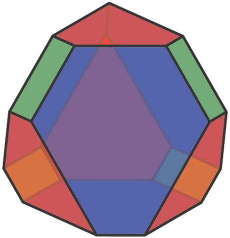
Spans and the Categorified Heisenberg Algebra – 2

John Baez



for more, see:
http://math.ucr.edu/home/baez/spans/

Last time I stated this theorem:

Theorem (Alex Hoffnung, Mike Stay)

There is a symmetric monoidal bicategory with:

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *maps of spans as 2-morphisms*

What is a symmetric monoidal bicategory?

k-tuply monoidal n-categories

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	bicategories
$k = 1$	monoids	monoidal categories	monoidal bicategories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal bicategories
$k = 3$	‘’	symmetric monoidal categories	sylleptic monoidal bicategories
$k = 4$	‘’	‘’	symmetric monoidal bicategories
$k = 5$	‘’	‘’	‘’

The chart goes on, but we only need the second column:

- bicategories
- monoidal bicategories
- braided monoidal bicategories
- sylleptic monoidal bicategories
- symmetric monoidal bicategories

Let me explain these concepts, using beautiful pictures taken from here:

- Mike Stay, Compact closed bicategories, arXiv:1301.1053.

Read this paper for more details!

For starters, a **bicategory** $\mathcal{B}$ consists of:

- a collection of **objects**, which we write as $A, B, C, \ldots \in \mathcal{B}$.
- for any pair of objects $A, B \in \mathcal{B}$, a category $\mathcal{B}(A, B)$.
- for each triple $A, B, C \in \mathcal{B}$, a **composition** functor

$$\circ \colon \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C).$$

- for each object A in $\mathcal{B}$, an **identity** 1-morphism $1_A \colon A \to A$.

An object $f \in \mathcal{B}(A, B)$ is called a **1-morphism** and written $f \colon A \to B$. A morphism α in $\mathcal{B}(A, B)$ is called a **2-morphism** and written like $\alpha \colon f \Rightarrow g$. An invertible 2-morphism is called a **2-isomorphism** and written $\alpha \colon f \stackrel{\sim}{\Rightarrow} g$.

A bicategory also has:

- given $f\colon C \to D$, $g\colon B \to C$, $h\colon A \to B$, a natural 2-isomorphism called the **associator for composition**:

$$\alpha_{f,g,h}\colon (f \circ g) \circ h \stackrel{\sim}{\Rightarrow} f \circ (g \circ h)$$

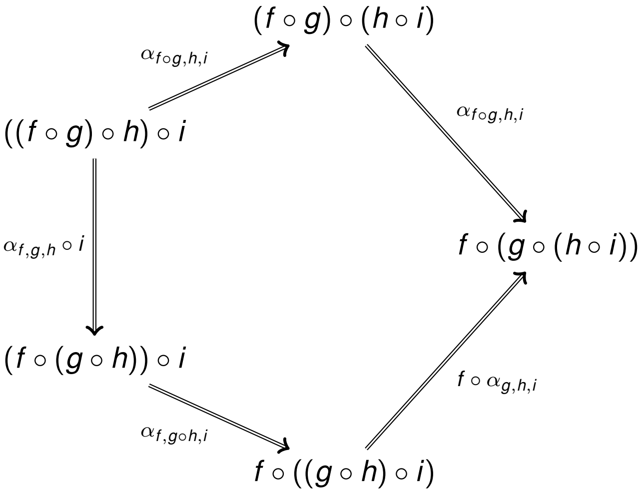
- given $f\colon A \to B$, natural 2-isomorphisms called **left and right unitors for composition**:

$$\lambda_f\colon 1_B \circ f \stackrel{\sim}{\Rightarrow} f$$

$$\rho_f\colon f \circ 1_A \stackrel{\sim}{\Rightarrow} f$$

Finally, we require that α, λ, and ρ satisfy some equations...

- The **pentagon identity**, saying this diagram commutes:



- The **triangle identities**, saying these diagrams commute:

$$\begin{array}{ccc} (1\circ f)\circ g & \xRightarrow{\alpha_{1,f,g}} & 1\circ(f\circ g) \\ \lambda_f\circ g \Downarrow & \swarrow_{\lambda_{f\circ g}} & \\ f\circ g & & \end{array} \qquad \begin{array}{ccc} (f\circ g)\circ 1 & \xRightarrow{\alpha_{f,g,1}} & f\circ(g\circ 1) \\ \rho_{f\circ g} \Downarrow & \swarrow_{f\circ\rho_g} & \\ f\circ g & & \end{array}$$

$$\begin{array}{ccc} (f\circ 1)\circ g & \xRightarrow{\alpha_{f,1,g}} & f\circ(1\circ g) \\ \rho_f\circ g \Downarrow & \swarrow_{f\circ\lambda_g} & \\ f\circ g & & \end{array}$$

(We only need the last, but it's better to use all three.)

We have:

- functors between bicategories,
- pseudonatural transformations between such functors,
- modifications between pseudonatural transformations.

For these read:

- Tom Leinster, Basic bicategories, arXiv:math/9810017.

A pseudonatural transformation $\alpha\colon F \Rightarrow G$ is like a natural transformation, but this square commutes *up to a 2-isomorphism*:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha_A}\Big\downarrow & \overset{\alpha_f}{\swarrow}\,\wr & \Big\downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

An **equivalence** $f\colon A \xrightarrow{\sim} B$ in a bicategory is a morphism equipped with a **weak inverse** $f^{-1}\colon B \to A$ and 2-isomorphisms

$$\alpha\colon f^{-1} \circ f \overset{\sim}{\Longrightarrow} 1_A \qquad \beta\colon f \circ f^{-1} \overset{\sim}{\Longrightarrow} 1_B$$

A **pseudonatural equivalence** $\alpha\colon F \xrightarrow{\sim} G$ is a pseudonatural transformation where each morphism $\alpha_A\colon F(A) \xrightarrow{\sim} G(A)$ is an equivalence.

A **monoidal bicategory** $\mathcal{B}$ is a bicategory with a tensor product. It consists of the following:

- A bicategory $\mathcal{B}$.
- A **tensor product** functor $\otimes\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$.
- A pseudonatural equivalence called the **associator for the tensor product**:

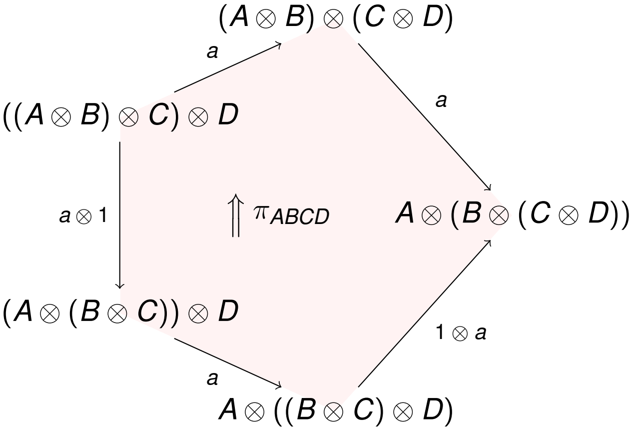
$$a_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

- An object $I \in \mathcal{B}$, the **unit for the tensor product.**
- Pseudonatural equivalences called the **left and right unitors for the tensor product**:

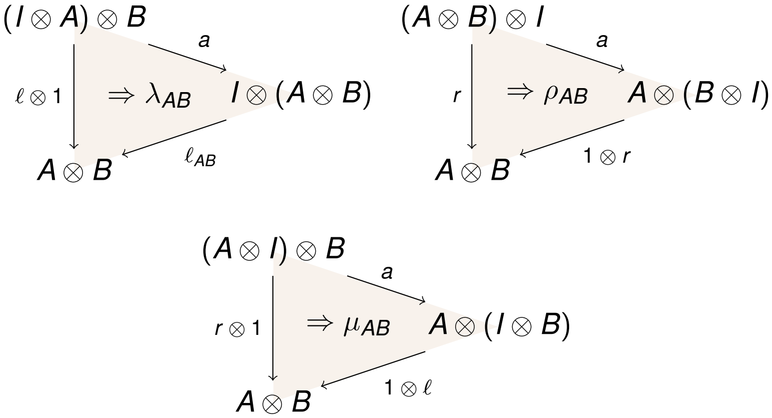
$$\ell_A\colon I \otimes A \xrightarrow{\sim} A$$

$$r_A\colon A \otimes I \xrightarrow{\sim} A$$

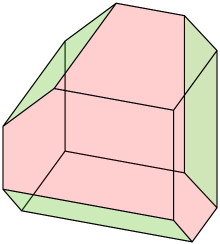
• The **pentagonator** 2-isomorphism π_{ABCD}, giving a modification between pseudonatural transformations:



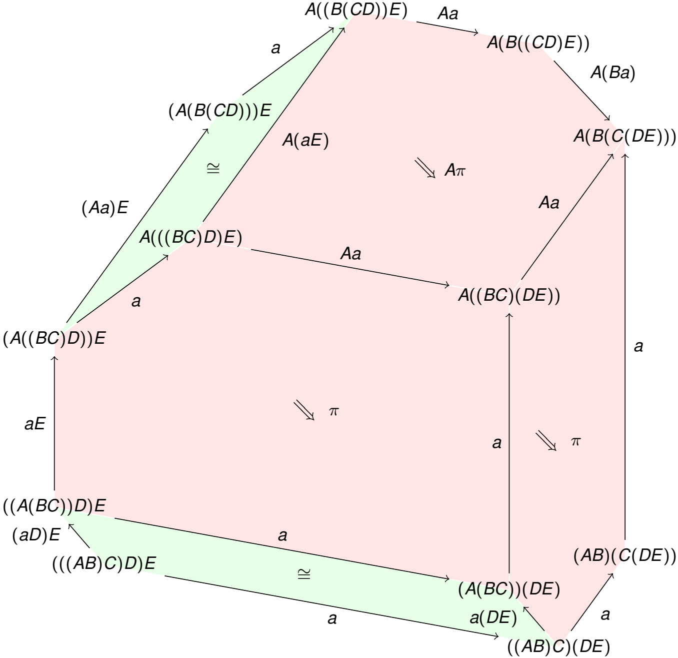
• The **left, middle and right 2-unitor** 2-isomorphisms λ_{AB}, μ_{AB} and ρ_{AB}, also giving modifications:

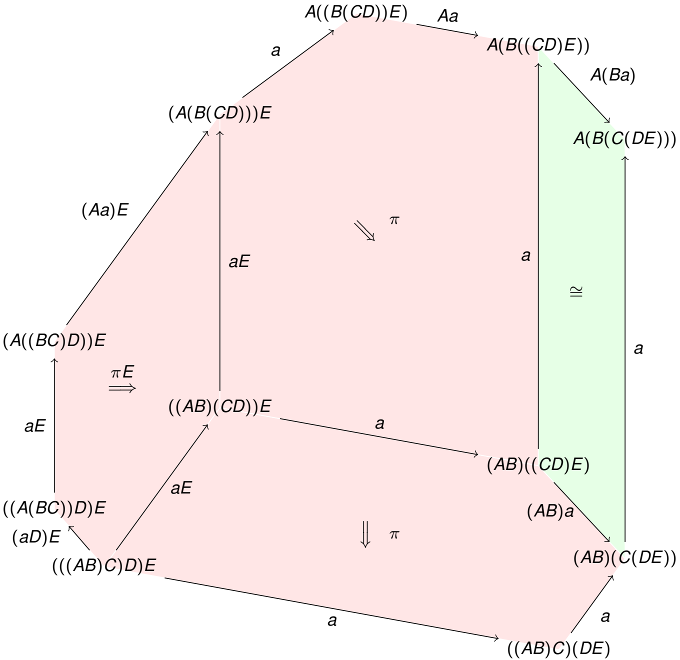


• The pentagonator obeys Stasheff's **associahedron axiom.** This polyhedron has one vertex for each way of putting parentheses in $A \otimes B \otimes C \otimes D \otimes E$:



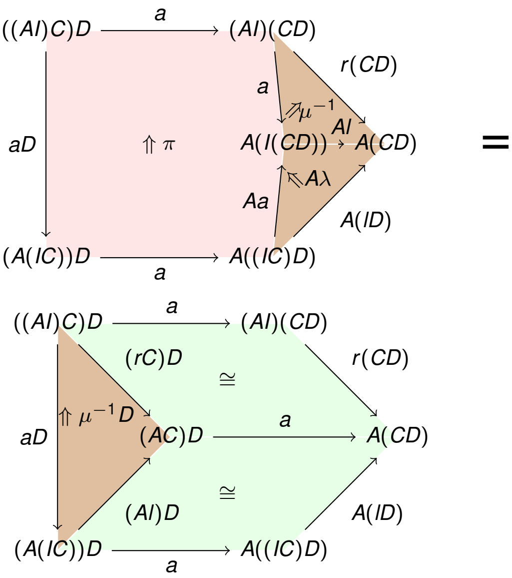
We see pink pentagonators, but also green rectangles arising from associator pseudonatural equivalences. Let's look at the front and back in detail, omitting $\otimes$ symbols and writing identity morphisms like 1_A simply as A:

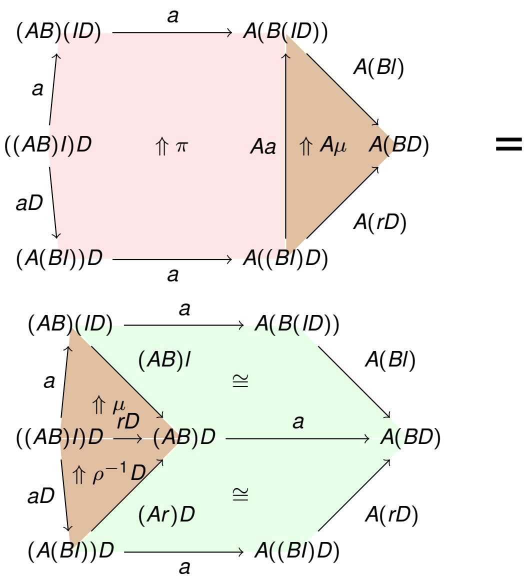

A((B(CD))E)
Aa
A(B((CD)E))
A(Ba)
a
(A(B(CD)))E
A(aE)
A(B(C(DE)))
≅
Aπ
(Aa)E
Aa
A(((BC)D)E)
Aa
a
A((BC)(DE))
(A((BC)D))E
a
aE
π
a
π
((A(BC))D)E
(aD)E
a
(((AB)C)D)E
≅
(AB)(C(DE))
(A(BC))(DE)
a
a(DE)
a
((AB)C)(DE)



A((B(CD))E)
Aa
A(B((CD)E))
a
A(Ba)
(A(B(CD)))E
A(B(C(DE)))
(Aa)E
π
aE
a
≅
(A((BC)D))E
a
πE
((AB)(CD))E
a
aE
(AB)((CD)E)
((A(BC))D)E
aE
(AB)a
(aD)E
π
(AB)(C(DE))
(((AB)C)D)E
a
a
((AB)C)(DE)

• The 2-unitors obey four equations... but those with the unit object I at front or back are automatic, so we only need two:





a
(AB)(ID)
A(B(ID))
a
A(BI)
((AB)I)D
⇑ π
Aa
⇑ Aμ
A(BD)
=
aD
A(rD)
(A(BI))D
A((BI)D)
a
a
(AB)(ID)
A(B(ID))
(AB)I
A(BI)
a
≅
⇑ μ
rD
a
((AB)I)D
(AB)D
A(BD)
⇑ ρ⁻¹D
aD
≅
(Ar)D
A(rD)
(A(BI))D
A((BI)D)
a

A **braided** monoidal bicategory $\mathcal{B}$ consists of:

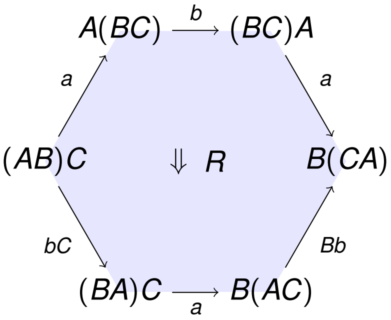
- A monoidal bicategory $\mathcal{B}$.
- A pseudonatural equivalence called the **braiding**:

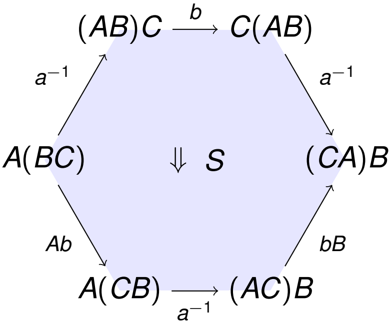
$$b_{AB} \colon A \otimes B \xrightarrow{\sim} B \otimes A$$

- Invertible modifications R and S called **hexagonators**:

  R relates the two ways to braid A over $B \otimes C$.

  S relates the two ways to braid $A \otimes B$ over C.


A(BC)
b
(BC)A
a
a
(AB)C
⇓ R
B(CA)
bC
Bb
(BA)C
B(AC)
a


(AB)C
b
C(AB)
a⁻¹
a⁻¹
A(BC)
⇓ S
(CA)B
Ab
bB
A(CB)
(AC)B
a⁻¹

• The hexagonators obey three equations relating ways to shuffle 4 objects:

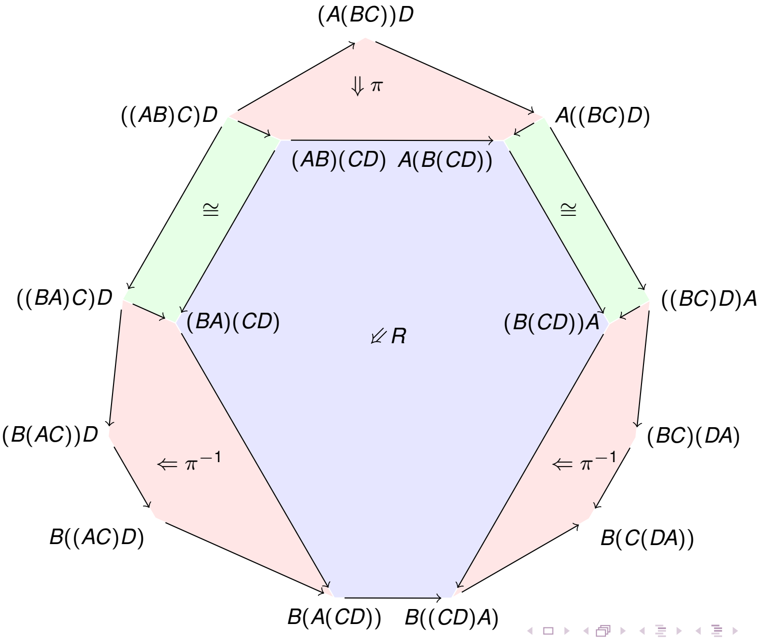
$$A \otimes (B \otimes C \otimes D) \stackrel{\sim}{\longrightarrow} (B \otimes C \otimes D) \otimes A$$

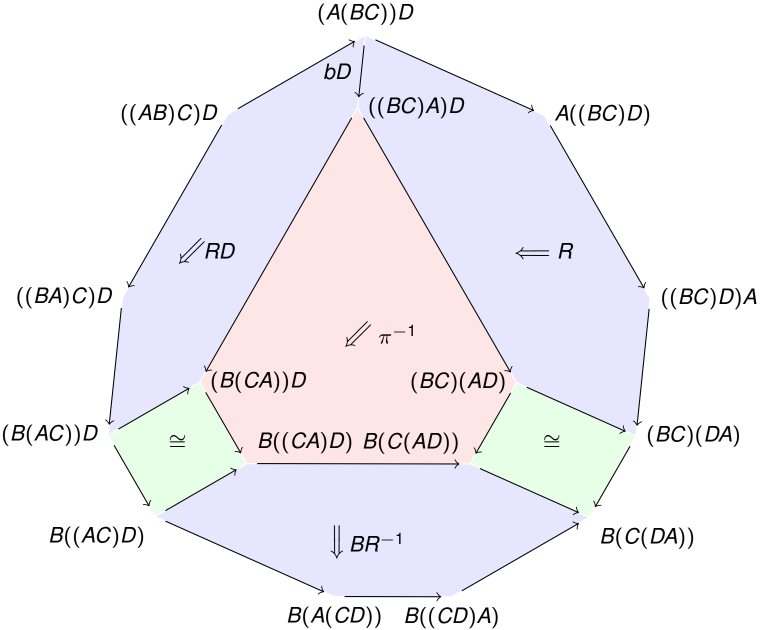
$$(A \otimes B \otimes C) \otimes D \stackrel{\sim}{\longrightarrow} D \otimes (A \otimes B \otimes C)$$

$$(A \otimes B) \otimes (C \otimes D) \stackrel{\sim}{\longrightarrow} (C \otimes D) \otimes (A \otimes B)$$

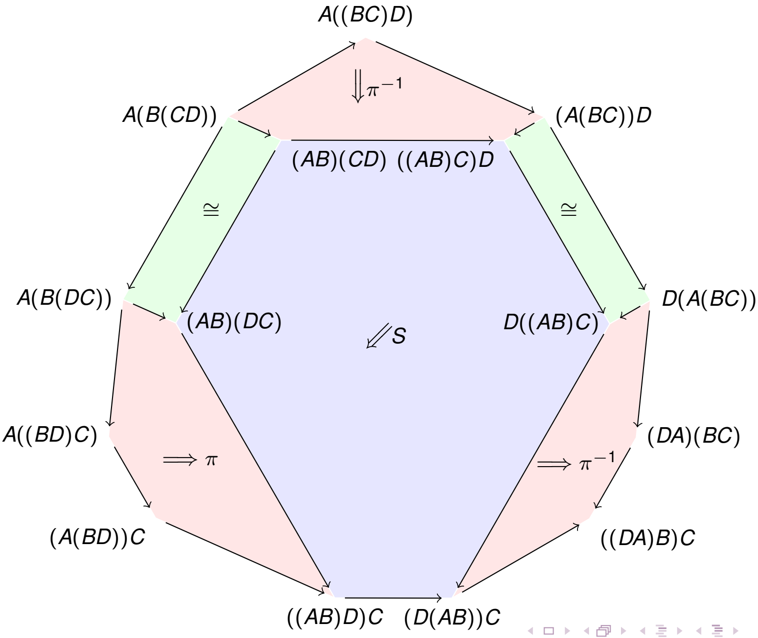
Let's look at the front and back of each. All morphisms are built from associators and braidings in obvious ways, so we don't need to label them!

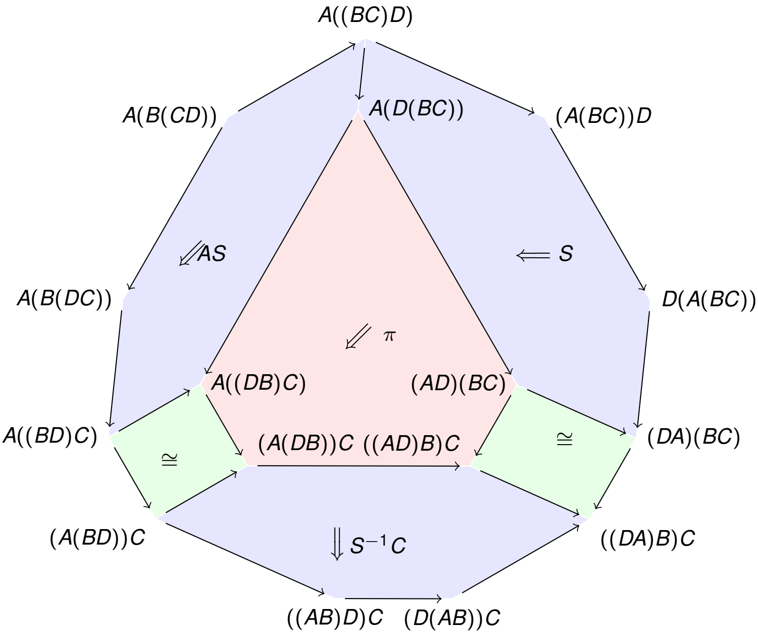
$$A \otimes (B \otimes C \otimes D) \xrightarrow{\sim} (B \otimes C \otimes D) \otimes A$$




(A(BC))D
bD
((AB)C)D
((BC)A)D
A((BC)D)
RD
R
((BA)C)D
((BC)D)A
π^{-1}
(B(CA))D
(BC)(AD)
(B(AC))D
≅
B((CA)D)
B(C(AD))
≅
(BC)(DA)
B((AC)D)
BR^{-1}
B(C(DA))
B(A(CD))
B((CD)A)

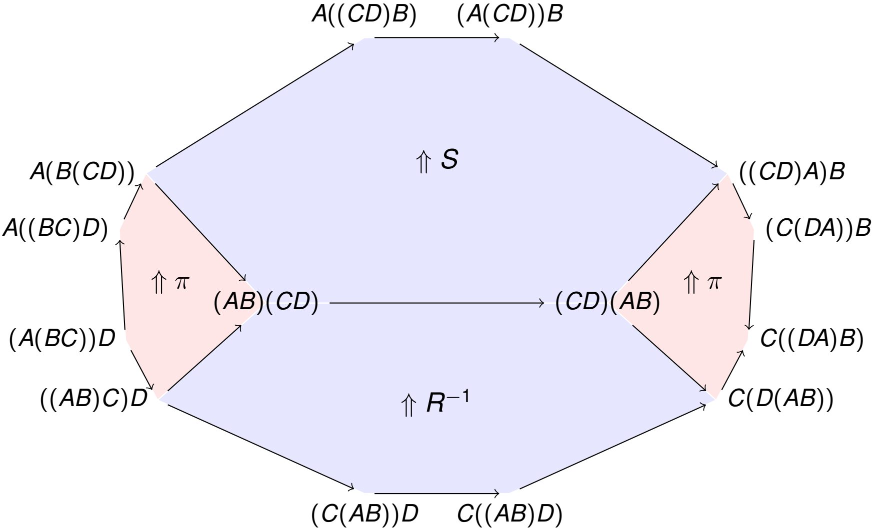
$$(A \otimes B \otimes C) \otimes D \xrightarrow{\sim} D \otimes (A \otimes B \otimes C)$$

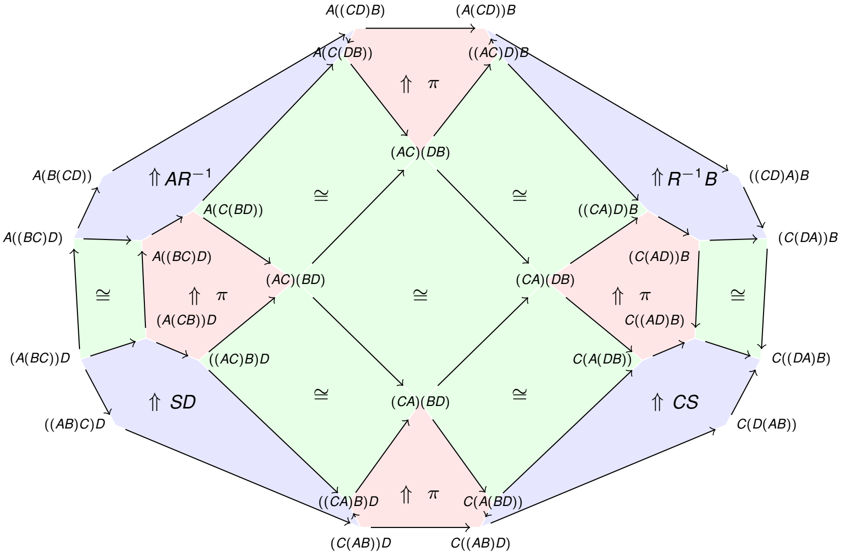




A((BC)D)
A(B(CD))
A(D(BC))
(A(BC))D
AS
S
A(B(DC))
D(A(BC))
π
A((DB)C)
(AD)(BC)
A((BD)C)
≅
(A(DB))C
((AD)B)C
≅
(DA)(BC)
(A(BD))C
S⁻¹C
((DA)B)C
((AB)D)C
(D(AB))C

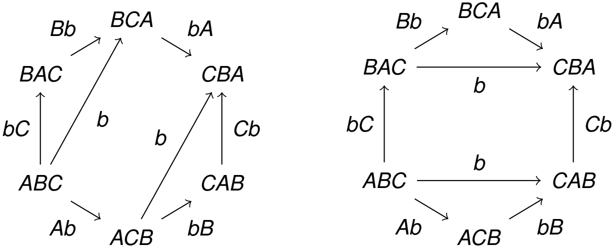
$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\sim} (C \otimes D) \otimes (A \otimes D)$$



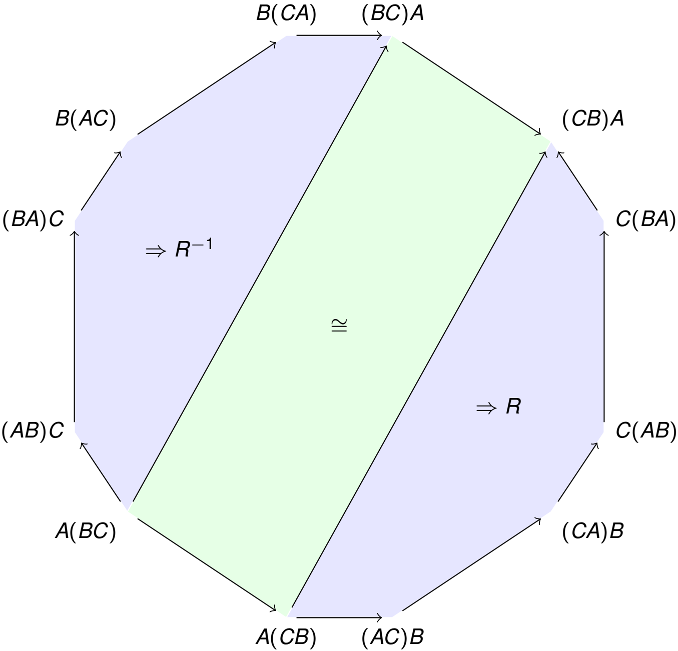


$A((CD)B)$
$(A(CD))B$
$A(C(DB))$
$((AC)D)B$
$\Uparrow \pi$
$(AC)(DB)$
$A(B(CD))$
$\Uparrow AR^{-1}$
$\Uparrow R^{-1}B$
$((CD)A)B$
$A(C(BD))$
$((CA)D)B$
$A((BC)D)$
$(C(DA))B$
$(C(AD))B$
$(AC)(BD)$
$(CA)(DB)$
$\cong$
$(A(CB))D$
$C((AD)B)$
$(A(BC))D$
$((AC)B)D$
$C(A(DB))$
$C((DA)B)$
$\Uparrow SD$
$(CA)(BD)$
$\Uparrow CS$
$((AB)C)D$
$C(D(AB))$
$((CA)B)D$
$C(A(BD))$
$(C(AB))D$
$C((AB)D)$

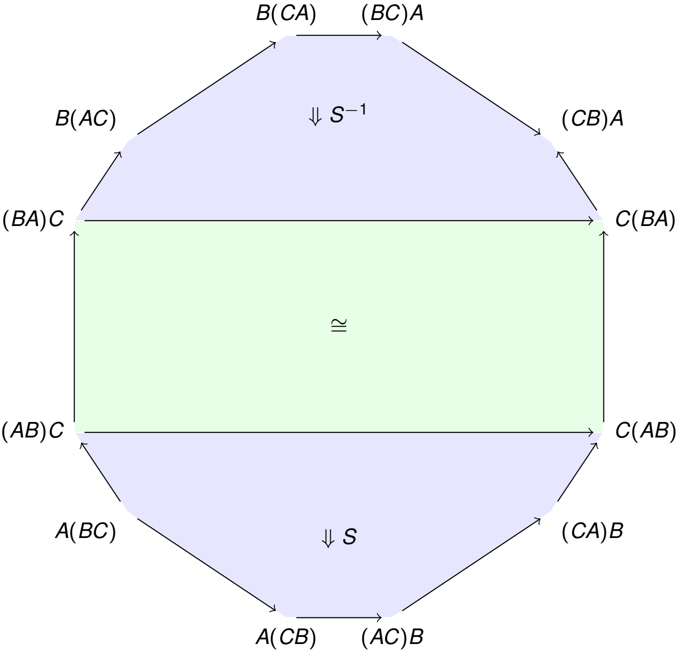
• Finally, there is a law noticed by Lawrence Breen. In a braided monoidal category, there are two proofs of the **Yang–Baxter equations**. Omitting associators and $\otimes$ symbols, they are:



In a braided monoidal bicategory these become 2-morphisms. Breen's law says they're equal! Let's look at front and back...



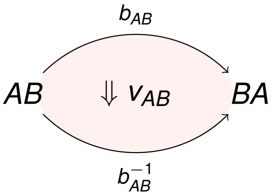
B(CA)
(BC)A
B(AC)
(CB)A
(BA)C
C(BA)
⇒ R^{−1}
≅
⇒ R
(AB)C
C(AB)
A(BC)
(CA)B
A(CB)
(AC)B



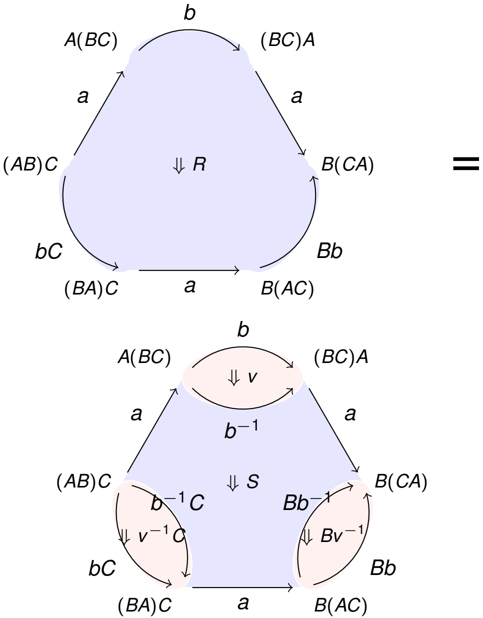
B(CA)
(BC)A
B(AC)
⇓ S^{−1}
(CB)A
(BA)C
C(BA)
≅
(AB)C
C(AB)
A(BC)
⇓ S
(CA)B
A(CB)
(AC)B

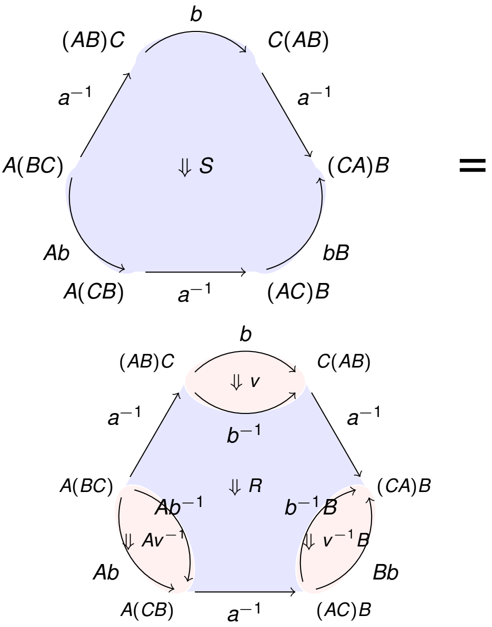
A **sylleptic** monoidal bicategory consists of:

- A braided monoidal bicategory $\mathcal{B}$.
- An invertible modification called the **syllepsis**, going between the braiding and inverse braiding:

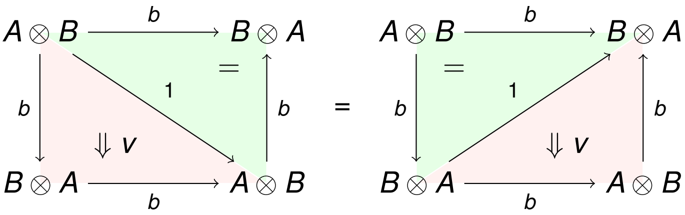


- The syllepsis obeys two equations...


b
A(BC)
(BC)A
a
a
(AB)C
⇓ R
B(CA)
=
bC
Bb
(BA)C
a
B(AC)
b
A(BC)
(BC)A
⇓ v
a
a
b^{−1}
(AB)C
⇓ S
B(CA)
b^{−1}C
Bb^{−1}
⇓ v^{−1}C
⇓ Bv^{−1}
bC
Bb
(BA)C
a
B(AC)


b
(AB)C
C(AB)
a⁻¹
a⁻¹
A(BC)
⇓ S
(CA)B
=
Ab
bB
A(CB)
a⁻¹
(AC)B
b
(AB)C
C(AB)
⇓ v
a⁻¹
a⁻¹
b⁻¹
A(BC)
⇓ R
(CA)B
Ab⁻¹
b⁻¹B
⇓ Av⁻¹
⇓ v⁻¹B
Ab
Bb
A(CB)
a⁻¹
(AC)B

Finally, a **symmetric** monoidal bicategory is a sylleptic monoidal bicategory $\mathcal{B}$ such that for all $A, B \in \mathcal{B}$, the following diagram commutes:



Note: here and elsewhere we are using the same names (like v) for for 'rotated' versions of these 2-isomorphisms. Mike Stay explains this idea but uses more careful notation.

So, now you've seen the $n = 2$ column of the periodic table:

***k*-tuply monoidal *n*-categories**

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	bicategories
$k = 1$	monoids	monoidal categories	monoidal bicategories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal bicategories
$k = 3$	‘’	symmetric monoidal categories	sylleptic monoidal bicategories
$k = 4$	‘’	‘’	symmetric monoidal bicategories
$k = 5$	‘’	‘’	‘’

Next time we will work with it *informally* and explain the categorified Heisenberg algebra using spans of groupoids.