Geometric Representation Theory seminar, Lecture 29

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1 Hall Module

We have an abelian category

$$A = Rep_F(A_2).$$

Our quiver looks like

$$A_2 = \cdot \rightarrow \cdot$$

and we consider [monomorphisms over constant reps]$_0$

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "\sim"] & R' \arrow[d, "\sim"] \\ R'' \arrow[r, tail] & R''' \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, tail] \arrow[d] & Z \arrow[d] \\ V \arrow[r, "1"'] & V \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, tail] \arrow[d, "\sim"] & Z \arrow[r, tail] \arrow[d, "\sim"] & V \arrow[d, "1"] \\ Y' \arrow[r, tail] & Z' \arrow[r, tail] & V \end{tikzcd}$$

We define $V = \underline{VSP_F}$ and $D =$ Young diagram.

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ {[\underline{Mon}_{Const}(A)]_0} \arrow[dr, "\text{``The constant''}"'] & & {[\underline{F-VSP\ with\ D-flag}]_0} \arrow[dl, "\text{Forget the D-flag}"] \\ & {[\underline{VSP}_F]_0} & \end{tikzcd}$$

We are going to take the 'homotopy pullback' or 'weak pullback' of this diagram P.

Equpping a vector space with 2 structures:

1. "Quiver rep embedded in it"
2. D-flag

Let's think about "homotopy pullbacks" (same as weak pullback in this context). This is based on the notion of pullback which is sometimes called "fiber product". So we can say the homotopy pullback is the "[homotopy fiber] product". The "fiber of the pullback = product of the fibers of the factors".

$$\begin{tikzcd}
 & P \arrow[dl] \arrow[dd] \arrow[dr] & \\
T \arrow[dr] & & T' \arrow[dl] \\
 & B &
\end{tikzcd}$$

A fiber is given by the inverse image over a point $1 \to B$. A homotopy fiber of a functor

$$G_1 \xrightarrow{F} G_2$$

between groupouids is the "moral" fiber.

$$\begin{tikzcd}
\underline{Ring} \arrow[r, equal] \arrow[d, "U"'] & G_1 \arrow[d, "F"'] \\
\underline{Set} & G_2
\end{tikzcd}$$

The concept of homotopy fiber includes any object not just in the inverse, but any object sitting over something isomorphic to the object in the base.

(All of this is getting us very close to earlier conversations about Hecke operators and classifying relationships between pairs of flags on a vector space.)

Here is a brief description of the tri-span which give the action of the groupoid A_0 that defines the Hall algebra on the Hall module P, which was our homotopy pullback above.

$$\begin{tikzcd}
P & & A_0 \\
 & SES(P,A) \arrow[ul] \arrow[ur] \arrow[d] & \\
 & P &
\end{tikzcd}$$

The short exact sequence looks like:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, tail] & X \arrow[d, "1"] \\
C \arrow[r, tail] \arrow[d, two heads] & D \arrow[r, tail] \arrow[d, two heads] & X \\
E \arrow[r] \arrow[d] & F \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

A_2
$SL(3)$
We want to classify "vector spaces equipped with a D-flag" where D has two boxes in first row and 1 in second row. This is a "3D vector space equipped with 2D subspace". There is 1 isomosphism class of this sort. The other structure that we are interested in is the quiver representations. This is a comparable pair of subspaces, one inside the other, inside a vector space. This is called a "vector space equipped with an A_2-representation embedded in it". There is no constraint on dimension of vector space on this side. Then we take the pullback and we want to count the isomorphism classes of "3D vector space V equipped with A_2-representation

$$A \rightarrowtail B \rightarrowtail V$$

embedded in it and a 2d subspace

$$C \rightarrowtail V\text{"}$$

dim$A$	dim$B$	
0	0	(only 1 way)
0	1	$\dim(B \cap C) = 0, 1$
0	2	$\dim(B \cap C) = 1, 2$
0	3	$\dim(B \cap C) = 2$
1	1	2 ways
1	2	$\dim(B \cap C) = 1, 2$ (total of 3 ways)
1	3	2 ways
2	2	2 ways
2	3	2 ways
3	3	1 ways

This is a total of 18 ways!