

# Geometric Representation Theory seminar, Lecture 29

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## 1 The Harmonic Oscillator with $n$ Degrees of Freedom

I have been talking about the Harmonic oscillator with one degree of freedom. Now we are going to generalize this to  $n$  degrees of freedom. This is important because in real life we have things which can wiggle around in  $n$  ways.

Before we looked at the groupoid of finite sets. Now we will look at the groupoid of  $n$  copies of finite sets.

$$\begin{aligned} H_0(\mathit{FinSet}_o^n) &\cong H_0(\mathit{FinSet}_0)^{\otimes n} \\ &\cong k[z]^{\otimes n} \\ &\cong k[z_1, \dots, z_n] \end{aligned}$$

where the isomorphism class of

$$(k_1, k_2, \dots, k_n) \in \mathit{FinSet}_0^n$$

corresponds to

$$z_1^{k_1} \dots z_n^{k_n} \in k[z_1, \dots, z_n]$$

We have  $n$  creation and annihilation operators, whose groupoidified versions are spans:

$A_i^*$ ,  $i = 1, \dots, n$

$$\begin{array}{ccc} & \mathit{FinSet}_0^n & \\ & \swarrow \quad \searrow & \\ \mathit{FinSet}_0^n & & \mathit{FinSet}_0^n \\ & \xleftarrow{1} & \xrightarrow{(k_1, \dots, k_n) \mapsto (k_1, \dots, k_{i+1}, \dots, k_n)} \end{array}$$

This “creates a particle of type  $i$ ”.

$A_i^*$ ,  $i = 1, \dots, n$

$$\begin{array}{ccc} & \mathit{FinSet}_0^n & \\ & \swarrow \quad \searrow & \\ \mathit{FinSet}_0^n & & \mathit{FinSet}_0^n \\ & \xleftarrow{(k_1, \dots, k_n) \mapsto (k_1, \dots, k_{i+1}, \dots, k_n)} & \xrightarrow{1} \end{array}$$

This “annihilates a particle of type  $i$ ”.

These give operators:

$$\tilde{A}_i = a_i, \tilde{A}_i^* = a_i^*$$

from the “Fock space”  $k[z_1, \dots, z_n]$  to itself, and explicitly:

$$a_i = \frac{\partial}{\partial z_i}$$

$$a_i^* = m_{z_i},$$

where  $m_{z_i}$  is multiplication by  $z_i$ .

These satisfy:

$$a_i a_j = a_j a_i$$

$$a_i^* a_j^* = a_j^* a_i^*$$

$$a_i a_j^* = a_j^* a_i + \delta_{ij} 1$$

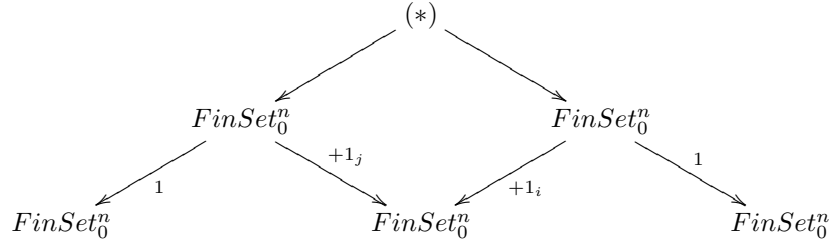
These are called the “CCR” - “canonical commutation relations”.  
So , we hope:

$$A_i A_j \cong A_j A_i$$

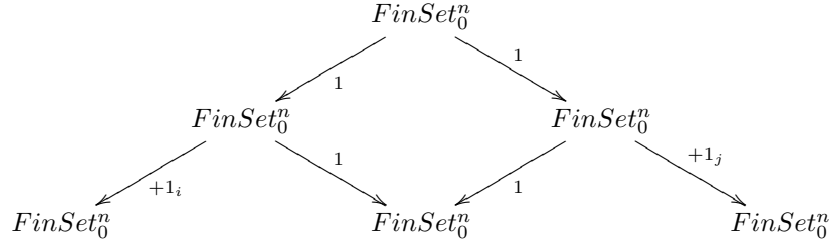
$$A_i^* A_j^* \cong A_j^* A_i^*$$

$$A_i A_j^* \cong A_j^* A_i + \delta_{ij} 1$$

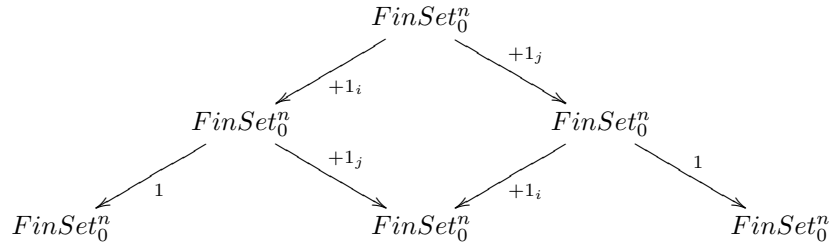
and it's true.  
 $A_i A_j^*$



where  $(+1)_i: (k_1, \dots, k_n) \mapsto (k_1, \dots, k_{i+1}, \dots, k_n)$  and  $(*) = [(k_1, \dots, k_n), (l_1, \dots, l_n), \alpha_j: k_j + 1 \rightarrow l_j, \alpha_i: k_i \rightarrow l_i + 1, \alpha_s: k_s \rightarrow l_s, s \neq i, j]$   
 $A_j^* A_i$



$(*)$  should really be a fancy way of giving an  $n$ -tuple of finite sets. Another way to describe this weak pullback is  $A_i A_j^*$



So we have groupoidified all these things:

1. The associative algebra generated by  $a_i, a_i^*, (i = 1, \dots, n)$  is called the **Weyl algebra**  $W_n$ . This has a god-given representation on  $k[z_1, \dots, z_n]$  with

$$a_i \mapsto \frac{\partial}{\partial z_i}$$

$$a_i^* \mapsto m_{z_i}.$$

A typical element, in this representation, is like:

$$z_1^3 \frac{\partial}{\partial z_2} + z_2 z_3 \frac{\partial^2}{\partial z_2 \partial z_1}.$$

So  $W_n = \{\text{polynomial coefficient differential operators}\}$ . This becomes a Lie algebra with

$$[a, b] = ab - ba$$

and then it has lots of Lie subalgebras, including:

2. {Polynomial coefficient vector fields} = {polynomial coefficient 1<sup>st</sup>-order differential operators}
3. {Homogeneous linear coefficient vector fields}  $\cong \mathfrak{gl}(n) = \{n \times n \text{ matrices}\}$
4. The **Heisenberg Lie algebra**,  $\mathfrak{h}_n$ , all linear combinations of  $a_i, a_i^*, 1$ . This is a  $(2n + 1)$ -dimensional Lie algebra.
5. {Homogeneous quadratic expressions in  $a_i$  and  $a_i^*$ , plus constants}  
This is closed under  $[\cdot, \cdot]$ . We have:

$$a_i^* a_j, i_j = 1, \dots, n$$

$$a_i a_j, i < j$$

$$a_i^* a_j^*, i < j$$

$$1$$

Then  $\dim = n^2 + n(n - 1) + 1 = 2n^2 - n + 1$ .

Homework: Work out:

$$[a_i^* a_j, a_k^* a_l] = ?$$

$$[a_i a_j, a_k^* a_l^*] = ?$$

Are these linear combinations of  $a_i^* a_j, a_i a_j, a_i^* a_j^*$  or do we get constant terms?