored sets], aka FinSet⁶.

Example:



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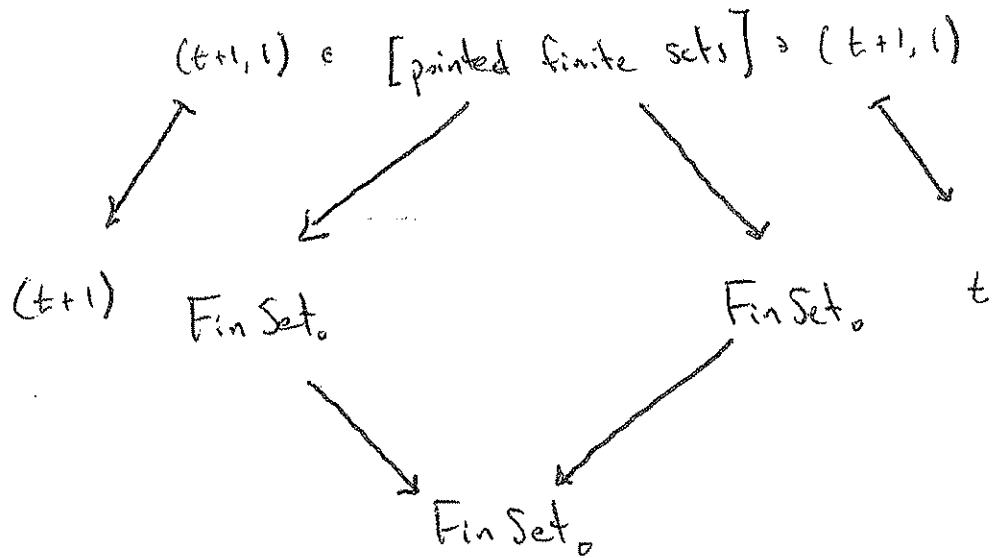
An object in the weak pullback is a pair of finite sets $s \sqcup t$ equipped with an isomorphism

$$s \xrightarrow[\sim]{\alpha} t + 1.$$

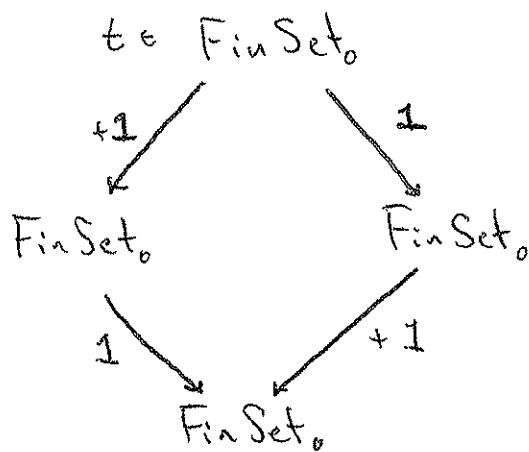
So an answer is $[\text{pointed finite sets}] \simeq \text{FinSet}_*$.

$$(t+1, 1) \longleftrightarrow t$$

$$(s, *) \longmapsto s - \{*\}$$



And equivalently,

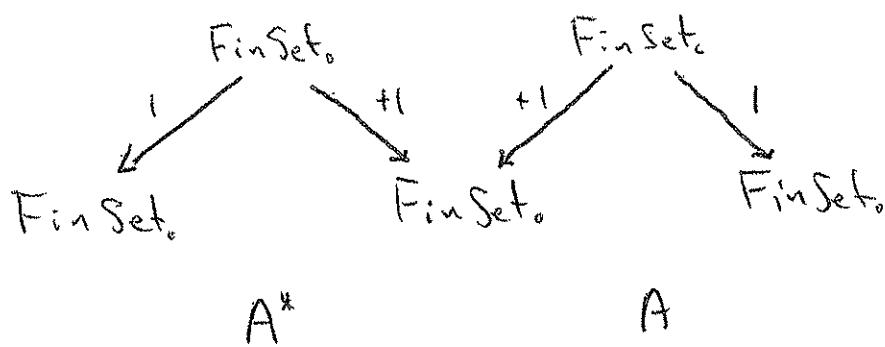
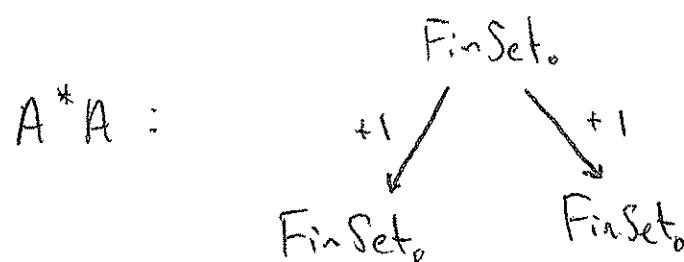
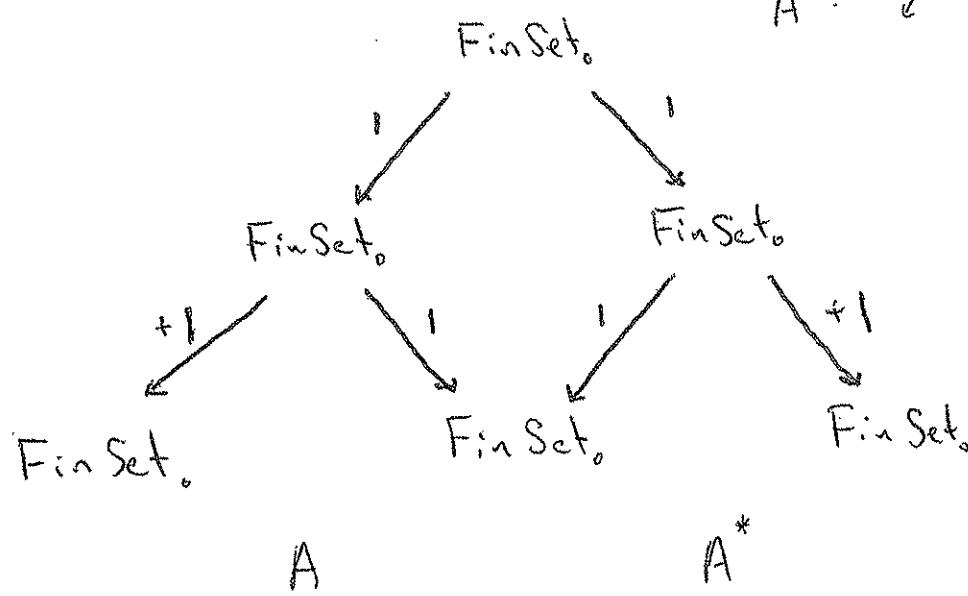


(7)

Let's compose $A \dot{\in} A^*$

$$A^*: \downarrow \downarrow^{+1}$$

$$A: +\downarrow \downarrow$$



An object in the weak pullback is :

$$s, t \in \text{FinSet.} \quad s \dot{\in} s+1 \xrightarrow{\cong} t+1.$$

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There are two choices: either α preserves the point
or not.

In the first case, we really have just a finite set. (s , say)

In the second case, we have a finite set w. 2
different marked points!

S_0 : the weak pullback is

[finite set s equipped with 2 points $x, y \in s$].