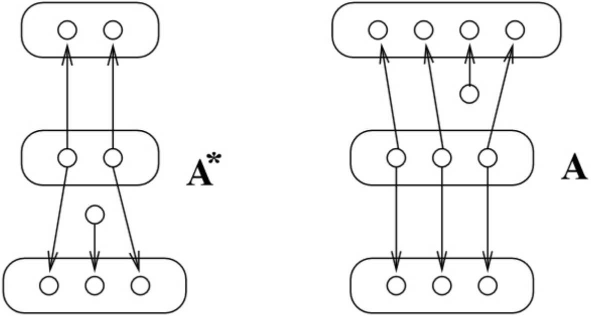
Spans and the Categorified Heisenberg Algebra – 1

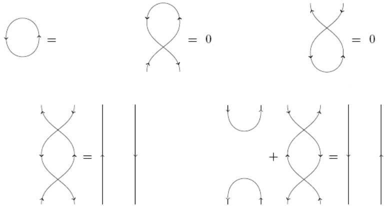
John Baez



for more, see:

http://math.ucr.edu/home/baez/spans/

In 2010, Mikhail Khovanov defined a ‘categorified Heisenberg algebra’ obeying some relations that look strange at first:



He motivated these relations using sophisticated concepts from algebra.

But Jeffrey Morton and Jamie Vicary showed they arise from simple ideas about creating and annihilating particles! This sheds new light on the combinatorics of quantum field theory.

Schrödinger described the position q and momentum p of a quantum particle on a line as operators on $L^2(\mathbb{R})$:

$$(q\psi)(x) = x\psi(x)$$

$$(p\psi)(x) = -i\frac{d}{dx}\psi(x)$$

They do not commute; instead

$$[p, q] := pq - qp = -i$$

Our goal is to understand this *combinatorially*, in terms of structures on finite sets.

Heisenberg made the first step, by doing a change of basis.

He introduced the **annihilation operator** and its adjoint, the **creation operator**:

$$a = \frac{q + ip}{\sqrt{2}} \qquad a^\dagger = \frac{q - ip}{\sqrt{2}}$$

Note that

$$[a, a^\dagger] = i[p, q] = 1$$

More abstractly, we can define the **Heisenberg algebra** to be the algebra over $\mathbb{C}$ generated by two elements $a, a^\dagger$ obeying the **canonical commutation relation**:

$$aa^\dagger = a^\dagger a + 1$$

Following Heisenberg, we define the **number operator** by

$$N = a^\dagger a$$

Suppose we have a representation of the Heisenberg algebra with a **vacuum vector** ψ_0 such that

$$a\psi_0 = 0$$

If we recursively define

$$\psi_{n+1} = a^\dagger \psi_n$$

we can inductively show

$$N\psi_n = n\psi_n$$

Here's how:

$$\begin{aligned} N\psi_{n+1} &= a^\dagger a a^\dagger \psi_n \\ &= a^\dagger (a^\dagger a + 1)\psi_n \\ &= a^\dagger (n+1)\psi_n \\ &= (n+1)\psi_{n+1} \end{aligned}$$

What this means:

- ψ_n is a state where there are *n* ‘quanta’. In quantum field theory these are *particles!*
- The number operator counts these quanta:

$$N\psi_n = n\psi_n$$

- The creation operator $a^\dagger$ creates one quantum:

$$a^\dagger\psi_n = \psi_{n+1}$$

What does the annihilation operator do? We have

$$a\psi_0 = 0$$

and for $n > 0$ we have

$$\begin{aligned} a\psi_n &= aa^\dagger \psi_{n-1} \\ &= (a^\dagger a + 1)\psi_{n-1} \\ &= (n - 1 + 1)\psi_{n-1} \\ &= n\psi_{n-1} \end{aligned}$$

Thus

$$a\psi_n = n\psi_{n-1}$$

for all $n = 0, 1, 2, 3, \ldots$

What does it *mean* that

$$a\psi_n = n\psi_{n-1}?$$

It means the annihilation operator is a sum over all ways of choosing one quantum and then annihilating it. We get a factor of n because there are n choices.

What does the canonical commutation relation

$$aa^\dagger = a^\dagger a + 1$$

really mean?

There is 1 more way to

create a quantum and then annihilate one,

than to

annihilate a quantum and then create one.

The reason: if you create one *first*, there's 1 more choice of which quantum to annihilate.

How can make this idea precise? We use the groupoid of finite sets, **S**:

- an object of **S** is a finite set s
- a morphism in **S** is a bijection $\alpha \colon s \to t$.

There is a functor

$$\begin{array}{rccc} +1\colon & \mathbf{S} & \to & \mathbf{S} \\ & s & \mapsto & s+1 \end{array}$$

sending each finite set s to its disjoint union with a chosen one-element set, called 1.

This is the real idea behind the creation operator: it adds one element to a finite set of 'quanta'.

But what about the annihilation operator? There is no functor $f\colon \mathbf{S} \to \mathbf{S}$ that takes a finite set and *removes* an element.

There are n different ways to remove an element from an n-element set. There is no functorial way to choose one. But we don't want to choose one. We want to consider *all possible ways*.

We can do this using spans of groupoids.

A **span** is a diagram shaped like this:

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

In a **span of sets**, $p\colon S \to X$ and $q\colon S \to Y$ are functions between sets.

$$
\begin{tikzcd}
 & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Y & & X
\end{tikzcd}
$$

A span of sets gives a matrix of sets:

$$S_{ji} = \{s \colon\; q(s) = j,\; p(s) = i\} \qquad i \in X, j \in Y$$

In physics, S_{ji} is the set of ways for a physical system to go from state i to state j. Physicists call these ways **paths** or **histories.**

Spans are closely connected to Heisenberg's **matrix mechanics**, where S_{ji} is a matrix of *numbers* describing the 'amplitude' for the system to go from i to j.

We compose spans of sets by taking a **pullback**:

$$
\begin{tikzcd}
 & & TS \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & & \\
 & T \arrow[dl, "q_T"'] \arrow[dr, "p_T"] & & S \arrow[dl, "q_S"'] \arrow[dr, "p_S"] & \\
Z & & Y & & X
\end{tikzcd}
$$

where

$$TS = \{(t, s) \in T \times S : \ p_T(t) = q_S(s)\}$$

Here we are doing matrix multiplication:

$$(TS)_{ki} = \sum_{j \in Y} T_{kj} \times S_{ji}$$

where $\sum$ is disjoint union and $\times$ is cartesian product. This is a baby 'path integral' where we sum over paths from i to some intermediate state j and then to k.

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

In a **span of groupoids**, $p\colon S \to X$ and $q\colon S \to Y$ are functors between groupoids. Any functor $f\colon X \to Y$ gives a span from X to Y:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\ Y & & X \end{tikzcd}$$

but we can also turn it around and get a span from Y back to X:

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

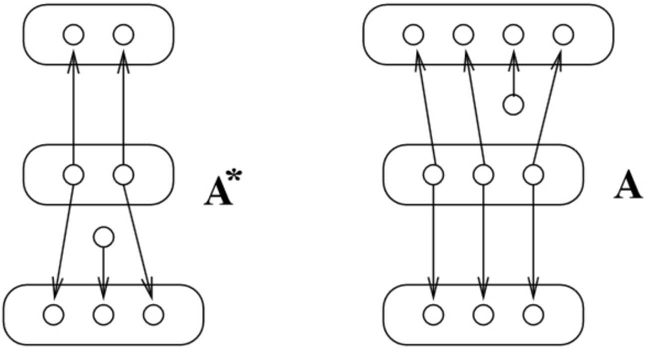
So, if **S** is the groupoid of finite sets, we have spans called the **annihilation operator** A:

$$\begin{tikzcd} & \mathbf{S} \arrow[dl, "1_{\mathbf{S}}"'] \arrow[dr, "+1"] & \\ \mathbf{S} & & \mathbf{S} \end{tikzcd}$$

and **creation operator** $A^\dagger$:

$$\begin{tikzcd} & \mathbf{S} \arrow[dl, "+1"'] \arrow[dr, "1_{\mathbf{S}}"] & \\ \mathbf{S} & & \mathbf{S} \end{tikzcd}$$

From an earlier 2006 paper by Jeffrey Morton:



We compose spans of groupoids by taking a **weak pullback**:

$$
\begin{tikzcd}
 & & TS \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & & \\
 & T \arrow[dl, "q_T"'] \arrow[dr, "p_T"] \arrow[rr, phantom, "\stackrel{\sim}{\Longrightarrow}"] & & S \arrow[dl, "q_S"'] \arrow[dr, "p_S"] & \\
Z & & Y & & X
\end{tikzcd}
$$

Now TS is the groupoid whose objects are triples

$$\big(t \in T,\ s \in S,\ \alpha\colon p_T(t) \xrightarrow{\sim} q_S(s)\big)$$

and the diamond commutes *up to natural isomorphism*.

$$
\begin{tikzcd}
 & & AA^\dagger \arrow[dl] \arrow[dr] & & \\
 & \mathbf{S} \arrow[dl, "1_{\mathbf{S}}"'] \arrow[dr, "+1"] \arrow[rr, phantom, "\overset{\sim}{\Longrightarrow}"] & & \mathbf{S} \arrow[dl, "+1"'] \arrow[dr, "1_{\mathbf{S}}"] & \\
t \in \mathbf{S} & & \mathbf{S} & & s \in \mathbf{S}
\end{tikzcd}
$$

An object of the weak pullback $AA^\dagger$ is a way to first add one element to a finite set and then remove one:

$$\big(t \in \mathbf{S},\, s \in \mathbf{S},\, \alpha \colon s + 1 \xrightarrow{\sim} t + 1\,\big)$$

We can think of this as a little 'history' starting at s and ending at t.

$$
\begin{tikzcd}
 & & A^\dagger A \arrow[dl] \arrow[dr] & & \\
 & \mathbf{S} \arrow[dl, "+1"'] \arrow[dr, "1_{\mathbf{S}}"] \arrow[rr, phantom, "\overset{\sim}{\Longrightarrow}"] & & \mathbf{S} \arrow[dl, "1_{\mathbf{S}}"'] \arrow[dr, "+1"] & \\
t+1 \in \mathbf{S} & & \mathbf{S} & & s+1 \in \mathbf{S}
\end{tikzcd}
$$

An object of the weak pullback $A^\dagger A$ is a way to remove one element from a finite set and then add one:

$$\big(\ t \in \mathbf{S},\ s \in \mathbf{S},\ \alpha\colon s \xrightarrow{\sim} t\ \big)$$

How can we relate $AA^\dagger$ and $A^\dagger A$?

A **map of spans** is roughly a diagram of groupoids and functors like this:

$$
\begin{tikzcd}
 & S \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\
Y \arrow[r, phantom, "\overset{\sim}{\Searrow}"] & \arrow[r, phantom, "\overset{\sim}{\Swarrow}"] & X \\
 & S' \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}
$$

where the triangles commute up to chosen natural isomorphisms. (It's really an equivalence class of these.)

An **isomorphism of spans** is a map of spans where f is an equivalence of groupoids.

Since there's 1 more way to *add an element to a finite set and then remove one* than to *remove an element and then add one*, we get an isomorphism of spans:

$$
\begin{tikzcd}
& A^\dagger A + \mathbf{S} \arrow[dl] \arrow[dr] \arrow[dd, "\wr"] & \\
\mathbf{S} \arrow[r, phantom, "\overset{\sim}{\Searrow}"] & \arrow[r, phantom, "\overset{\sim}{\Swarrow}"] & \mathbf{S} \\
& AA^\dagger \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Here $A^\dagger A + \mathbf{S}$ is the disjoint union or **coproduct** of the groupoids $A^\dagger A$ and **S.**

Even better, for any groupoid X there is an **identity span** $1_X \colon X \rightsquigarrow X$ given by

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

We can also **add** two spans

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

getting a span

$$\begin{tikzcd} & S+T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

where $S + T$ is the coproduct of the groupoids S and T.

Using these ideas, our isomorphism of spans

$$
\begin{tikzcd}
& A^\dagger A + \mathbf{S} \arrow[dl] \arrow[dr] \arrow[dd, "\wr"] & \\
\mathbf{S} \arrow[r, phantom, "\overset{\sim}{\Searrow}"] & \arrow[r, phantom, "\overset{\sim}{\Swarrow}"] & \mathbf{S} \\
& AA^\dagger \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

can be written as

$$f\colon\ A^\dagger A + 1_{\mathbf{S}} \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

where the double arrow denotes a map of spans.

This isomorphism of spans

$$f\colon A^\dagger A + 1_{\mathsf{S}} \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

is the categorified version of the canonical commutation relation.

James Dolan and I noticed this in 2000, and used it to describe the combinatorics of Feynman diagrams using spans of groupoids. Jeffrey Morton developed this further, showing how to include complex numbers. But this is just the beginning!

Morton and Vicary showed f obeys certain nontrivial *equations* — which are precisely the relations in Khovanov's categorified Heisenberg algebra!

To see these relations, we need to go beyond maps of spans. We need **spans of spans**:

$$
\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
Y \arrow[r, phantom, "\Downarrow\wr"] & Z \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow\wr"] & X \\
 & S' \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Just as any functor gives a span of groupoids, any map of spans gives a span of spans.

But a span of spans $Z\colon S \Rightarrow S'$ can be ‘flipped’ to give a span of spans $Z^\dagger\colon S' \Rightarrow S$, just by turning the diagram upside down.

So, the isomorphism of spans

$$f\colon A^\dagger A + 1_{\mathbf{S}} \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

gives two 'inclusions'

$$i\colon A^\dagger A \Longrightarrow AA^\dagger$$

$$j\colon 1_{\mathbf{S}} \Longrightarrow AA^\dagger$$

and we can flip these to get 'projections'

$$i^\dagger\colon AA^\dagger \Longrightarrow A^\dagger A$$

$$j^\dagger\colon AA^\dagger \Longrightarrow 1_{\mathbf{S}}$$

which are spans of spans.

Just as a span of groupoids

$$\begin{tikzcd}
& \mathbf{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & \mathbf{S}
\end{tikzcd}$$

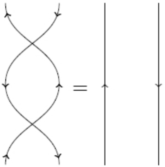
is a collection of 'histories' $x \in X$ that start at some finite set $p(x)$ and end at some finite set $q(x)$...

... a span of spans

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
\mathbf{S} \arrow[r, phantom, "\Downarrow\wr"] & Z \arrow[u, "f"'] \arrow[d, "g"] \arrow[r, phantom, "\Downarrow\wr"] & \mathbf{S} \\
& X' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

is a collection of 'histories of histories' $z \in Z$ going from some history $f(z)$ to some history $g(z)$.

Here is one of Khovanov's relations, as he drew it and as a commutative triangle where the double arrows are spans of spans:



$$\begin{tikzcd} & A^\dagger A \arrow[dl, Rightarrow, "i"'] \arrow[dd, Rightarrow, "1"] \\ AA^\dagger \arrow[dr, Rightarrow, "i^\dagger"'] & \\ & A^\dagger A \end{tikzcd}$$

What does this mean? We begin with a way to remove one element x from a finite set and then add one element y. $i^\dagger$ relates this to a way to add y and then remove x. i relates this back to a way where we remove x and add y. This gives the identity: we come back to the same 'history'!

To set this work in a good context, we should prove this:

Conjecture (Morton and Vicary)

There is a symmetric monoidal bicategory with:

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *spans of spans as 2-morphisms*

Here is a big step toward proving the conjecture:

Theorem (Alex Hoffnung and Mike Stay)

There is a symmetric monoidal bicategory with:

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *maps of spans as 2-morphisms*

Next time I'll explain what a 'symmetric monoidal bicategory' is!