GAUGE THEORY ON A GRAPH

For us, a <u>graph</u> γ is a finite set E of <u>edges</u>, a finite set V of <u>vertices</u>, and maps $s, t : E \to V$.

$$s(e) \xrightarrow{e} t(e)$$

Fixing a group G, a <u>connection</u> on γ is a map

$$A : E \to G$$

assigning to each edge e a <u>holonomy</u> $A_e \in G$.

A <u>gauge transformation</u> on γ is a map

$$g : V \to G$$

Gauge transformations act on connections via:

$$(gA)_e = g_{t(e)} A \, g_{s(e)}^{-1}$$

$$g_{s(e)} \xrightarrow{A_e} g_{t(e)}$$

Let

$$\mathcal{A} = \{\text{connections on } \gamma\} = G^E$$

$$\mathcal{G} = \{\text{gauge transformations on } \gamma\} = G^V$$

If G has a left- and right-invariant measure (Haar measure), $\mathcal{A}$ gets a $\mathcal{G}$-invariant measure, so $\mathcal{G}$ acts as unitary operators on $L^2(\mathcal{A})$.

Via

$$\begin{tikzcd} \mathcal{A} \arrow[d, dashed] \\ \mathcal{A}/\mathcal{G} \end{tikzcd}$$

the measure on $\mathcal{A}$ pushes forward to a measure on $\mathcal{A}/\mathcal{G}$, and:

$$L^2(\mathcal{A}/\mathcal{G}) \cong \{\psi \in L^2(\mathcal{A}) : g\psi = \psi\}$$

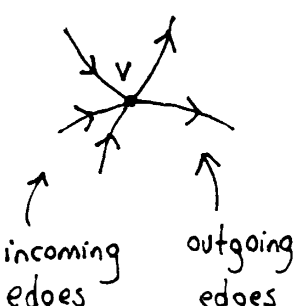
$$\begin{tikzcd} \mathcal{A} \arrow[d] \arrow[dr, dashed, "\psi"] & \\ \mathcal{A}/\mathcal{G} \arrow[r] & \mathbb{C} \end{tikzcd}$$

If G is compact, it has a unique Borel measure μ that is left- and right-invariant and has $\int_G \mu = 1$ — normalized Haar measure. Then we can describe $L^2(\mathcal{A})$ using the Peter-Weyl theorem:

$$L^2(G) \cong \bigoplus_{\rho \in \mathrm{Irrep}(G)} \rho \otimes \rho^*$$

so

$$\begin{aligned}
L^2(\mathcal{A}) &\cong L^2(G^E) \\
&\cong \bigotimes_{e \in E} L^2(G) \\
&\cong \bigotimes_{e \in E} \bigoplus_{\rho \in \mathrm{Irrep}(G)} \rho \otimes \rho^* \\
&\cong \bigoplus_{\rho : E \to \mathrm{Irrep}(G)} \bigotimes_{e \in E} \rho_e \otimes \rho_e^* \\
&\cong \bigoplus_{\rho : E \to \mathrm{Irrep}(G)} \bigotimes_{v \in V} \Big[\bigotimes_{e : t(e) = v} \rho_e^* \otimes \bigotimes_{e : s(e) = v} \rho_e \Big]
\end{aligned}$$



incoming edges   outgoing edges

$G \times G$ acts on $L^2(G)$ by left/right translations; Peter-Weyl says how:

$$L^2(G) \cong \bigoplus_{\rho \in \mathrm{Irrep}(G)} \underbrace{\rho \otimes \rho^*}_{\text{irrep of } G \times G}$$

This says how $\mathcal{G}$ acts on $L^2(\mathcal{A})$:

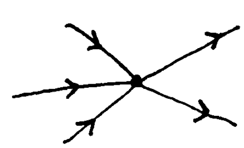
$$L^2(\mathcal{A}) \cong \bigoplus_{\rho : E \to \mathrm{Irrep}(G)} \; \bigotimes_{v \in V} \underbrace{\left[\bigotimes_{e : t(e) = v} \rho_e \otimes \bigotimes_{e : s(e) = v} \rho_e^* \right]}$$

If $g \in \mathcal{G}$, g_v acts on this factor in the obvious way! Thus:

$$L^2(\mathcal{A}/\mathcal{G}) \cong \bigoplus_{\rho : E \to \mathrm{Irrep}(G)} \; \bigotimes_{v \in V} \mathrm{Inv}\left[\bigotimes_{e : t(e) = v} \rho_e \otimes \bigotimes_{e : s(e) = v} \rho_e^* \right]$$

$$\cong \bigoplus_{\rho : E \to \mathrm{Irrep}(G)} \; \bigotimes_{v \in V} \mathrm{Hom}\left(\bigotimes_{e : s(e) = v} \rho_e , \bigotimes_{e : t(e) = v} \rho_e \right)$$

intertwiners



SPIN NETWORKS

Theorem – If γ is a graph & G is a compact group, an orthonormal basis for $L^2(\mathcal{A}/\mathcal{G})$ is given by all ways of labelling edges of γ by irreps $\rho_e \in \mathrm{Irrep}(G)$ and vertices by intertwiners

$$\iota_v : \bigotimes_{e:\, t(e)=v} \rho_e \to \bigotimes_{e:\, s(e)=v} \rho_e$$

running over any orthonormal basis of such intertwiners.

$\Psi = (\gamma, \rho, \iota)$ is called a <u>spin network</u>.

Proof –

$$L^2(\mathcal{A}/\mathcal{G}) = \bigoplus_{\rho: E \to \mathrm{Irrep}(G)} \bigotimes_{v \in V} \mathrm{Hom}\Big(\bigotimes_{e:\, t(e)=v} \rho_e \,,\, \bigotimes_{e:\, s(e)=v} \rho_e \Big)$$

EXAMPLES:

$G = U(1)$ — electromagnetism!

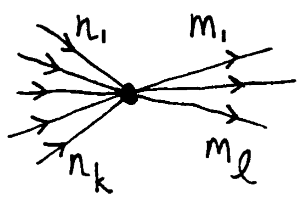
$\mathrm{Irrep}(G) \cong \mathbb{Z}$ — "charges"

All irreps are 1-dimensional: $\rho_n(e^{i\theta}) = e^{in\theta}$, $n \in \mathbb{Z}$.

$$\rho_n \otimes \rho_m \cong \rho_{n+m}$$

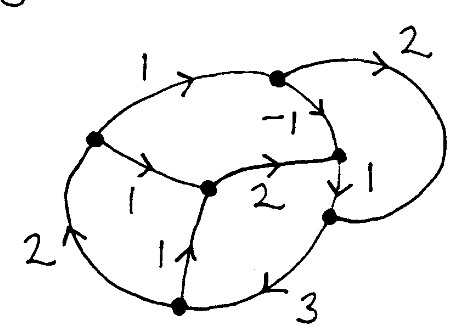
$$\rho_n^* \cong \rho_{-n}$$

Space of intertwiners:



is 1-dimensional if $n_1 + \cdots + n_k = m_1 + \cdots + m_\ell$, 0-dimensional otherwise.

$L^2(\mathcal{A}/\mathcal{G})$ has basis of "flux networks":



Faraday's electric field lines! $\vec{\nabla} \cdot \vec{E} = 0$!

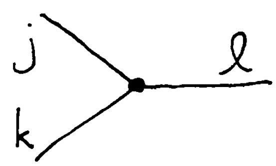
$G = SU(2)$ — weak force, gravity!

$\mathrm{Irrep}(G) \cong \frac{\mathbb{N}}{2}$ — "spins" $j = 0, \frac{1}{2}, 1, \ldots$

$$\rho_j \otimes \rho_k \cong \underbrace{\rho_{|j-k|} \oplus \cdots \oplus \rho_{j+k}}_{\text{integer steps}}$$

$$\rho_j^* \cong \rho_j$$

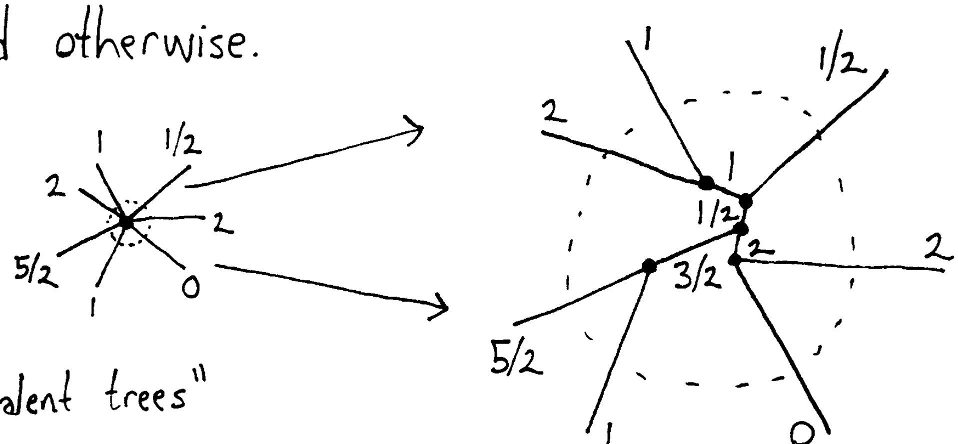
Space of intertwiners:



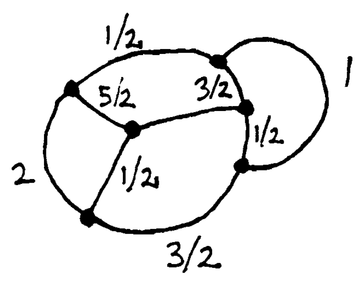
is 1d if $j+k+\ell \in \mathbb{N}$ & $|j-k| \leq \ell \leq j+k$,

0d otherwise.

Basis of intertwiners given by "virtual trivalent trees" with edges labelled by spins satisfying above constraints.



$L^2(\mathcal{A}/\mathcal{G})$ has basis of spin networks:



Given any graph γ with edges real-analytically embedded in M, let

$$\mathcal{A}_\gamma = \{\text{connections on } \gamma\}$$
$$\mathcal{G}_\gamma = \{\text{gauge transformations on } \gamma\}$$

defined as before!

$$\mu_\gamma = \mathcal{G}_\gamma\text{-invariant measure on } \mathcal{A}_\gamma \text{ with } \int_{\mathcal{A}_\gamma} \mu_\gamma = 1$$

If we arbitrarily trivialize P at each point of M, we get onto maps:

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{A}_\gamma & & \mathcal{G}_\gamma \end{array}$$

Idea: form $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ as an inverse limit of these $\mathcal{A}_\gamma$, $\mathcal{G}_\gamma$.

GAUGE THEORY ON A REAL-ANALYTIC MANIFOLD

Let

M be a real-analytic manifold

G a compact connected Lie group

$P \to M$ a smooth principal G-bundle

$\mathcal{A} = \{$smooth connections on $P\}$

$\mathcal{G} = \{$smooth gauge transformations of $P\}$

We want to define $L^2(\mathcal{A}/\mathcal{G})$ but there's no good measure. So: define $L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}})$ where

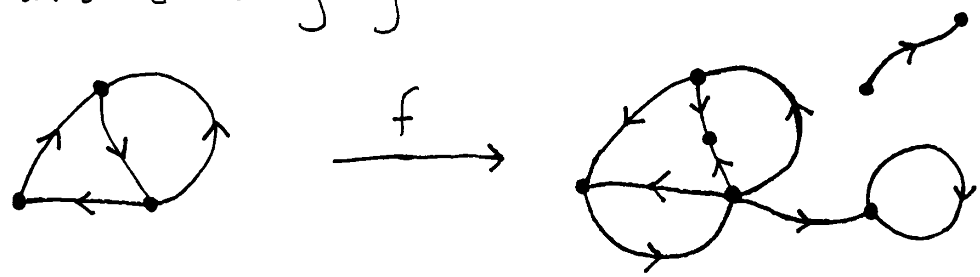
$$\mathcal{A} \xhookrightarrow{\text{dense}} \overline{\mathcal{A}}$$

$$\mathcal{G} \xhookrightarrow{\text{dense}} \overline{\mathcal{G}}$$

and $\overline{\mathcal{A}}$ has a $\overline{\mathcal{G}}$-invariant measure.

HOW?

Graphs real-analytically embedded in M form a category:



where morphisms are:

- adding new edges & vertices
- subdividing edges with new vertices
- reversing orientation of edges

Any morphism $f : \gamma \to \gamma'$ induces:

$$f^* : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$$ ← measure-preserving continuous onto map: $f^* \mu_{\gamma'} = \mu_\gamma$

$$f^* : \mathcal{G}_{\gamma'} \to \mathcal{G}_\gamma$$ ← onto homomorphism of Lie groups

Let

$$\bar{\mathcal{A}} = \varprojlim_\gamma \mathcal{A}_\gamma$$

$$\bar{\mathcal{G}} = \varprojlim_\gamma \mathcal{G}_\gamma$$

$$\begin{tikzcd}
 & \bar{\mathcal{A}} \arrow[d] \arrow[dd, bend left] \\
\mathcal{A} \arrow[ur, dashed] \arrow[r] \arrow[dr] & \mathcal{A}_{\gamma'} \arrow[d] \\
 & \mathcal{A}_\gamma
\end{tikzcd}$$

& also for $\mathcal{G}$

Theorem – $\overline{\partial}$ is a compact Hausdorff space ; $\overline{\mathcal{G}}$ is a compact Hausdorff group acting continuously on $\overline{\partial}$. We have inclusions :

$$\partial \xhookrightarrow{\text{dense}} \overline{\partial}$$

$$\mathcal{G} \xhookrightarrow{\text{dense}} \overline{\mathcal{G}}$$

$\overline{\partial}$ has a $\overline{\mathcal{G}}$-invariant Borel measure μ with $\int_{\overline{\partial}} \mu = 1$. $\partial \subseteq \overline{\partial}$ is contained in a set of measure zero.

Key lemma : given γ, γ' there exists γ'' with

$$\begin{tikzcd} & \gamma'' & \\ \gamma \arrow[ur, "f"] & & \gamma' \arrow[ul, "f'"'] \end{tikzcd}$$

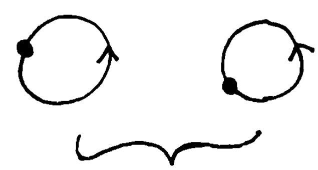
<u>Not</u> true in smooth category : 

Nonetheless this <u>theorem</u> is true in smooth category.

Corollary – $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$ has an orthonormal basis given by spin networks $\psi = (\gamma, \rho, \iota)$, where:

- γ ranges over graphs analytically embedded in M, without lone vertices: 

  unnecessary vertices: 

  or redundancies: 

  pick one

  pick one

- ρ ranges over labellings of edges by nontrivial irreps of G

- ι ranges over labellings of vertices by intertwiners chosen from an orthonormal basis.