Spin Foam Models

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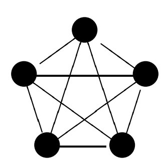
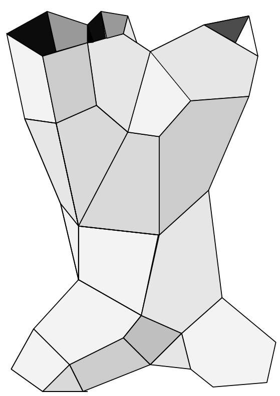


The Idea: spacetime and everything in it is a quantum superposition of 'spin foams'. A spin foam is a generalized Feynman diagram where instead of a graph we use a higher-dimensional complex:



A spin foam model specifies a class of complexes and labels for vertices, edges, faces, etc. It also says how to calculate an amplitude for any such spin foam — typically as a product of vertex amplitudes, edge amplitudes, face amplitudes, etc.

Many of the basic questions about *how to do physics with spin foams* remain unanswered!

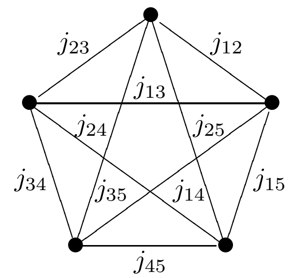
The Barrett–Crane Model

In this model we use 2d spin foams lying in the 'dual 2-skeleton' of a triangulated 4-manifold:

- one **spin foam vertex** in each **4-simplex**
- one **spin foam edge** intersecting each **tetrahedron**
- one **spin foam face** intersecting each **triangle**

We label each spin foam face by a number describing its area: a spin $j = 0, \frac{1}{2}, 1, \ldots$ in the Riemannian case, or an arbitrary number $a \geq 0$ in the Lorentzian case.

Each spin foam vertex touches ten spin foam faces labelled by numbers:



The vertex amplitude is a certain function of these numbers: the **10j symbol**.

Different versions of the model make difference choices of edge and face amplitudes. Numerical calculations show that for the DePietri-Freidel-Krasnov-Rovelli choice of edge and face amplitudes, the sum over spin foams dual to a given triangulation *diverges*. For the Perez-Rovelli choice, it *converges* so rapidly that in the Riemannian case only the lowest allowed spins make a significant contribution.

Q: Which choice is best? Are divergences bad? What do they mean?

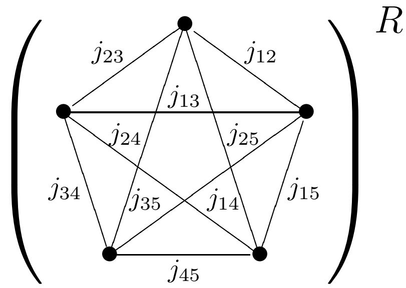
J	$Z_J(M)$
0	$1.000 \cdot 10^0$
1/2	$3.722 \cdot 10^5$
1	$7.812 \cdot 10^9$
3/2	$2.128 \cdot 10^{13}$
2	$1.345 \cdot 10^{16}$

S^4 partition function —
DFKR model with spin cutoff J

J	$Z_J(M)$
0	1.000000000000
1/2	1.000014319178
1	1.000014323656
3/2	1.000014323670
2	1.000014323670

S^4 partition function —
Perez–Rovelli model with spin cutoff J

Motivated by the Einstein-Hilbert Lagrangian and defined using group representation theory, the Riemannian $10j$ symbols work out to this:

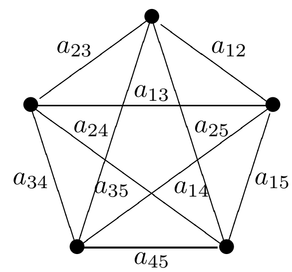


$$= \int_{(S^3)^5} \prod_{k<l} K^R_{2j_{kl}+1}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2}$$

where the unit sphere $S^3 \subset \mathbb{R}^4$ is equipped with its usual measure (total volume $2\pi^2$), ϕ_{kl} is the angle between the unit vectors h_k and h_l, and

$$K^R_a(\phi) = \frac{\sin a\phi}{\sin \phi}.$$

The Lorentzian $10j$ symbols are ‘morally’ given by the same sort of integral:



$$\left(\quad \right)^L =$$

$$\int_{(H^3)^5} \prod_{k<l} K^L_{a_{kl}}(\phi_{kl}) \; \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2}$$

where the hyperbolic space

$$H^3 = \{t^2 - x^2 - y^2 - z^2 = 1, \; t > 0\}$$

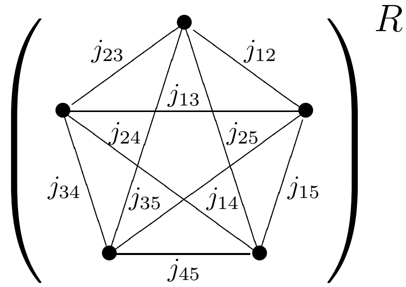
is equipped with its usual measure, ϕ_{kl} is the hyperbolic distance between points h_k and h_l, and

$$K^L_a(\phi) = \frac{\sin a\phi}{\sinh \phi}.$$

However, this integral diverges! So, we ‘gauge-fix’ it, holding one point h_k fixed and integrating only over the rest.

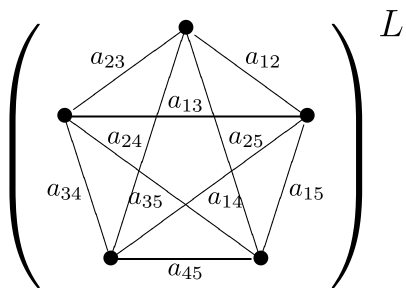
Some surprises...

Theorem:



$$\geq 0$$

Conjecture:

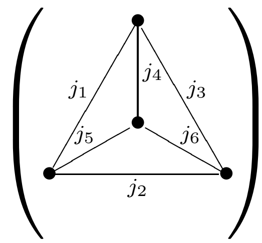


$$\geq 0$$

This conjecture is backed by considerable numerical evidence, but the Lorentzian $10j$ symbol is currently very hard to compute.

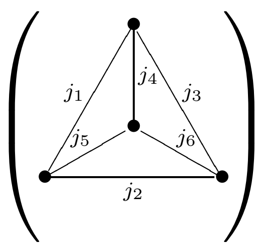
Q: What is the physical meaning of this positivity? It doesn't happen for the $6j$ symbols!

In the spin foam model for Riemannian $3d$ quantum gravity, the vertex amplitude is given by the $6j$ symbol:



$$\left(\cdot \right)^R$$

Regge and Ponzano used a stationary phase approximation to argue that

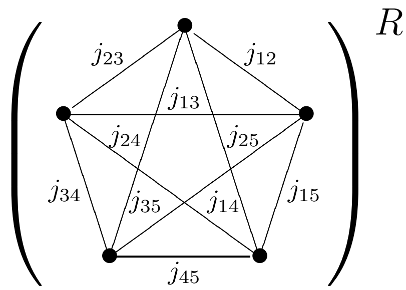


$$\left(\cdot \right)^R \sim \cos(S + \frac{\pi}{4}) \sqrt{\frac{2}{3\pi V}}$$

where S is the Regge–Ponzano action of the dual tetrahedron with edge lengths $2j_k + 1$, and V is its volume. A rigorous proof was given in 1999 by Justin Roberts.

We hoped a similar stationary phase approximation would relate the $10j$ symbols to the Regge action for 4d gravity. But...

The Riemannian $10j$ symbols are *not* well approximated by stationary phase! Instead, they are dominated by configurations where all 5 points on S^3 are very near — or nearly antipodal. Then the angles ϕ_{kl} between these points are all nearly 0 or π, and the integrand in



$$= \int_{(S^3)^5} \prod_{k<l} K^R_{2j_{kl}+1}(\phi_{kl}) \; \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2}$$

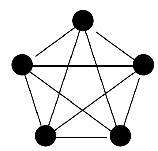
can be very large, since we have

$$K^R_{2j_{kl}+1}(\phi_{kl}) = \frac{\sin(2j_{kl}+1)\phi_{kl}}{\sin \phi_{kl}} \simeq \pm(2j_{kl}+1)$$

For such configurations the integrand is always *positive* — consistent with the positivity of the $10j$ symbols. Such configurations correspond to **degenerate 4-simplices**, whose tetrahedral faces are all almost parallel.

If we assume that degenerate 4-simplices dominate the asymptotics of the Riemannian $10j$ symbols, a calculation gives:

Conjecture: If the ten spins j_{kl} are admissible and we rescale the areas $2j_{kl}+1$ by λ, the $\lambda \to \infty$ asymptotics of the Riemannian $10j$ symbols are:

$$\Bigg(\quad$$

$$\quad \Bigg)^R \sim$$

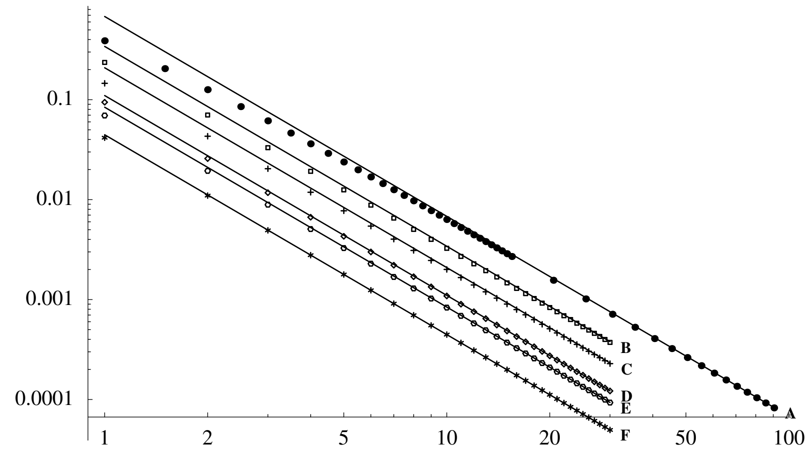
$$16\lambda^{-2} \int_{(\mathbb{R}^3)^4} \prod_{k<l} K^D_{2j_{kl}+1}(|y_k - y_l|) \, \frac{dy_2}{2\pi^2} \cdots \frac{dy_5}{2\pi^2}$$

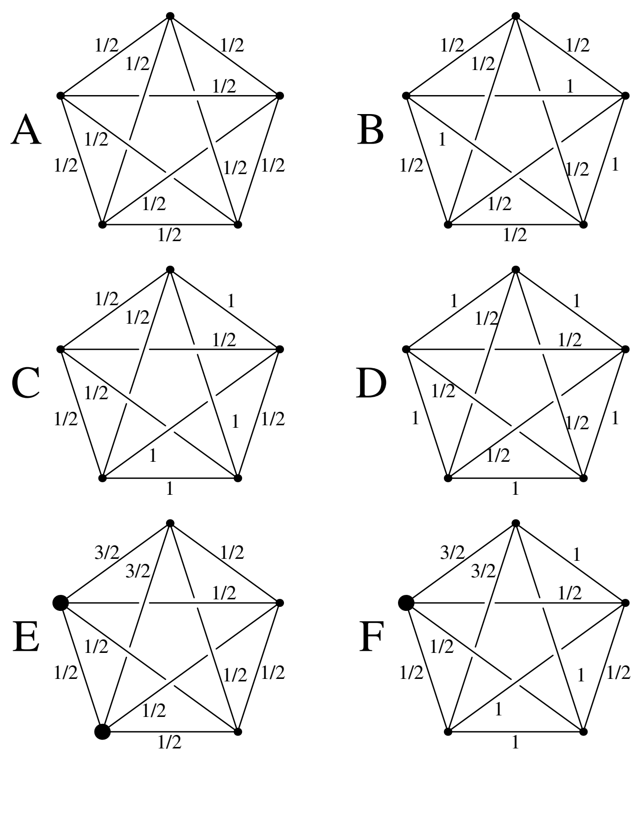
where

$$K^D_a(\phi) = \frac{\sin a\phi}{\phi}.$$

Verified by computer calculations and further work by Barrett/Steele and Freidel/Louapre. Certainly true, but still no rigorous proof!

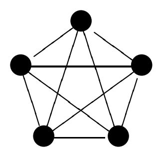
Riemannian $10j$ symbols: numerical calculations vs. predicted asymptotics





A similar calculation suggests that the Lorentzian $10j$ symbols have the same asymptotics as the Riemannian ones, but without the factor of 16:

Conjecture: If the ten areas a_{kl} are admissible and we multiply them all by λ, the $\lambda \to \infty$ asymptotics of the Lorentzian $10j$ symbols are:



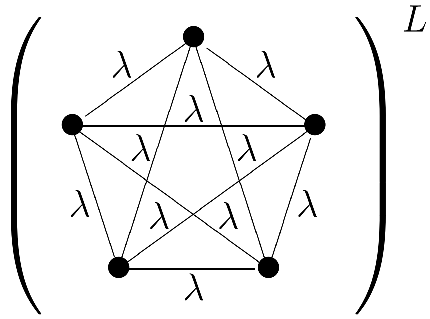
$$\left(\quad \right)^L \sim$$

$$\lambda^{-2} \int_{(\mathbb{R}^3)^4} \prod_{k<l} K^D_{a_{kl}}(|y_k - y_l|) \, \frac{dy_2}{2\pi^2} \cdots \frac{dy_5}{2\pi^2}$$

where

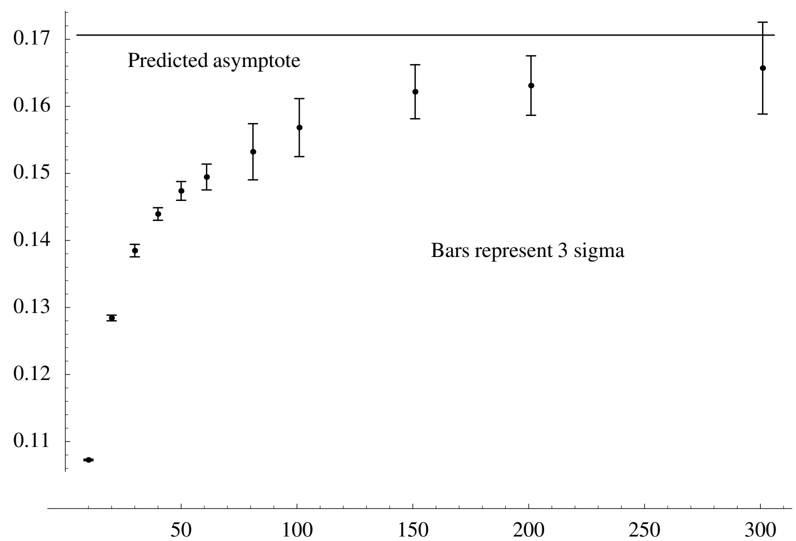
$$K^D_a(\phi) = \frac{\sin a\phi}{\phi}.$$

For example, this conjecture implies:



$$\sim .1706\,\lambda^{-2}$$

Here are some numerical calculations of λ^2 times this $10j$ symbol as a function of λ:



Q: Do these results mean the Barrett–Crane model is 'unphysical'?

Degenerate 4-simplices dominate the $10j$ symbols in the limit where all the triangles in our triangulated spacetime have *large* area... but why should this limit be relevant to physics? Don't we want discrete geometry only at the Planck scale?

In the Perez–Rovelli version of the Barrett–Crane model, only triangulations with mainly *small* triangles contribute much to the partition function.

If *small* triangles are what matter, asymptotics of $10j$ symbols are irrelevant. Instead, we need to understand the sum over spin foams with many vertices, edges, and faces. Hints of the Einstein–Hilbert action need only emerge at scales much bigger than the Planck length.