Comparing Operator Topologies

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Problem 1. If $R\colon \ell^2 \to \ell^2$ is the right shift operator, show that the sequence R^i converges weakly to 0, but not strongly.

Proof. Recall that a sequence of operators $T_i\colon H \to H$ converges weakly to T if

$$\langle \Phi, (T_i - T)\Psi \rangle \to 0$$

for all $\Phi, \Psi \in H$. So, we need to show that for any $\Phi, \Psi \in \ell^2$ we have $\langle \Phi, R^i\Psi \rangle \to 0$. Since the right shift operator is the adjoint of the left shift operator L, we have

$$\langle \Phi, R^i\Psi \rangle = \langle L^i\Phi, \Psi \rangle$$

and thus

$$|\langle \Phi, R^i\Psi \rangle| \leq \|L^i\Phi\| \|\Psi\|$$

by the Cauchy–Schwarz inequality. By Problem 2, $L^i \to 0$ strongly, which means that $\|L^i\Phi\| \to 0$ for all Φ. So,

$$\langle \Phi, R^i\Psi \rangle \to 0$$

as desired.

Next, recall that a sequence of operators T_i converges strongly to T if $\|T_i\Phi - T\Phi\| \to 0$ for all $\Phi \in H$. To see that R^i does not converge strongly to 0, we only need to find a vector $\Phi \in \ell^2$ such that $\|R^i\Phi\|$ does not go to zero. Let $\Phi = (1, 0, 0, \ldots)$. Notice that this is a unit vector in ℓ^2. Also note that R^i preserves the norm on any vector in ℓ^2. This gives us that $\|R^i\Phi\| = 1$ for all i, and so $\|R^i\Phi\| \to 1 \neq 0$. Thus R^i does not converge strongly to 0.

$\square$

Problem 2. If $L = R^* \colon \ell^2 \to \ell^2$ is the left shift operator, show that the sequence L^i converges strongly to 0 (and thus weakly), but does not converge to 0 in norm.

Proof. In general, we can think of the left shift operators L^i as "removing" the first i terms of a vector $\Phi \in \ell^2$. So to see that L^i converges strongly to 0, we will write the norm of $L^i\Phi$ as the difference of two sums:

$$\left\|L^i\Phi\right\|^2 = \sum_{j=1}^{\infty} \Phi_j^2 - \sum_{j=1}^{i} \Phi_j^2.$$

If we consider the limit as $i \to \infty$ this norm goes to 0. Thus L^i converges strongly to 0.

Finally, recall that a sequence of operators T_i converges to T in norm if $\|T_i - T\| \to 0$. So we need to show that $\|L^i\|$ does not converge to zero. By definition, $\left\|L^i\right\| = \sup_{\|\Phi\|=1} \left\|L^i\Phi\right\|$. But for any i consider the vector $E_{i+1} = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the ith coordinate. We see that $\|E_{i+1}\| = 1$ and $\|L^i E_{i+1}\| = 1$. This gives us that $\|L^i\| \geq 1$ for all i. Thus the sequence $\|L^i\|$ cannot possibly converge to 0. Thus L^i does not converge to 0 in norm. $\square$