Suppose $\mathfrak{g}$ is a simple (complex) Lie algebra. There is a simple Lie group $G$ associated to it — i.e., having no normal subgroups except $1 \leq G$.

Pick a maximal abelian subgroup $H \leq G$ (any two are isomorphic). This has a Lie algebra $\mathfrak{h} \leq \mathfrak{g}$. $\mathfrak{h}$ has a god-given inner product on it, i.e., a lattice $L \leq \mathfrak{h}$ given by:

$$L = \ker (\exp : \mathfrak{h} \to H \leq G)$$

Sometimes $L$ has a basis of vectors that are all the same length and at angles 90° or 120°. Then you can draw a "simply-laced Dynkin diagram":

```
x1 -- x2 -- x3
```

An edge between dots corresponding to basis vectors $x_i$, $x_j$ if they are at 120° angle; no edge otherwise.
The only diagrams you get this way are $A_n$, $D_n$, $E_6$, $E_7$, $E_8$.

Example: $g = sl(n, \mathbb{C})$

Here $h = \{\text{diagonal matrices w. trace zero}\}$ and

$$\exp: h \rightarrow H$$

$$\begin{pmatrix} \alpha_1 & \cdots & 0 \\ 0 & \cdots & \alpha_n \end{pmatrix} \rightarrow \begin{pmatrix} e^{\alpha_1} & \cdots & 0 \\ 0 & \cdots & e^{\alpha_n} \end{pmatrix}$$

So

$$L = \left\{ \begin{pmatrix} \alpha_1 & \cdots & 0 \\ 0 & \cdots & \alpha_n \end{pmatrix} : \alpha_1 + \cdots + \alpha_n = 0 \right\}$$

If $g = sl(3)$ then $L$ has a basis

$$\begin{pmatrix} 2\pi \\ -2\pi \\ 0 \end{pmatrix} \quad \begin{pmatrix} \pi \\ 2\pi \\ -2\pi \end{pmatrix}$$

which are at a $120^\circ$ angle with respect to

$$\langle x, y \rangle = \text{ctr}(xy)$$
In this general situation we always have:

$$g = n_- \oplus h \oplus n_+$$

as vector spaces where $h$ is maximal abelian ("Cartan") and $n_\pm$ are maximal nilpotent.

Example: If $g = \mathfrak{sl}(n)$ then

$$g = \begin{cases} 
\begin{pmatrix} 0 \\ \ast & \ast \end{pmatrix} 
\oplus 
\begin{pmatrix} 0 \\ \ast \end{pmatrix} 
\oplus 
\begin{pmatrix} 0 \\ \ast \end{pmatrix} 
\oplus 
\begin{pmatrix} 0 \\ \ast \end{pmatrix} 
\end{cases} 
\oplus 
\begin{pmatrix} 0 \\ \ast \end{pmatrix} 
$$
In this example
\[ [h, n^+] \leq n^+ \]

and this holds in general.

In this example of \( g = SL(n) \), \( n^+ \) has a basis of elementary matrices \( e_{ij} \) \((i < j)\)

Ringel-Hall - If \( Q = A_n, D_n, E_6, E_7, E_8 \) then for any prime power we have an isomorphism of associative algebras:

\[ \text{Hall}(\text{Rep}_q(Q)) \cong U_q n^+ \]

where

\[ \text{Rep}_q(Q) = \text{hom}(Q, \text{Fin Vect}_{F_q}) \]

and \( U_q n^+ \) is the "\( q \)-deformed universal enveloping algebra of \( n^+ \)" part of the "quantum group" \( U_q g \).
But what is $\mathfrak{u}_n$? First, what is $\mathfrak{u}_1$?

$\mathfrak{u}$ means "universal enveloping algebra" - the trick for turning Lie algebras into associative algebras. There

is a functor

\[
\begin{array}{ccc}
\text{Assoc Alg} & \xrightarrow{\quad} & \mathfrak{u} \text{- Alg} \\
\downarrow & & \downarrow F \\
\text{Lie Alg} & & \\
\end{array}
\]

where given $A \in \text{AssocAlg}$, $FA$ has the same underlying vector space with Lie bracket:

$$[a,b] = ab - ba.$$ 

This functor has a left adjoint:

\[
\begin{array}{ccc}
\mathfrak{u} \text{- Alg} & \xleftarrow{\quad} & \text{Assoc Alg} \\
& & \downarrow F \\
& & \text{Lie Alg} \\
\end{array}
\]
i.e. given \( A \triangleleft \text{Assoc Alg} \) \& \( L \triangleleft \text{Lie Alg} \)

\[
\text{hom} (L, FA) \equiv \text{hom} (UL, A)
\]

(set of Lie algebra homomorphisms) (set of assoc.
alg. homomorphisms)

Concretely, \( UL \) is the assoc. alg. freely generated
by elements of \( L \) modulo relations:

\[
\begin{align*}
\xi (\alpha y + \beta z) &= \alpha xy + \beta xz \\
(\alpha x + \beta y)z &= \alpha xz + \beta yz \\
xy - yx &= [x, y]
\end{align*}
\]

Then given Lie algebra homomorphism \( f: L \rightarrow FA \)

it extends uniquely to an assoc. alg. homomorphism

\[
\tilde{f}: UL \rightarrow A
\]

\[
\begin{array}{ccc}
UL & \xrightarrow{\tilde{f}} & A \\
\downarrow & & \downarrow \\
L & \xrightarrow{f} & FA
\end{array}
\]
Example: \[ L = \mathfrak{n}_+ \leq \mathfrak{g}_f = \mathfrak{sl}(n, C) \]
\[ A = \text{End}(C^n) \]

Then \[ FA = \mathfrak{gl}(n, C) \] - all non-\ncommutative \n\[ \text{w. } [a, b] = ab - ba \]

\[ UL = U \mathfrak{n}_+ \]

We have an inclusion of Lie algebras,

\[ f: L \rightarrow FA \]
\[ \downarrow \]
\[ \mathfrak{n}_+ \rightarrow \mathfrak{gl}(n, C) \]

So we get an assoc. alg. homo:

\[ \tilde{f}: UL \rightarrow A \]
\[ \downarrow \]
\[ U\mathfrak{n}_+ \rightarrow \text{End}(C^n) \]

where \( \tilde{f} \) sends any \( x \in \mathfrak{n}_+ \) to the element of \( \text{End}(C^n) \) that it secretly is!
We also have an assoc. alg. homomorphism

$$\text{End}(C^n) \rightarrow \text{End}(SC^n)$$

where $SC^n$ is the commutative assoc. algebra generated by $C^n$ — i.e., $C[x_1, \ldots, x_n]$ — polynomials in $n$ variables. This is the "Fock space" we have been talking about all along! So we get

$$U_n^+ \rightarrow \text{End}(C^n) \rightarrow \text{End}(SC^n)$$

which sends $e_{ij} \in U_+$ to the "transmutation operator" $a_i^* a_j$. This is the thing we would like to categorify (we have done that, getting groupoidified transmutation operators $A_i^* A_j$ for all $i, j$) i.e., q-deform (but here we will be stuck with $i \leq j$, since we are only using $U_n^+$ instead of $U_{q^n}$).

Now, what is $U_n^+ q$? As a vector space, it is isomorphic to $U_n^+$, but it has a different product, i.e., depending on $q \in \mathbb{K} \subseteq C$, with
Next time we will assume

\[ U_{q^n} \subseteq \text{Hall}(\text{Rep}_q(Q)) \]

Show

\[ U_{q^n} = U_{n^q} \]

as vector spaces in some nice way.

Work out the product in \( U_{q^n} \) and show

\[ x \cdot_q y \rightarrow xy \quad \text{as} \quad q \rightarrow 1. \]