# Loop Groups and Lie 2-Algebras

Alissa S. Crans

Joint work with:

John Baez
Urs Schreiber
& Danny Stevenson

## in honor of Ross Street's 60th birthday

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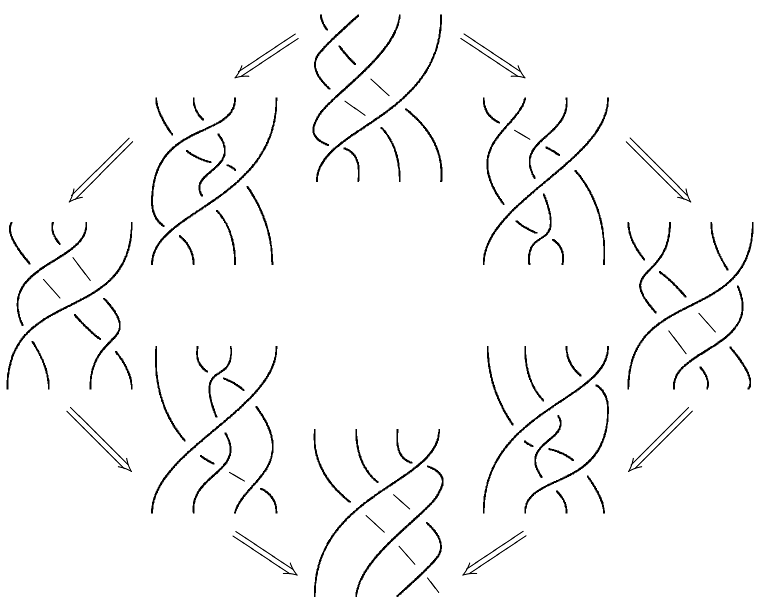


# Lie 2-Algebras

A **2-vector space** $L$ is a category in Vect, the category of vector spaces.

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

**Theorem.** The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

**Moral:** Homological algebra is secretly categorified linear algebra!

A **Lie 2-algebra** consists of:

- a 2-vector space $L$

equipped with:

- a functor called the **bracket**:

$$[\cdot,\cdot]\colon L \times L \to L$$

  bilinear and skew-symmetric as a function of objects and morphisms,

- a natural isomorphism called the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

  trilinear and antisymmetric as a function of the objects $x, y, z$,

such that:

- the **Jacobiator identity** holds, meaning the following diagram commutes:

$$\begin{tikzcd}
& {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y},z]}"'] \arrow[dr, "1"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],y],z]} \arrow[d, "{J_{[w,x],y,z}}"] \\
\begin{array}{c}[[[w,y],z],x]+[[w,y],[x,z]]\\ +[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[d, "{[J_{w,y,z},x]}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]}"] \\
\begin{array}{c}[[[w,z],y],x]+[[w,[y,z]],x]\\ +[[w,y],[x,z]]+[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[dr, "{[w,J_{x,y,z}]}"'] & & \begin{array}{c}[[w,[x,z]],y]\\ +[[w,x],[y,z]]+[[[w,z],x],y]\end{array} \arrow[dl, "{\begin{array}{c}J_{w,[x,z],y}\\ +J_{[w,z],x,y}+J_{w,x,[y,z]}\end{array}}"] \\
& \begin{array}{c}[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]]\\ +[w,[[x,z],y]]+[[w,[y,z]],x]+[w,[x,[y,z]]]\end{array} &
\end{tikzcd}$$

Given a vector space $V$ and an isomorphism

$$B\colon V \otimes V \to V \otimes V,$$

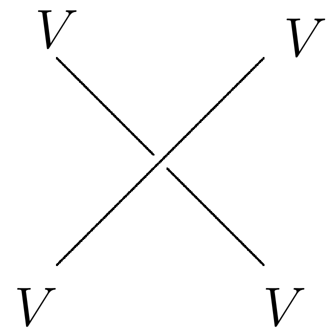
we say $B$ is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

$$(B\otimes 1)(1\otimes B)(B\otimes 1) = (1\otimes B)(B\otimes 1)(1\otimes B),$$

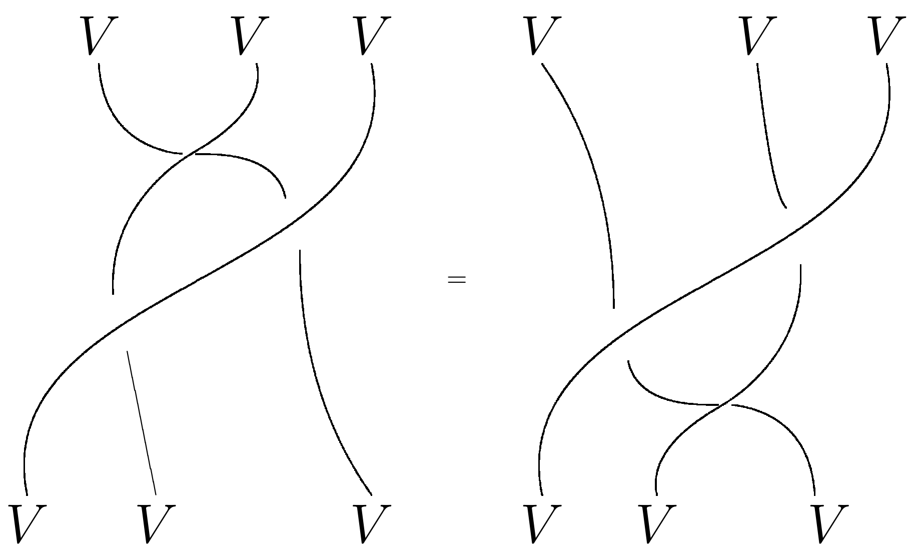
or in other words, that this diagram commutes:

$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "1\otimes B"'] \arrow[dr, "B\otimes 1"] & \\
V \otimes V \otimes V \arrow[d, "B\otimes 1"'] & & V \otimes V \otimes V \arrow[d, "1\otimes B"] \\
V \otimes V \otimes V \arrow[dr, "1\otimes B"'] & & V \otimes V \otimes V \arrow[dl, "B\otimes 1"] \\
 & V \otimes V \otimes V &
\end{tikzcd}$$

If we draw $B\colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 


the Yang–Baxter equation says that:



**Proposition.** Let $L$ be a vector space over $k$ equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L.$$

Let $L' = k \oplus L$ and define the isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x)+(1,0)\otimes(0,[x,y]).$$

Then $B$ is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

# Zamolodchikov tetrahedron equation

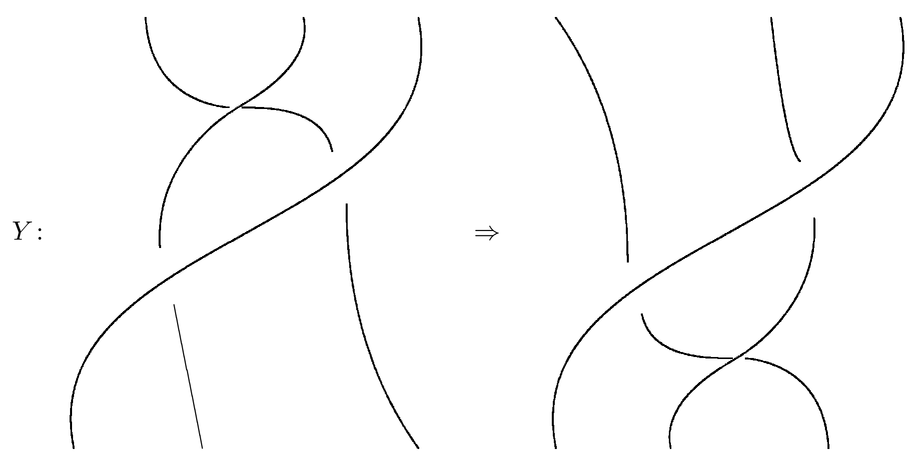
Given a 2-vector space $V$ and an invertible linear functor $B \colon V \otimes V \to V \otimes V$, a linear natural isomorphism

$$Y \colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

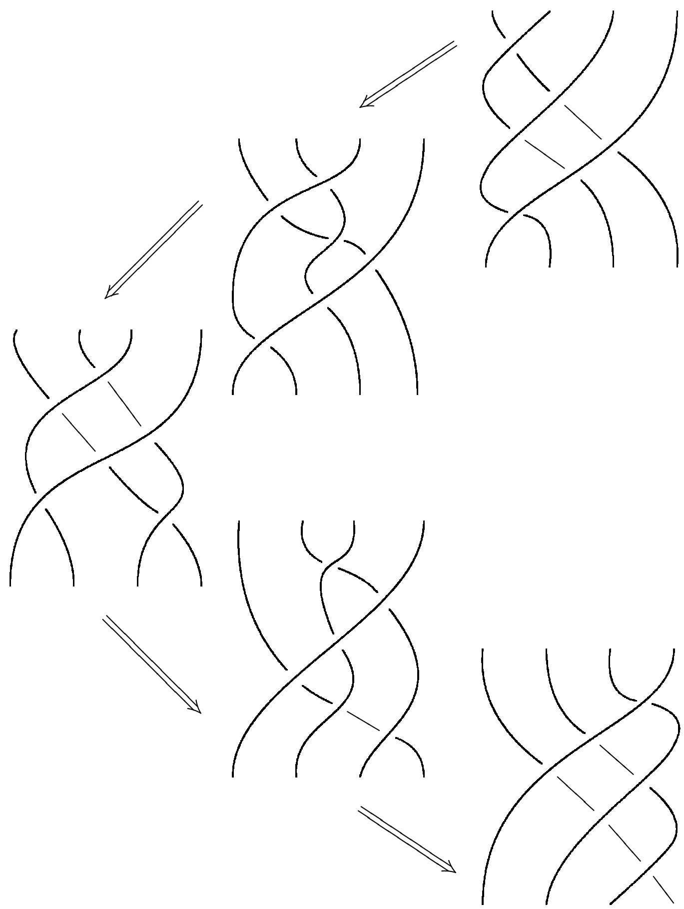
satisfies the **Zamolodchikov tetrahedron equation** if:

$$\begin{gathered}
[Y \circ (1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1)][(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1) \circ Y \circ (B \otimes 1 \otimes 1)] \\
[(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B) \circ Y \circ (1 \otimes 1 \otimes B)][Y \circ (B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B)] \\
= \\
[(B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)(1 \otimes 1 \otimes B) \circ Y][(B \otimes 1 \otimes 1) \circ Y \circ (B \otimes 1 \otimes 1)(1 \otimes B \otimes 1)] \\
[(1 \otimes 1 \otimes B) \circ Y \circ (1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)][(1 \otimes 1 \otimes B)(1 \otimes B \otimes 1)(B \otimes 1 \otimes 1) \circ Y]
\end{gathered}$$

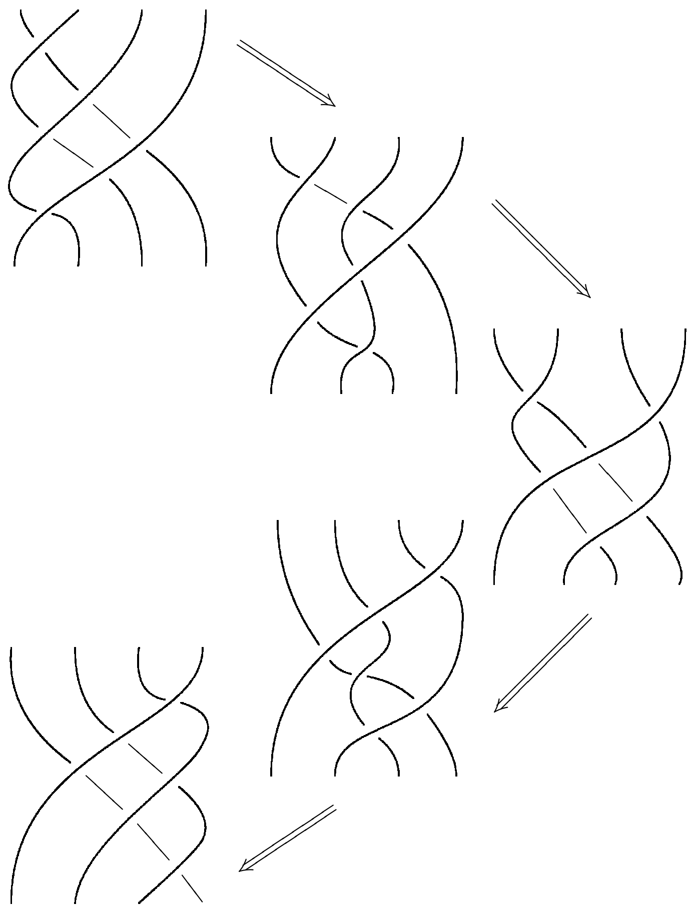
We should think of $Y$ as the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



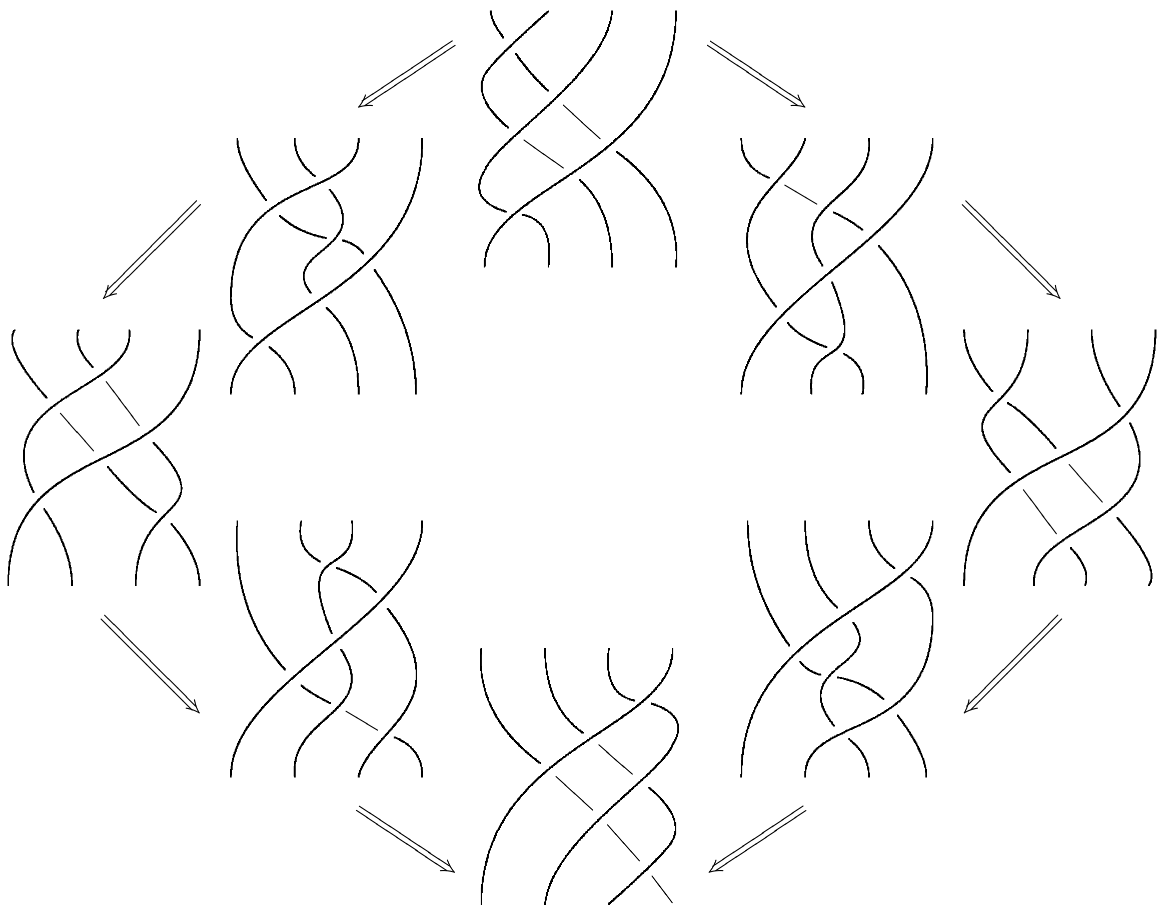
Left side of Zamolodchikov tetrahedron equation:



Right side of Zamolodchikov tetrahedron equation:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



**Theorem.** Let $L$ be a 2-vector space, let $[\cdot,\cdot]\colon L \times L \to L$ be a skew-symmetric bilinear functor, and let $J$ be a completely antisymmetric trilinear natural transformation with

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

Let $L' = K \oplus L$, where $K$ is the categorified ground field.
Let $B\colon L' \otimes L' \to L' \otimes L'$ be defined as follows:

$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y])$$

whenever $(a,x)$ and $(b,y)$ are both either objects or morphisms in $L'$. Finally, let

$$Y\colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

be defined as follows:

$$Y = \begin{tikzcd}
L' \otimes L' \otimes L' \arrow[d, "p\otimes p\otimes p"] \\
\underset{(x,y,z)}{L \otimes L \otimes L} \arrow[d, bend right=60, "{[[x,y],z]}"'] \arrow[d, bend left=60, "{[x,[y,z]]+[[x,z],y]}"] \arrow[d, phantom, "\overset{J}{\Longrightarrow}"] \\
\underset{a}{L} \arrow[d, hook, "j"] \\
\underset{(1,0)\otimes(1,0)\otimes(0,a)}{L' \otimes L' \otimes L'}
\end{tikzcd}$$

where $a$ is either an object or morphism of $L$. Then $Y$ is a solution of the Zamolodchikov tetrahedron equation if and only if $J$ satisfies the Jacobiator identity.

# Hierarchy of Higher Commutativity

| **Topology** | **Algebra** |
|---|---|
| Crossing | Commutator |
| Crossing of crossings | Jacobi identity |
| Crossing of crossing of crossings<br>⋮ | Jacobiator identity<br>⋮ |

We can define **homomorphisms** between Lie 2-algebras, and **2-homomorphisms** between these.

**Theorem.** The 2-category of Lie 2-algebras, homomorphisms and 2-homomorphisms is equivalent to the 2-category of:

- 2-term $L_\infty$-algebras,
- $L_\infty$-homomorphisms between these,
- $L_\infty$-2-homomorphisms between these.

The Lie 2-algebras $L$ and $L'$ are **equivalent** if there are homomorphisms

$$f: L \to L' \qquad \bar{f}: L' \to L$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

**Theorem.** Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

# The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra having $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that $\rho$ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. By Whitehead's lemma, this only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle.$$

The Lie algebra $\mathfrak{g}$ together with the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and $k$ times the above 3-cocycle give the Lie 2-algebra $\mathfrak{g}_k$.

The 2-term $L_\infty$-algebra $V$ corresponding to the Lie 2-algebra $\mathfrak{g}_k$ is given by:

- $V_0 =$ the Lie algebra $\mathfrak{g}$,
- $V_1 = \mathbb{R}$,
- $d\colon V_1 \to V_0$ is the zero map,
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $\mathfrak{g}$:
$$l_2(x, y) = [x, y],$$
  and $l_2\colon V_0 \times V_1 \to V_1$ given by the trivial representation $\rho$ of $\mathfrak{g}$ on $\mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ given by:
$$l_3(x, y, z) = k\langle [x, y], z\rangle$$
  for all $x, y, z \in \mathfrak{g}$.

In summary: *every simple Lie algebra $\mathfrak{g}$ gives a one-parameter family of Lie 2-algebras, $\mathfrak{g}_k$, which reduces to $\mathfrak{g}$ when $k = 0$!*

**Puzzle:** Does $\mathfrak{g}_k$ come from a Lie 2-group?

Suppose we try to copy the construction of $\mathfrak{g}_k$ for a particularly nice kind of Lie group. Let $G$ be a simply-connected compact simple Lie group whose Lie algebra is $\mathfrak{g}$. We have

$$H^3(G, \mathrm{U}(1)) \stackrel{\iota}{\hookleftarrow} \mathbb{Z} \hookrightarrow \mathbb{R} \cong H^3(\mathfrak{g}, \mathbb{R})$$

Using the classification of 2-groups, we can build a skeletal 2-group $G_k$ for $k \in \mathbb{Z}$:

- $G$ as its group of objects,
- $\mathrm{U}(1)$ as the group of automorphisms of any object,
- the trivial action of $G$ on $\mathrm{U}(1)$,
- $[a] \in H^3(G, \mathrm{U}(1))$ given by $k\, \iota[\nu]$, which is nontrivial when $k \neq 0$.

**Question:** Can $G_k$ be made into a Lie 2-group?

Here's the bad news:

**(Bad News) Theorem.** Unless $k = 0$, there is no way to give the 2-group $G_k$ the structure of a Lie 2-group for which the group $G$ of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.

**(Good News) Theorem.** For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f\colon [0, 2\pi] \to G$ starting at the identity. A morphism from $f_1$ to $f_2$ is an equivalence class of pairs $(D, \alpha)$ consisting of a disk $D$ going from $f_1$ to $f_2$ together with $\alpha \in \mathrm{U}(1)$.

For any two such pairs $(D_1, \alpha_1)$ and $(D_2, \alpha_2)$ there is a 3-ball $B$ whose boundary is $D_1 \cup D_2$, and the pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where $\nu$ is the left-invariant closed 3-form on $G$ with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that $\nu$ gives an integral cohomology class.

# The Lie 2-Algebra $\mathcal{P}_k\mathfrak{g}$

$\mathcal{P}_k G$ is a particularly nice kind of Lie 2-group: a *strict* one! Thus, its Lie 2-algebra is easy to compute.

The 2-term $L_\infty$-algebra $V$ corresponding to the Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is given by:

- $V_0 = P_0\mathfrak{g}$

- $V_1 = \widehat{\Omega_k\mathfrak{g}} \cong \Omega\mathfrak{g} \oplus \mathbb{R}$,

- $d\colon V_1 \to V_0$ equal to the composite
$$\widehat{\Omega_k\mathfrak{g}} \to \Omega\mathfrak{g} \hookrightarrow P_0\mathfrak{g}\,,$$

- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $P_0\mathfrak{g}$:
$$l_2(p_1, p_2) = [p_1, p_2],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the action $d\alpha$ of $P_0\mathfrak{g}$ on $\widehat{\Omega_k\mathfrak{g}}$, or explicitly:
$$l_2(p, (\ell, c)) = \Big([p, \ell],\ 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle \, d\theta\ \Big)$$
  for all $p \in P_0\mathfrak{g}$, $\ell \in \Omega G$ and $c \in \mathbb{R}$,

- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ equal to zero.

The 2-term $L_\infty$-algebra $V$ corresponding to the Lie 2-algebra $\mathfrak{g}_k$ is given by:

- $V_0 =$ the Lie algebra $\mathfrak{g}$,
- $V_1 = \mathbb{R}$,
- $d\colon V_1 \to V_0$ is the zero map,
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $\mathfrak{g}$:

  $$l_2(x, y) = [x, y],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the trivial representation $\rho$ of $\mathfrak{g}$ on $\mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ given by:

  $$l_3(x, y, z) = k\langle [x, y], z\rangle$$

  for all $x, y, z \in \mathfrak{g}$.

# The Equivalence $\mathcal{P}_k\mathfrak{g} \simeq \mathfrak{g}_k$

We describe the two Lie 2-algebra homomorphisms forming our equivalence in terms of their corresponding $L_\infty$-algebra homomorphisms:

- $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ has:

$$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell, c) &= c \end{aligned}$$

  where $p \in P_0\mathfrak{g}$, $\ell \in \Omega\mathfrak{g}$, and $c \in \mathbb{R}$.

- $\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$ has:

$$\begin{aligned} \psi_0(x) &= xf \\ \psi_1(c) &= (0, c) \end{aligned}$$

  where $x \in \mathfrak{g}$, $c \in \mathbb{R}$, and $f\colon [0, 2\pi] \to \mathbb{R}$ is a smooth function with $f(0) = 0$ and $f(2\pi) = 1$.

**Theorem.** With the above definitions we have:

- $\phi \circ \psi$ is the identity Lie 2-algebra homomorphism on $\mathfrak{g}_k$, and
- $\psi \circ \phi$ is isomorphic, as a Lie 2-algebra homomorphism, to the identity on $\mathcal{P}_k\mathfrak{g}$.

# What's Next?

We know how to get Lie $n$-algebras from Lie algebra cohomology! We should:

- Classify their representations
- Find their corresponding Lie $n$-groups
- Understand their relation to higher braid theory

Moreover, many other questions remain:

- Weak $n$-categories in Vect?
- Weakening laws governing addition and scalar multiplication?
- Weakening the antisymmetry of the bracket in the definition of Lie 2-algebra?
- What's a free Lie 2-algebra on a 2-vector space?
- Lie 2-algebra cohomology? $L_\infty$-algebra cohomology?
- Deformations of Lie 2-algebras?