

# Geometric Representation Theory seminar, Lecture 29

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## 1 Hall Module

We have an abelian category

$$A = \text{Rep}_F(A_2).$$

Our quiver looks like

$$A_2 = \cdot \rightarrow \cdot$$

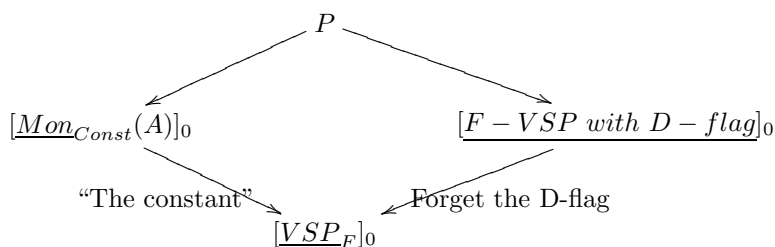
and we consider [monomorphisms over constant reps]<sub>0</sub>

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow \sim & & \downarrow \sim \\ R'' & \longrightarrow & R''' \end{array}$$

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ V & \xrightarrow{1} & V \end{array}$$

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & V \\ \downarrow \sim & & \downarrow \sim & & \downarrow 1 \\ Y' & \longrightarrow & Z' & \longrightarrow & V \end{array}$$

We define  $V = \underline{VSP}_F$  and  $D =$  Young diagram.



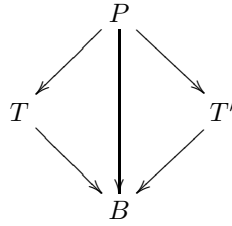
We are going to take the ‘homotopy pullback’ or ‘weak pullback’ of this diagram  $P$ .

Equipping a vector space with 2 structures:

1. “Quiver rep embedded in it”
2. D-flag

Let’s think about “homotopy pullbacks” (same as weak pullback in this context). This is based on the notion of pullback which is sometimes called “fiber product”. So we can say the homotopy

pullback is the “[homotopy fiber] product”. The “fiber of the pullback = product of the fibers of the factors”.



A fiber is given by the inverse image over a point  $1 \rightarrow B$ . A homotopy fiber of a functor

$$G_1 \xrightarrow{F} G_2$$

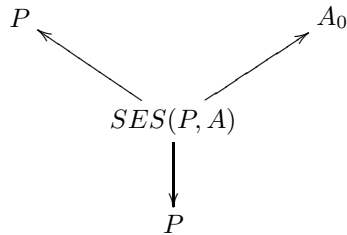
between groupoids is the “moral” fiber.

$$\begin{array}{ccc}
 \underline{Ring} & \xlongequal{\quad} & G_1 \\
 U \downarrow & & F \downarrow \\
 \underline{Set} & & G_2
 \end{array}$$

The concept of homotopy fiber includes any object not just in the inverse, but any object sitting over something isomorphic to the object in the base.

(All of this is getting us very close to earlier conversations about Hecke operators and classifying relationships between pairs of flags on a vector space.)

Here is a brief description of the tri-span which give the action of the groupoid  $A_0$  that defines the Hall algebra on the Hall module  $P$ , which was our homotopy pullback above.



The short exact sequence looks like:

$$\begin{array}{ccccc}
 0 & & 0 & & \\
 \downarrow & & \downarrow & & \\
 A & \longrightarrow & B & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow 1 \\
 C & \longrightarrow & D & \longrightarrow & X \\
 \downarrow & & \downarrow & & \\
 E & \longrightarrow & F & & \\
 \downarrow & & \downarrow & & \\
 0 & & 0 & &
 \end{array}$$

$A_2$   
 $SL(3)$

We want to classify “vector spaces equipped with a D-flag” where  $D$  has two boxes in first row and 1 in second row. This is a “3D vector space equipped with 2D subspace”. There is 1 isomorphism class of this sort. The other structure that we are interested in is the quiver representations. This is a comparable pair of subspaces, one inside the other, inside a vector space. This is called a “vector space equipped with an  $A_2$ -representation embedded in it”. There is no constraint on dimension of vector space on this side. Then we take the pullback and we want to count the isomorphism classes of “3D vector space  $V$  equipped with  $A_2$ -representation

$$A \longrightarrow B \longrightarrow V$$

embedded in it and a 2d subspace

$$C \longrightarrow V''$$

$\dim A$	$\dim B$	
0	0	(only 1 way)
0	1	$\dim(B \cap C) = 0, 1$
0	2	$\dim(B \cap C) = 1, 2$
0	3	$\dim(B \cap C) = 2$
1	1	2 ways
1	2	$\dim(B \cap C) = 1, 2$ (total of 3 ways)
1	3	2 ways
2	2	2 ways
2	3	2 ways
3	3	1 ways

This is a total of 18 ways!