Last time we saw:

$$H_0(\mathrm{FinSet}_0) \cong k[z]$$
$$[n] \longmapsto z^n$$

& the functor

$$\mathrm{FinSet}_0 \xrightarrow{+1} \mathrm{FinSet}_0$$
$$S \longmapsto S+1$$

gives the creation operator, a^* (covariant)

$$(+1)_* : k[z] \longrightarrow k[z]$$
$$z^n \longmapsto z^{n+1}$$

& the annihilation operator, a (contravariant)

$$(+1)^! : k[z] \longrightarrow k[z]$$
$$z^n \longmapsto n z^{n-1}.$$

Can we groupoidify the commutation relation

$$a a^* - a^* a = 1 \ ?$$

Both a & a^* come from spans of groupoids. Recall: given a span of groupoids:

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

we get an operator

$$\begin{tikzcd} & H_0(S) \arrow[dr, "q_*"] & \\ H_0(X) \arrow[ur, "p^!"] & & H_0(Y) \end{tikzcd}$$

To get the creation operator this way, use:

$$\begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd} \qquad A^*$$

This span really is the groupoidified version of the creation operator, A^*. To get the annihilation operator, use:

$$\begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd} \qquad A$$

We've groupoidified a, a^* to get A & A^*; now can we show:

$$AA^* \cong A^*A + 1$$

First question: how do we compose spans of groupoids? Second: how do we add them?

Watch the video to see JB perform a marker experiment in quantum mechanics!

Given composable spans

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

how do we compose them? We take the "weak pullback"

$$\begin{tikzcd}
 & & S \times_Y T \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

Given sets & functions:

$$\begin{tikzcd} S \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

the <u>pullback</u> is

$$S \times_Y T = \{(s,t) \in S \times T : f(s) = g(t)\}$$

Given groupoids & functors:

$$\begin{tikzcd} S \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

we let the <u>weak pullback</u> be the groupoid

$$S \times_Y T = \left[(s,t) \in S \times T \text{ equipped with } f(s) \xrightarrow[\sim]{\alpha} g(t)\right]$$

A morphism in $S \times_Y T$ looks like:

$$\begin{tikzcd} f(s) \arrow[r, "\alpha", "\sim"'] \arrow[d, "h"'] & g(t) \arrow[d, "k"] \\ f(s') \arrow[r, "\alpha'", "\sim"'] & g(t') \end{tikzcd}$$

a commutative square in Y.

Example:

$$\begin{tikzcd} {[2\text{-colored finite sets}]} \arrow[dr, "f"'] & & {[3\text{-colored finite sets}]} \arrow[dl, "g"] \\ & \mathrm{FinSet}_0 & \end{tikzcd}$$

What's the weak pullback?

An object is a 2-colored set X & 3-colored set y equipped with $f(x) \xrightarrow{\alpha} g(y)$, i.e. a set with 2-coloring AND 3-coloring, i.e. a 6-colored set.

So the weak pullback is:

[6-colored sets], aka FinSet_0^6.

Example:

$$\begin{tikzcd} \mathrm{FinSet}_0 \arrow[dr, "1"'] & & \mathrm{FinSet}_0 \arrow[dl, "+1"] \\ & \mathrm{FinSet}_0 & \end{tikzcd}$$

An object in the weak pullback is a pair of finite sets s & t equipped with an isomorphism

$$s \xrightarrow[\sim]{\alpha} t+1 .$$

So an answer is $[\text{pointed finite sets}] \simeq \mathrm{FinSet}_0$

$$(t+1, 1) \longleftarrow\!\shortmid\; t$$

$$(s, *) \longmapsto s - \{*\}$$

$$\begin{tikzcd}
(t+1,1) \in [\text{pointed finite sets}] \ni (t+1,1) \arrow[d, leftrightarrow] \arrow[dl] \arrow[dr] \arrow[d, leftrightarrow] \\
(t+1) \arrow[dr] & \mathrm{FinSet}_0 \arrow[dr] & \mathrm{FinSet}_0 \arrow[dl] & t \\
& \mathrm{FinSet}_0 &
\end{tikzcd}$$

And equivalently,

$$\begin{tikzcd}
& t \in \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\
\mathrm{FinSet}_0 \arrow[dr, "1"'] & & \mathrm{FinSet}_0 \arrow[dl, "+1"] \\
& \mathrm{FinSet}_0 &
\end{tikzcd}$$

Let's compose A & A^*

A^* : $\swarrow$ $\searrow^{+1}$

A : $\swarrow^{+1}$ $\searrow$

$$\begin{tikzcd}
 & & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
 & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

$A \qquad\qquad A^*$

A^*A :

$$\begin{tikzcd}
 & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "+1"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+1"] & & \mathrm{FinSet}_0 \arrow[dl, "+1"'] \arrow[dr, "1"] & \\
\mathrm{FinSet}_0 & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

$A^* \qquad\qquad A$

An object in the weak pullback is:

$s, t \in \mathrm{FinSet}_0$ & $s+1 \xrightarrow{\cong} t+1$.

There are two choices: either α preserves the point or not.

In the first case, we really have just a finite set. (S, s_{xy})

In the second case, we have a finite set w. 2 different marked points!

So: the weak pullback is

[finite set S equipped with 2 points $x, y \in S$].