EINSTEIN'S EQUATIONS:
THE NEW VARIABLES

Canonical quantum gravity simplifies if for the kinematical phase space we use, not $T^*(\text{Met}(S)) \ni (\varrho, p)$:

\[ \varrho_{ij} = \text{metric on } S \]
\[ p_{ij} = \sqrt{\det(\varrho)} \left( K_{ij} - K \varrho_{ij} \right) \]

extrinsic curvature of $S$

but instead $T^*\mathfrak{a}$, where $\mathfrak{a}$ is the space of smooth connections on the spin bundle $P \rightarrow S$ (a principal $SU(2)$ bundle).
Meaning of the new variables \((A, E) \in T^*\mathcal{A}\):

- \(A \in \mathcal{A}\) is a connection on \(P\) given by:

\[
A = \Gamma' - K
\]

\(i^j\) : tangent space associated to metric \(q\)

\(a, b\) : \(\text{Ad}(P)\) or \(\text{``}U(2)\text{''}\)

- \(E \in T^*_A \mathcal{A}\) is a "densitized frame field:"

\[
\begin{align*}
T_A \mathcal{A} & \cong \mathcal{L}^1(S) \otimes \text{Ad}(P) \\
T^*_A \mathcal{A} & \cong \text{Vect}(S) \otimes \text{Ad}(P)^* \otimes \mathcal{L}^3(S) \\
\end{align*}
\]

Frame fields densities

Let \(e \in \text{Vect}(S) \otimes \text{Ad}(P)\) have

\[
e^i_a e^j_b = q^{ij} \delta_{ab}
\]

and set

\[
E = e \otimes \text{vol}
\]
As a cotangent bundle, $T^{*}\mathcal{A}$ has the usual Poisson structure:

$$\{ E^i_a(x), A^b_j(y) \} = \delta^i_j \delta^b_a \delta(x,y)$$

and there are Poisson maps:

$$\begin{array}{ccc}
T^{*}\mathcal{A} & \xrightarrow{\text{dense}} & T^{*}(\mathcal{M}_{\text{et}}(S)) \\
\downarrow & & \leftarrow \text{open} \\
T^{*(\mathcal{A}/\mathcal{O}_g)} & \xrightarrow{\text{partial}} & T^{*(\mathcal{M}_{\text{et}}(S))}
\end{array}$$

defined by Poisson reduction & relations between $(A, E)$ & $(g, p)$. Inverse $T^{*(\mathcal{A}/\mathcal{O}_g)} \xrightarrow{\text{partial}} T^{*(\mathcal{M}_{\text{et}}(S))}$

defined only for $[(A, E)]$ such that

$$(\text{det } g) \ q^{ij} = E^i_a E^j_b \delta^{ab}$$

gives nondegenerate $q^{ij}$. 
To form the physical phase space, we do Poisson reduction thrice:

\[ T^* \mathcal{A} \]

\[ \xrightarrow{\text{Gauss constraint}} \]

\[ T^*(\mathcal{A}/\partial \mathcal{G}) = T^*(\text{Met}(S)) \]

\[ \xrightarrow{\text{Diffeomorphism constraint}} \]

\[ T^*(\mathcal{A}/\text{Aut}(P)) \]

\[ \xrightarrow{\text{Hamiltonian constraint}} \]

\[ \times: \text{physical phase space} \]

\[ D_i E_a^i = 0 \]
Now we get further with quantization:

\[ T^* \mathcal{A} \xrightarrow{\text{QUANTIZE}} L^2(\overline{\mathcal{A}}) \]

\[ T^*(\mathcal{A}/\mathcal{O}_0) \xrightarrow{\text{QUANTIZE}} L^2(\overline{\mathcal{A}}/\overline{\mathcal{O}_0}) \]

\[ T^*(\mathcal{A}/\text{Aut}(\mathcal{P})) \xrightarrow{\text{QUANTIZE}} \mathcal{H}_{\text{diff}} \]

\[ X \xrightarrow{\text{QUANTIZE}} \mathcal{H}_{\text{phys}} \]

--- solid ---→ controversial

.... → still intractable
HILBERT SPACES FOR QUANTUM GRAVITY

We've seen that the gauge-invariant Hilbert space $L^2(\Omega/\Omega_f)$ has a basis of spin networks:

Similarly, $H_{\text{diff}}$ has a basis given by diffeomorphism equivalence classes of spin networks. But to understand the physical meaning of these spin network states, let's focus on $L^2(\Omega/\Omega_f)$. We need to introduce observables....
GAUGE-ININVARIANT
OBSERVABLES

On $L^2(\mathbb{R}/G)$, certain gauge-invariant functions of $A$ act as multiplication operators. These capture information about parallel transport.

**EXAMPLE**: Wilson loops.

$$W(\gamma)(A) = \text{tr}(\rho_{1/2}(e^{i\oint_{\gamma} A}))$$

The trace of the holonomy of $A$ around $\gamma$ is a Wilson loop: multiplication by this function acts as a bounded self-adjoint operator on $L^2(\mathbb{R}/G)$.

**Theorem**: Finite linear combinations of spin networks form an algebra, generated by Wilson loops.
On $L^2(\mathbb{R}^3 \setminus \mathcal{D})$, certain gauge-invariant functions of $E$ act as (pseudo) differential operators. Heuristically,

$$
\hat{E}^j_\alpha(x) = \frac{1}{i} \frac{\partial}{\partial A^\alpha_j(x)}
$$

These operators capture information about the metric.

**EXAMPLE**: Volume operators.

**EXAMPLE**: Area operators.

Classically, area of oriented surface $\Sigma$ is:

$$A(\Sigma) = \gamma \int_{\Sigma} \sqrt{E \cdot E} \omega = \gamma \int_{\Sigma} \sqrt{\varepsilon^a \varepsilon_a} \omega$$

where $E = e \omega$

Ad($\mathfrak{p}$)-valued 2-form:

$\text{Vect}(S) \otimes \mathfrak{g}(\mathfrak{m}) \otimes L^2(S)$

Quantizing....
QUANTIZATION OF AREA

If $\psi$ intersects $\Sigma$ transversely, the area operator $\hat{A}(\Sigma)$ has

$$\hat{A}(\Sigma) \psi = \gamma \int_{\Sigma} \sqrt{\mathbf{E} \cdot \mathbf{E}} \, \psi$$

$$= \gamma \sum_{\text{puncture } \Sigma} \sqrt{\mathbf{e}(\mathbf{e} + \mathbf{1})} \, \psi$$

So: spin network edges represent field lines of $\mathbf{E}$ field, & give area to surfaces they puncture. Minimal unit of area is

$$\frac{8\pi G \hbar}{c^3} \gamma \sqrt{\frac{1}{2} (\frac{1}{2} + 1)} = \sqrt{3} \pi \gamma \ell_p^2$$

we'd been using units where this equals 1.

$\sim 3 \times 10^{-70}$ meter$^2$
NONCOMMUTATIVITY OF AREA OPERATORS

Subtler phenomena occur in nongeneric cases, e.g.:

\[ \hat{A}(\Sigma)\psi = \gamma \sqrt{j_5 (j_5 + 1)} \psi \]

(back to units where \( 8\pi G = c = \hbar = 1 \))
\[
\hat{A}(\Sigma) \Psi = \gamma \sqrt{\frac{j_5(j_5+1)}{2}} \Psi
\]

\[
\hat{A}(\Sigma') \Psi = \gamma \sqrt{\frac{j_6(j_6+1)}{2}} \Psi
\]

**Nontrivial change of basis:**

\[
\begin{align*}
\sum \left\{ \begin{array}{c} j_1, j_2, j_5 \\ j_3, j_4 \end{array} \right\} 
&= \sum \left\{ \begin{array}{c} j_4, j_3, j_5 \\ j_6 \end{array} \right\}
\end{align*}
\]

\[
\Rightarrow [\hat{A}(\Sigma), \hat{A}(\Sigma')] \neq 0
\]