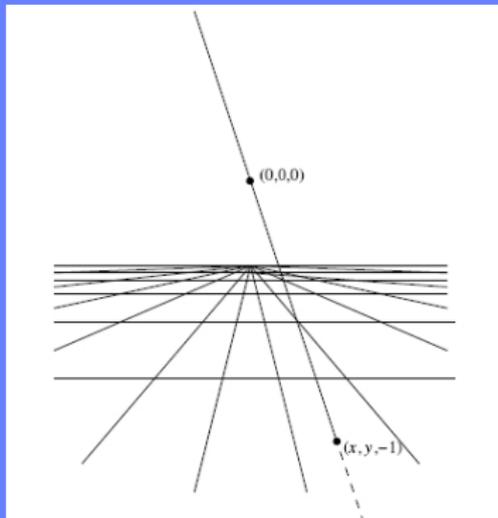# PROJECTIVE GEOMETRY AND
THE EXCEPTIONAL JORDAN ALGEBRA

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The Standard Model of particle physics has a symmetry group, or "gauge group", given by

$$S(U(2) \times U(3)) = \left\{ x \in SU(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\}$$

Nobody knows why! What is special about this group?

Dubois-Violette and Todorov had an idea:

- ▶ Michel Dubois-Violette, Exceptional quantum geometry and particle physics.

- ▶ Ivan Todorov and Michel Dubois-Violette, Exceptional quantum geometry and particle physics II.

- ▶ Ivan Todorov and Michel Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra.

They showed that $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is the intersection of two maximal subgroups of the exceptional Lie group F_4.

F_4 is the automorphism group of the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, which consists of 3×3 self-adjoint octonion matrices.

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Our goal today is to make this more appealing.

Jordan, Wigner and von Neumann argued that the 'observables' of any physical system should form a **Euclidean Jordan algebra**: a real vector space with a bilinear, commutative and power-associative product ∘ satisfying

$$a_1 \circ a_1 + \cdots + a_n \circ a_n = 0 \quad \Longrightarrow \quad a_1 = \cdots = a_n = 0$$

Theorem (Jordan–Wigner–von Neumann). Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, **where $n \leq 3$**.
- $\mathbb{R} \oplus \mathbb{R}^n$, with $(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x})$.

Main Conjecture. Suppose $A, B \subset \mathfrak{h}_3(\mathbb{O})$ are Jordan subalgebras such that

$$A \cong h_2(\mathbb{O}) \qquad B \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B \cong \mathfrak{h}_2(\mathbb{C})$$

Then the group of automorphisms of $\mathfrak{h}_3(\mathbb{O})$ preserving both A and B is the gauge group of the Standard Model:

$$\mathrm{Stab}(A) \cap \mathrm{Stab}(B) \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

For example, since

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}, \ x, y, z \in \mathbb{O} \right\}$$

we can take

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{O} \right\} \quad \cong \quad \mathfrak{h}_2(\mathbb{O})$$

$$B = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}, \ x, y, z \in \mathbb{C} \right\} \quad \cong \quad \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{C} \right\} \quad \cong \quad \mathfrak{h}_2(\mathbb{C})$$

Dubois-Violette and Todorov showed that *in this example* we have

$$\mathrm{Stab}(A) \cong \mathrm{Spin}(9)$$

$$\mathrm{Stab}(B) \cong (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$$

and

$$\mathrm{Stab}(A) \cap \mathrm{Stab}(B) \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

So the Main Conjecture holds *in this example*.

To prove the Main Conjecture, it thus suffices to show that Dubois-Violette and Todorov's example is in fact universal, in the following sense.

Conjectured Lemma. The group F_4 acts transitively on the set of pairs of Jordan subalgebras $A, B \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$, $B \cong \mathfrak{h}_3(\mathbb{C})$ and $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$.

The conjectured lemma implies the Main Conjecture by reducing it to the example studied by Dubois-Violette and Todorov.

So far we have proved some preliminary results:

Lemma 1. The group F_4 acts transitively on the set of Jordan subalgebras $A \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$.

Lemma 2. The group F_4 acts transitively on the set of Jordan subalgebras $B \subset \mathfrak{h}_3(\mathbb{O})$ with $B \cong \mathfrak{h}_3(\mathbb{C})$.

But what is the *meaning* of the concepts in the Main Conjecture?

But what is the *meaning* of the concepts in the Main Conjecture?

There are at least three ways to answer this, involving:

- ▶ projective geometry
- ▶ quantum logic
- ▶ particle physics

They are related, but today I'll focus on the first and restate the Main Conjecture using projective geometry.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

A **projection** $p \in \mathfrak{h}_n(\mathbb{K})$ is an element with $p \circ p = p$.

We can define the **trace** $\mathrm{tr}: \mathfrak{h}_n(\mathbb{K}) \to \mathbb{R}$ by taking the sum of diagonal elements.

Any projection $p \in \mathfrak{h}_n(\mathbb{K})$ has trace $0, 1, 2, \ldots, n$. Projections with $\mathrm{tr}(p) = 1$ are called **points**, projections with $\mathrm{tr}(p) = 2$ are called **lines**, and so on. A point p **lies on** a line ℓ if $p \circ \ell = p$.

The points and lines for $\mathfrak{h}_3(\mathbb{K})$ obey the axioms of a **projective plane**:

- ▸ Any two distinct points lie on a unique line.
- ▸ Any two distinct lines have a unique point lying on both.

This projective plane is called $\mathbb{K}\mathrm{P}^2$.

Consider $\mathbb{K} = \mathbb{R}$, for example.

Any 1d linear subspace of $\mathbb{R}^3$ determines determines a projection $p \in \mathfrak{h}_3(\mathbb{R})$, namely orthogonal projection onto that subspace, and $\mathrm{tr}(p) = 1$, so p is a point in $\mathbb{R}\mathrm{P}^2$.
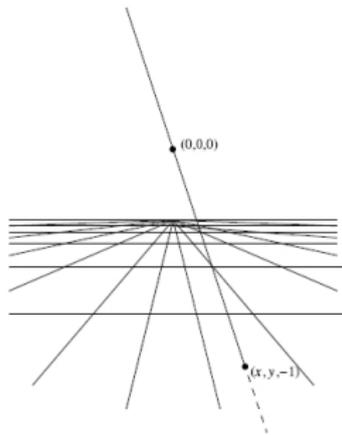
Similarly any 2d linear subspace of $\mathbb{R}^3$ determines a projection $\ell \in \mathfrak{h}_3(\mathbb{R})$ with $\mathrm{tr}(\ell) = 2$, so ℓ is a line in $\mathbb{R}\mathrm{P}^2$.

Similarly, 1d and 2d subspaces of $\mathbb{C}^3$ give points and lines in $\mathbb{C}\mathrm{P}^2$. But for $\mathbb{H}$ (noncommutative) and especially $\mathbb{O}$ (nonassociative), the Jordan algebra approach becomes very useful.

We get 4 projective planes:

- $\mathfrak{h}_3(\mathbb{R})$ gives $\mathbb{R}\mathrm{P}^2$, with a 2d manifold of points.
- $\mathfrak{h}_3(\mathbb{C})$ gives $\mathbb{C}\mathrm{P}^2$, with a 4d manifold of points.
- $\mathfrak{h}_3(\mathbb{H})$ gives $\mathbb{H}\mathrm{P}^2$, with an 8d manifold of points.
- $\mathfrak{h}_3(\mathbb{O})$ gives $\mathbb{O}\mathrm{P}^2$, with a 16d manifold of points.

Similarly, we define the **projective line** $\mathbb{K}P^1$ to be the manifold whose points are the projections $p \in \mathfrak{h}_2(\mathbb{K})$ with $\operatorname{tr}(p) = 1$.

- $\mathbb{R}P^1 \cong S^1$.
- $\mathbb{C}P^1 \cong S^2$.
- $\mathbb{H}P^1 \cong S^4$.
- $\mathbb{O}P^1 \cong S^8$.

Each line in $\mathbb{K}P^2$ is a copy of the projective line $\mathbb{K}P^1$.

$$\mathbb{C}P^1 =$$
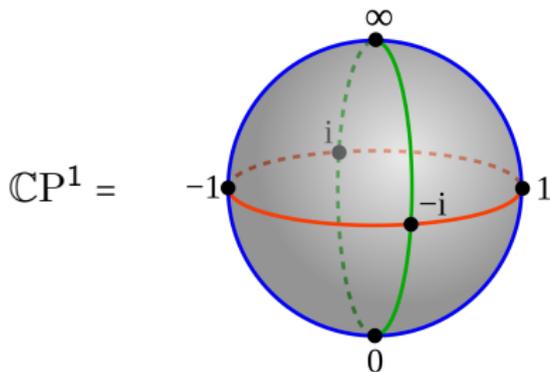
Thus we have *many choices* of inclusions

$$\begin{array}{ccccccc}
\mathbb{R}\mathrm{P}^1 & \hookrightarrow & \mathbb{C}\mathrm{P}^1 & \hookrightarrow & \mathbb{H}\mathrm{P}^1 & \hookrightarrow & \mathbb{O}\mathrm{P}^1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\mathrm{P}^2 & \hookrightarrow & \mathbb{C}\mathrm{P}^2 & \hookrightarrow & \mathbb{H}\mathrm{P}^2 & \hookrightarrow & \mathbb{O}\mathrm{P}^2
\end{array}$$

and this is rich playground for projective geometry.

If $A \subset \mathfrak{h}_3(\mathbb{O})$ is a Jordan subalgebra with $A \cong \mathfrak{h}_2(\mathbb{O})$, then its identity is a projection $\ell \in \mathfrak{h}_3(\mathbb{O})$ with $\mathrm{tr}(\ell) = 2$, so it defines a line in $\mathbb{O}\mathrm{P}^2$.

Let's call such ℓ an **octonionic projective line** in $\mathbb{O}\mathrm{P}^2$.

There is a map sending subalgebras $A \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$ to octonionic projective lines in $\mathbb{O}\mathrm{P}^2$. In fact this map is a bijection, and equivariant under the action of F_4.

Since Lemma 1 says F_4 acts transitively on the set of Jordan subalgebras $A \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$, we get:

Lemma 1′ (well-known). The group F_4 acts transitively on the set of octonionic projective lines in $\mathbb{O}\mathrm{P}^2$.

Example: the subalgebra

$$A = \left\{ \left(\begin{array}{ccc} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{array} \right) : \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

of $\mathfrak{h}_3(\mathbb{O})$ has identity

$$\ell = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

which is an octonionic projective line in $\mathbb{OP}^2$.

If $B \subset \mathfrak{h}_3(\mathbb{O})$ is a Jordan subalgebra with $B \cong \mathfrak{h}_3(\mathbb{C})$, then each projection $p \in B$ with $\mathrm{tr}(p) = 1$ defines a point in $\mathbb{O}\mathrm{P}^2$, and such points form a submanifold $X \subset \mathbb{O}\mathrm{P}^2$.

Let us call such X a **complex projective plane in** $\mathbb{O}\mathrm{P}^2$. Any such X has the structure of a projective plane isomorphic to $\mathbb{C}\mathrm{P}^2$.

We get a map sending Jordan subalgebras $B \subset \mathfrak{h}_3(\mathbb{O})$ with $B \cong \mathfrak{h}_3(\mathbb{C})$ to complex projective planes in $\mathbb{O}\mathrm{P}^2$. In fact this map is a bijection, and equivariant under the action of F_4.

Since we know F_4 acts transitively on the set of Jordan subalgebras $B \subset \mathfrak{h}_3(\mathbb{O})$ with $B \cong \mathfrak{h}_3(\mathbb{C})$, we get:

Lemma 2′. The group F_4 acts transitively on the set of complex projective planes in $\mathbb{O}\mathrm{P}^2$.

The Conjectured Lemma then implies this:

Main Conjecture: Projective Geometry Version. Suppose

- X is complex projective plane in $\mathbb{OP}^2$
- ℓ is an octonionic projective line in $\mathbb{OP}^2$
- $X \cap \ell$ is a complex projective line in X.

Then the group of automorphisms of $\mathfrak{h}_3(\mathbb{O})$ preserving both X and ℓ is the gauge group of the Standard Model:

$$\mathrm{Stab}(X) \cap \mathrm{Stab}(\ell) \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

Warning: I am *not* claiming this conjecture, if true, explains the Standard Model of particle physics. It could easily be a red herring.

However, it suggests some interesting lines of investigation, both in math and in physics.