Geometric Representation Theory seminar, Lecture 29

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1 The Harmonic Oscillator with n Degrees of Freedom

I have been talking about the Harmonic oscillator with one degree of freedom. Now we are going to generalize this to n degrees of freedom. This is important because in real life we have things which can wiggle around in n ways.

Before we looked at the groupoid of finite sets. Now we will look at the groupoid of n copies of finite sets.

$$\begin{aligned} H_0(FinSet_o^n) &\cong H_0(FinSet_0)^{\otimes n} \\ &\cong k[z]^{\otimes n} \\ &\cong k[z_1, \ldots, z_n] \end{aligned}$$

where the isomorphism class of

$$(k_1, k_2, \ldots k_n) \in FinSet_0^n$$

corresponds to

$$z_1^{k_1} \cdots z_n^{k_n} \in k[z_1, \ldots, z_n]$$

We have n creation and annihilation operators, whose groupoidified versions are spans:

A_i^*, $i = 1, \ldots, n$

$$\begin{tikzcd} & FinSet_0^n \arrow[dl, "1"'] \arrow[dr, "(k_1, \ldots k_n) \mapsto (k_1, \ldots, k_{i+1}, \ldots, k_n)"] & \\ FinSet_0^n & & FinSet_0^n \end{tikzcd}$$

This "creates a particle of type i".

A_i^*, $i = 1, \ldots, n$

$$\begin{tikzcd} & FinSet_0^n \arrow[dl, "(k_1, \ldots k_n) \mapsto (k_1, \ldots, k_{i+1}, \ldots, k_n)"'] \arrow[dr, "1"] & \\ FinSet_0^n & & FinSet_0^n \end{tikzcd}$$

This "annihilates a particle of type i".

These give operators:

$$\tilde{A}_i = a_i, \tilde{A}_i^* = a_i^*$$

from the "Fock space" $k[z_1, \ldots, z_n]$ to itself, and explicitly:

$$a_i = \frac{\partial}{\partial z_i}$$

$$a_i^* = m_{z_i},$$

where m_{z_i} is multiplication by z_i.

These satisfy:

$$a_i a_j = a_j a_i$$

$$a_i^* a_j^* = a_j^* a_i^*$$

$$a_i a_j^* = a_j^* a_i + \delta_{ij} 1$$

These are called the "CCR" - "canonical commutation relations".
So , we hope:

$$A_i A_j \cong A_j A_i$$

$$A_i^* A_j^* \cong A_j^* A_i^*$$

$$A_i A_j^* \cong A_j^* A_i + \delta_{ij} 1$$

and it's true.
$A_i A_j^*$

$$\begin{tikzcd}
& & (*) \arrow[dl] \arrow[dr] & & \\
& FinSet_0^n \arrow[dl, "1"'] \arrow[dr, "+1_j"] & & FinSet_0^n \arrow[dl, "+1_i"'] \arrow[dr, "1"] & \\
FinSet_0^n & & FinSet_0^n & & FinSet_0^n
\end{tikzcd}$$

where $(+1)_i\colon (k_1, \ldots, k_n) \mapsto (k_1, \ldots, k_{i+1}, \ldots, k_n)$ and $(*) = [(k_1, \ldots, k_n), (l_1, \ldots, l_n), \alpha_j\colon k_j + 1 \to l_j, \alpha_i\colon k_i \to l_i + 1, \alpha_s\colon k_s \to l_s, s \neq i, j]$
$A_j^* A_i$

$$\begin{tikzcd}
& & FinSet_0^n \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
& FinSet_0^n \arrow[dl, "+1_i"'] \arrow[dr, "1"] & & FinSet_0^n \arrow[dl, "1"'] \arrow[dr, "+1_j"] & \\
FinSet_0^n & & FinSet_0^n & & FinSet_0^n
\end{tikzcd}$$

(*) should really be a fancy way of giving an n-tuple of finite sets. Another way to describe this weak pullback is $A_i A_j^*$

$$\begin{tikzcd}
& & FinSet_0^n \arrow[dl, "+1_i"'] \arrow[dr, "+1_j"] & & \\
& FinSet_0^n \arrow[dl, "1"'] \arrow[dr, "+1_j"] & & FinSet_0^n \arrow[dl, "+1_i"'] \arrow[dr, "1"] & \\
FinSet_0^n & & FinSet_0^n & & FinSet_0^n
\end{tikzcd}$$

So we have groupoidified all these things:

1. The associative algebra generated by a_i, a_i^*, $(i = 1, \ldots, n)$ is called the **Weyl algebra** W_n. This has a god-given representation on $k[z_1, \ldots, z_n]$ with

$$a_i \mapsto \frac{\partial}{\partial z_i}$$

$$a_i^* \mapsto m_{z_i}.$$

A typical element, in this representation, is like:

$$z_1^3\frac{\partial}{\partial z_2}+z_2z_3\frac{\partial^2}{\partial z_2 \partial z_1}.$$

So $W_n = \{$polynomial coefficient differential operators$\}$. This becomes a Lie algebra with

$$[a,b]=ab-ba$$

and then it has lots of Lie subalgebras, including:

2. $\{$Polynomial coefficient vector fields$\}$ = $\{$polynomial coefficient 1^{st}-order differential operators$\}$

3. $\{$Homogeneous linear coefficient vector fields$\} \cong \mathfrak{gl}(n) = \{n \times n$ matrices$\}$

4. The **Heisenberg Lie algebra**, $\mathfrak{h}_n$, all linear combinations of a_i, a_i^*, 1. This is a $(2n+1)$-dimensional Lie algebra.

5. $\{$Homogeneous quadratic expressions in a_i *and* a_i^*, plus constants$\}$
   This is closed under $[\cdot,\cdot]$. We have:

$$a_i^* a_j, i_j = 1,\dots,n$$

$$a_i a_j, i<j$$

$$a_i^* a_j^*, i<j$$

$$1$$

Then dim $= n^2+n(n-1)+1 = 2n^2-n+1$.

Homework: Work out:

$$[a_i^* a_j, a_k^* a_l] = ?$$

$$[a_i a_j, a_k^* a_l^*] = ?$$

Are these linear combinations of $a_i^* a_j$, $a_i a_j$, $a_i^* a_j^*$ or do we get constant terms?