QUANTUM GEOMETRY & ITS APPLICATIONS

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Basic assumptions of loop quantum gravity:

1) We get <u>some</u> insight into QG by attempting to quantize GR without any special choice of matter fields.

2) The theory should be background-independent, but

3) some insight can be obtained making a 3+1 split of spacetime – "canonical quantization".

4) Basic geometric variables are holonomies of Ashtekar connection along paths in space.

$\Rightarrow$ QUANTUM GEOMETRY

THE BASIC FIELDS

- Ashtekar connection ($SU(2)$ gauge field) :

$$A_i^a = \Gamma_i^a - \gamma K_i^a$$

Levi-Civita connection

extrinsic curvature

Barbero-Immirzi parameter

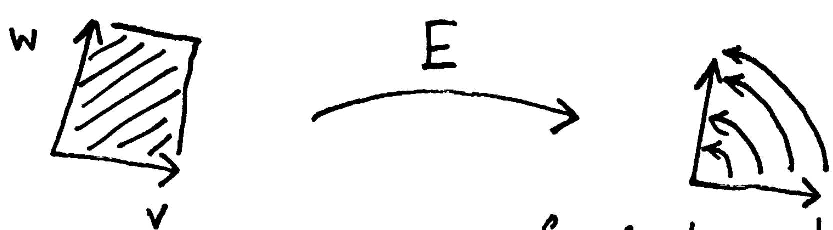
$i, j = 1,2,3$ (space)

$a, b = 1,2,3$ (internal)

- Area field ($\mathfrak{su}(2)$-valued 2-form) :

$$E_{jka} = \varepsilon_{abc}\, e_j^b\, e_k^c$$

"3-bein", aka "triad", aka "frame field"



$E(v,w) \in \mathfrak{su}(2)$

"infinitesimal parallelogram"

"infinitesimal rotation" with magnitude = <u>area</u> of parallelogram

In general relativity, A & E are <u>canonically</u> <u>conjugate</u>:

$$\{A_i^a(x), E_{jkb}(y)\} = 8\pi G \gamma \, \delta_b^a \, \varepsilon_{ijk} \, \delta(x-y)$$

They satisfy 3 <u>constraints</u>:

1) Gauss law (generates gauge transformations)

"$\mathrm{div}\, E = 0$"

2) Diffeomorphism constraint (generates diffeomorphisms of space)

"$G_{0i} = 8\pi G T_{0i}$"

3) Hamiltonian constraint (generates time evolution)

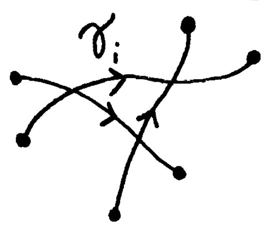
"$G_{00} = 8\pi G T_{00}$"

QUANTIZATION

A ~ "position" E ~ "momentum"

To quantize, we start with a Hilbert space of wavefunctions $\psi(A)$. Then we impose constraints.

Assumption: Allowed wavefunctions include those depending on holonomies (= parallel transport) along finitely many paths in space:



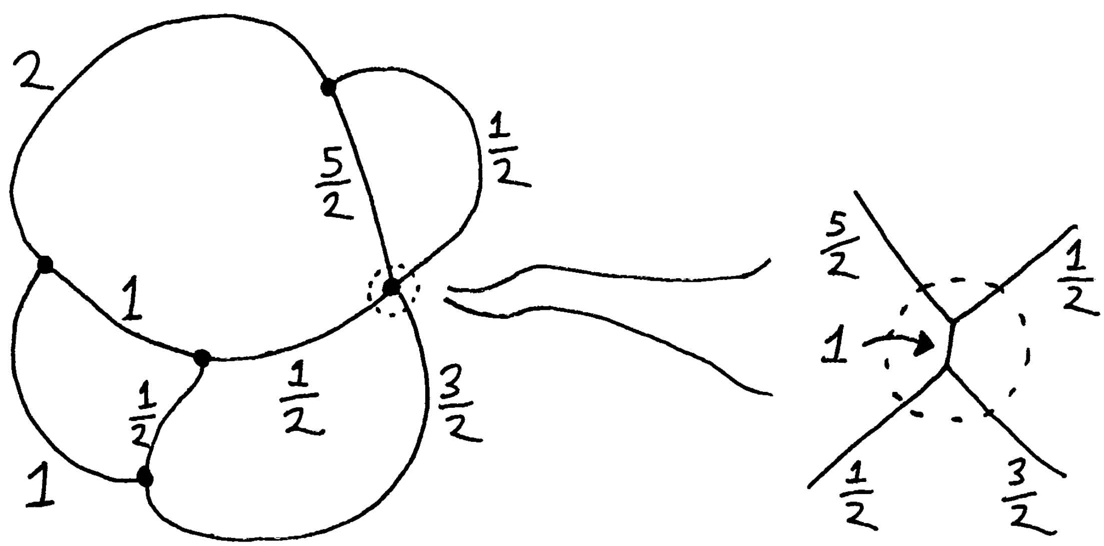
$$g_i = P e^{\int_{\gamma_i} A} \in SU(2)$$

$$\psi(A) = f(g_1, \ldots, g_n)$$

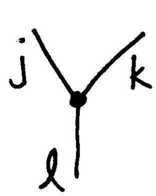
is an allowed wavefunction if

$$\int_{SU(2)^n} |f|^2 \, dg_1 \cdots dg_n < \infty$$

Completing, we obtain a Hilbert space. The Gauss law picks out states in here that are <u>gauge-invariant</u>, defining a subspace $L^2(\mathcal{A}/\mathcal{G})$. This has a basis given by "spin networks":



where each vertex has



$$\ell = |j-k|,\ |j-k|+1, \ldots, j+k$$

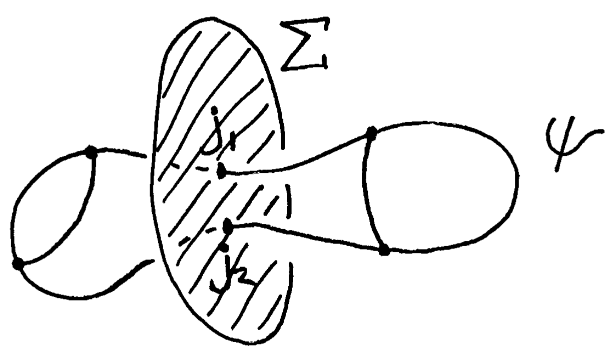
from "div $E = 0$".

OBSERVABLES

Spin networks describe quantum states of the geometry of space. To understand their meaning we need gauge-invariant "observables" – operators on $L^2(\mathcal{A}/\mathcal{G})$.

1) Area operators:

$\int_\Sigma |E|$ measures area of surface Σ:



$$\int_\Sigma |E| \psi = 8\pi \ell_p^2 \gamma \sum_i \sqrt{j_i(j_i+1)}\ \psi$$

(if ψ intersects Σ transversely)

Area is quantized!! Spin network edges are "flux tubes of area"!!

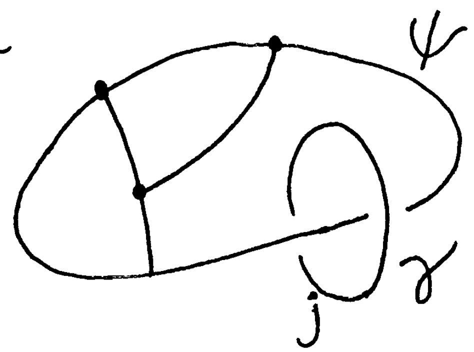
2) Volume operators:

Volume is quantized too. Volume comes from spin network <u>vertices</u>.

3) Wilson loops:

$\mathrm{tr}\,(e^{\oint_\gamma A})$ creates a flux loop of the E field:

in spin-j representation



$$\mathrm{tr}(e^{\oint_\gamma A})\,\psi = \psi \cup \gamma$$

(if ψ doesn't intersect γ)

All states can be built from the "empty state" by applying Wilson loop operators!!

BEYOND KINEMATICS

After finding states that satisfy the Gauss law we must go on to tackle:

1) Diffeomorphism constraint – easy.

Use spin networks mod diffeomorphisms.

2) Hamiltonian constraint – hard!

"Problem of time" makes it hard to see if proposed solutions are correct. So:

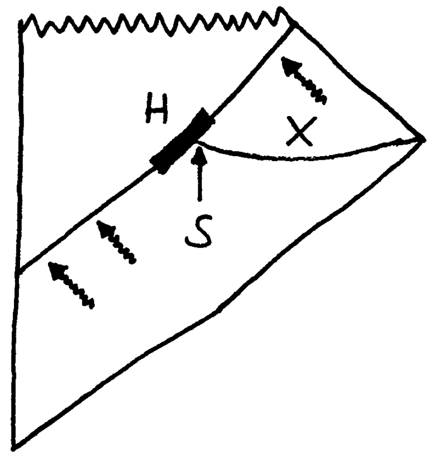
a) work hard! – see Lewandowski & Thiemann

b) try "spin foam models" – see Rovelli

c) study black holes

d) study quantum cosmology – see Bojowald

BLACK HOLES



Classically, an "isolated horizon" H is a surface in spacetime satisfying some conditions that <u>imply</u>:

1) H is null and $\cong \mathbb{R} \times S^2$

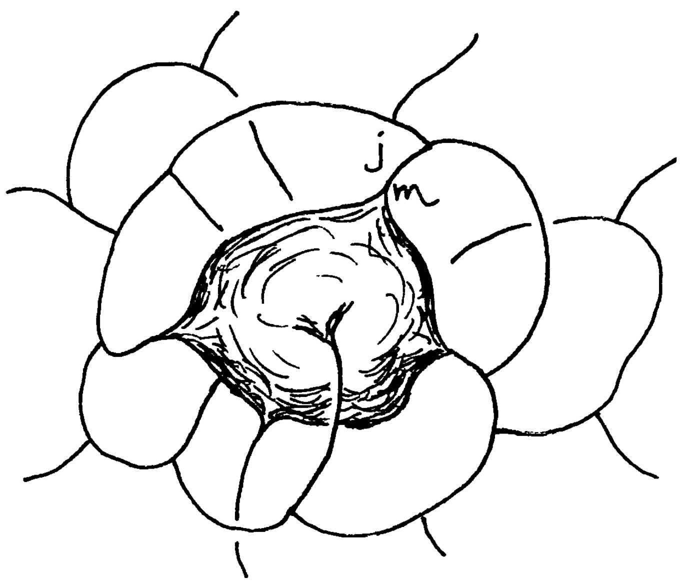
2) S is outer marginally trapped.

3) no gravitational radiation or matter falls in H.

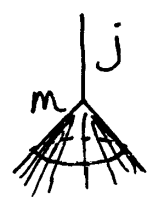
4) the area A of S is time-independent.

We can do loop quantum gravity on a slice X for which fields satisfy <u>isolated horizon boundary conditions</u> on S.

Before imposing Hamiltonian constraint, we get this picture of states:



Spin networks puncture the horizon at points labelled by numbers $m = -j, -j+1, \ldots, j-1, j$ which describe quantized angle deficits:



The curvature of the horizon is concentrated at these punctures.

Assuming Hamiltonian constraint has <u>at least</u> <u>one</u> solution for each list $j_1, \ldots, j_N, m_1, \ldots, m_N$ labelling punctures, we can count states of <u>horizon geometry</u> with area very near A:

$$A \cong 8\pi \ell_p^2 \gamma \sum_{i=1}^{N} \sqrt{j_i(j_i+1)}$$

The vast majority of these states have $j_i = \frac{1}{2}$, $m_i = \pm\frac{1}{2}$. For these

$$A \cong 8\pi \ell_p^2 \gamma \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)}\; N$$

$$= 4\pi\sqrt{3}\, \ell_p^2 \gamma N$$

so number of punctures is

$$N \cong \frac{1}{4\pi\sqrt{3}\gamma} \frac{A}{\ell_p^2}$$

and number of horizon states is about

$$2^N \cong 2^{\frac{1}{4\pi\sqrt{3}\gamma}\frac{A}{\ell_p^2}}$$

so entropy is

$$S \sim \frac{\ln 2}{4\pi\sqrt{3}\,\gamma} \frac{A}{\ell_p^2}$$

Now

$$S \sim \frac{\ln 2}{4\pi\sqrt{3}\,\gamma} \frac{A}{\ell_p^2}$$

agrees with Hawking's semiclassical

$$S = \frac{1}{4} \frac{A}{\ell_p^2}$$

if

$$\gamma = \frac{\ln 2}{\pi\sqrt{3}}$$

This gives a "quantum of area" — area of spin-$\frac{1}{2}$ puncture — equal to

$$8\pi\gamma\,\ell_p^2 \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)} = 4\ln 2\ \ell_p^2$$

In short, we've determined the Barbero-Immirzi parameter γ and found a black hole has one bit of information per quantum of area!!

$$S = \underbrace{\ln 2}_{\text{bit}} \cdot \frac{A}{\underbrace{4\ln 2\ \ell_p^2}_{\text{quantum of area}}}$$

QUANTUM COSMOLOGY

In quantum cosmology, we get around the problem of time by assuming the geometry of spacetime takes a special form (e.g. Friedmann-Robertson-Walker) before quantizing, reducing the problem to one with finitely many degrees of freedom, and using the size of the universe (a) as a clock. In the usual Wheeler-DeWitt approach, quantum cosmology is singular at the Big Bang. In loop quantum cosmology we can <u>extrapolate</u> <u>through</u>, essentially because the discreteness of quantum geometry gives a <u>difference</u> <u>equation</u> that "steps over $a=0$".