WHY IS CANONICAL QUANTUM GRAVITY HARD?

Consider the vacuum Einstein equations

$$G_{\mu\nu} = 0$$

where

S is a (compact) smooth 3-manifold - "space"

$M = \underset{\underset{t}{\Psi}}{\mathbb{R}} \times S$ - "spacetime"

g is a Lorentzian metric such that
$\{t=0\} \subseteq M$ is a Cauchy surface

$R_{\mu\nu}$ = Ricci tensor of g $\qquad R = R^{\mu}_{\mu}$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor

Naively speaking, in quantum gravity we'd like to make the space of solutions into a Poisson manifold & then quantize it. In <u>canonical</u> quantum gravity we describe solutions using initial data.

There are many tricks for quantizing Poisson manifolds. E.g. for a particle on the line:

$$\underbrace{T^*\mathbb{R}}_{\substack{\text{Poisson manifold or} \\ \text{"phase space"}}} \ni (q,p)$$

q = position
p = momentum

$\downarrow$ quantize

$$\underbrace{L^2(\mathbb{R})}_{\text{Hilbert space}}$$

$\hat{q} = M_q$
$\hat{p} = \frac{1}{i}\frac{d}{dq}$

For Einstein's equation this is hard because:

1) The phase space is infinite-dimensional, with singularities.

2) The phase space is not a cotangent bundle, due to <u>constraints</u>.

3) Problem of time: the phase space really consists of solutions <u>modulo</u> <u>diffeomorphisms</u> of spacetime.

CONSTRAINED SYSTEMS – CLASSICAL & QUANTUM

An easy example: the particle on a plane, constrained to lie on a line.

Classical particle on plane:

Configuration space $= \mathbb{R}^2 \ni (q_1, q_2)$

Phase space $= T^*\mathbb{R}^2 \ni (q_1, q_2, p_1, p_2)$

Poisson structure:

$$\{q_j, q_k\} = \{p_j, p_k\} = 0,$$

$$\{p_j, q_k\} = \delta_{jk}$$

Quantum particle on plane:

Hilbert space $= L^2(\mathbb{R}^2)$

Operators: $\hat{q}_j = M_{q_j}$, $\hat{p}_j = \frac{1}{i}\frac{\partial}{\partial q_j}$

NOW: Impose constraint $q_2 = 0$!

$$\begin{tikzcd}
T^*\mathbb{R}^2 \arrow[rr, "\text{QUANTIZE}"] \arrow[d, "\begin{array}{l} q_2 = 0; \\ \text{mod out by } p_2 \sim p_2 + \text{const.} \end{array}", "\begin{array}{c}\text{CONSTRAIN} \\ \text{(POISSON} \\ \text{REDUCTION)}\end{array}"'] & & L^2(\mathbb{R}^2) \arrow[d, "\text{CONSTRAIN}"', "\hat{q}_2 \psi = 0"] \\
T^*\mathbb{R} \arrow[rr, "\text{QUANTIZE}"'] & & L^2(\mathbb{R})
\end{tikzcd}$$

POISSON REDUCTION

Constraining classical systems is done via Poisson reduction: starting with a Poisson manifold X and smooth function $f : X \to \mathbb{R}$, we do a 2-step process:

$$
\begin{tikzcd}
X \arrow[r, "\text{FUNCTIONS ON...}"] \arrow[d, bend right, "\text{SUBSPACE}"'] & C^\infty(X) \ \text{Poisson algebra} \arrow[d, bend left, "\text{QUOTIENT ALGEBRA}"] \\
\{f=0\} \subseteq X \arrow[r] \arrow[d, bend right, "\text{QUOTIENT SPACE}"'] & C^\infty(X)/[f] \arrow[d, bend left, "\text{SUBALGEBRA}"] \\
\{f=0\}/\sim \arrow[r] & \{g \in C^\infty(X) : \{f,g\}=0\}/[f]
\end{tikzcd}
$$

X: "kinematical phase space"

$C^\infty(X)$: "kinematical observables"

$\{f=0\} \subseteq X$: "constraint surface"

Suppose $v_f = \{f, \cdot\}$ generates flow $F_t : X \to X$. Let

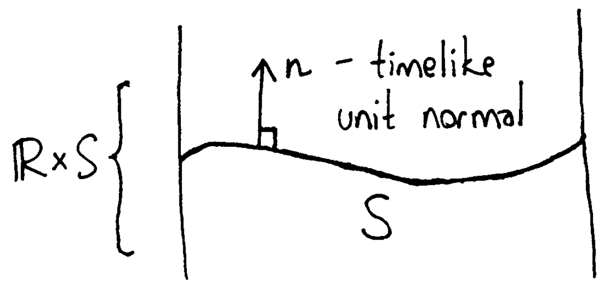
$$x \sim y \iff x = F_t y \quad \exists t \in \mathbb{R}$$

$\{f=0\}/\sim$: "physical phase space"

$\{g \in C^\infty(X) : \{f,g\}=0\}/[f]$: "physical observables"

Similarly for a <u>Lie algebra</u> of functions on X which generate a group action.

EINSTEIN'S EQUATIONS: THE INITIAL VALUE PROBLEM



$i, j = 1, 2, 3$

Kinematical configuration space $= \mathcal{M}et(S) \ni q$
(space of Riemannian metrics on S)

Kinematical phase space $= T^*(\mathcal{M}et(S)) \ni (q, p)$

where $p^{ij} = \sqrt{\det(q)}\,(K^{ij} - K\, q^{ij})$

$$K_i^j = \nabla_i n^j = \text{extrinsic curvature}$$

$$K = K_i^i$$

Poisson structure:

$$\{ p^{ij}(x), q_{k\ell}(y) \} = \delta_k^i \delta_\ell^j \, \delta(x, y)$$

4 of Einstein's 10 equations are constraints on the initial data (q,p):

- DIFFEOMORPHISM CONSTRAINT:

$G_{0i} = 0$ where $G_{0i} = -2 D_i p^i_j$.

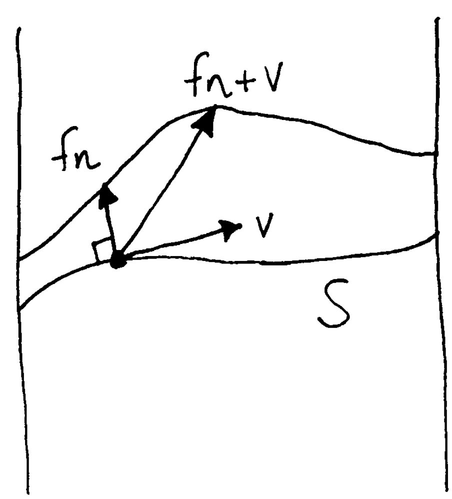
These generate action of $\mathrm{Diff}_0(S)$ on $T^*(\mathcal{M}et(S))$: $\int_S v^i G_{0i} \, \mathrm{vol}$ generates action of flow corresponding to vector field v.

- HAMILTONIAN CONSTRAINT:

$$G_{00} = 0 \quad \text{where} \quad G_{00} = -\sqrt{\det(q)}\, {}^3R + \frac{1}{\sqrt{\det(q)}}\left(p^{ij}p_{ij} - \tfrac{1}{2}(p^i_i)^2\right).$$

This generates action of time translation in n direction: $\int_S f\, G_{00} \, \mathrm{vol}$ generates action of flow corresponding to fn.

In short: in general relativity, unlike most field theories set on a manifold with a fixed metric, the <u>dynamics is generated by constraints</u>:



The function

$$\int_S (v^i G_{0i} + f\, G_{00})\, \mathrm{vol}$$

on $T^*(\mathrm{Met}(S))$ generates time evolution pushing S forwards in the direction $fn+v$.

So: to form the physical phase space, we take the kinematical phase space & do Poisson reduction twice:

$$\begin{array}{ll}
\text{Kinematical Phase Space} & T^*(\underbrace{\mathrm{Met}(S)}_{\text{symplectic Hilbert manifold}}) \\
\Big\downarrow \text{diffeo. constraint} & \\
\text{Diffeomorphism-Invariant Phase Space} & T^*(\mathrm{Met}(S)/\underbrace{\mathrm{Diff}_0(S)}) \\
\Big\downarrow \text{Hamiltonian constraint} & \\
\text{Physical Phase Space} & X
\end{array}$$

not a manifold, but a Hilbert manifold on an open dense set, away from metrics w. symmetry — $B\,\mathrm{Diff}_0(S)$!

$X = \{$maximally extended solns of Einstein's equation w. Cauchy data on $S\}$ / diffeomorphism

not a manifold, but a symplectic Hilbert manifold on an open dense set, away from solutions w. symmetry

DIFFICULTIES

$$
\begin{tikzcd}[column sep=large, row sep=huge]
T^*(\mathcal{M}et(S)) \arrow[r, dotted, "\text{QUANTIZE}"] \arrow[d, bend right, "\text{CONSTRAIN}"'] \arrow[dr, phantom, "\text{commutes?}"] & L^2(\mathcal{M}et(S)) \arrow[d, dotted, "\text{CONSTRAIN}"'] & \text{what measure?} \\
T^*(\mathcal{M}et(S)/\mathrm{Diff}_0(S)) \arrow[r, dotted, "\text{QUANTIZE}"] \arrow[d, bend right, "\text{CONSTRAIN}"'] \arrow[dr, phantom, "\text{commutes?}"] & L^2(\mathcal{M}et(S)/\mathrm{Diff}_0(S)) \arrow[d, dotted, "\text{CONSTRAIN}"'] & \text{how?} \\
X \arrow[r, dotted, "\text{QUANTIZE}"] & \mathcal{H}_{phys} & \text{how?}
\end{tikzcd}
$$

how?

what measure?

how?

- Operators like
$$\hat{G}_{00} = \sqrt{\det(\hat{q})}\ {}^3\hat{R} + \frac{1}{\sqrt{\det(\hat{q})}}\left(\hat{p}^{ij}\hat{p}_{ij} - \frac{1}{2}(\hat{p}^i_i)^2\right)$$
difficult to define.

- Problem of time: states in $\mathcal{H}_{phys}$ will be <u>diffeomorphism</u>-<u>invariant</u>, so dynamics is hidden.

A ROUTE TO PROGRESS

"loop quantum gravity"

1) Describe initial data using new variables: an $SU(2)$ connection on spin bundle $P \to S$ and an $\mathrm{Ad}(P)$-valued 2-form. Kinematical phase space becomes $T^*(\mathcal{A})$, where $\mathcal{A}$ is the space of connections on P.

2) Use new technology involving graphs to define completion $\bar{\mathcal{A}}$ and measure on $\bar{\mathcal{A}}$.

3) Try again:

$$\begin{tikzcd}
T^*(\mathcal{A}) \arrow[r] \arrow[d, swap, "\text{Gauss constraint}"] & L^2(\bar{\mathcal{A}}) \arrow[d] \\
T^*(\mathcal{A}/\mathcal{G}) \arrow[r] \arrow[d, swap, "\text{diffeo. constraint}"] & L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}}) \arrow[d] \\
T^*(\mathcal{A}/\mathrm{Aut}\,P) \arrow[r, dotted] \arrow[d, swap, "\text{Hamiltonian constraint}"] & \mathcal{H}_{diff} \arrow[d, dashed, "\longleftarrow \text{ controversial!}"] \\
X \arrow[r, dotted] & \mathcal{H}_{phys}
\end{tikzcd}$$