LOOP GROUPS AND CATEGORIFIED GEOMETRY

Notes for talk at Streetfest

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Lie 2-groups

A (strict) **Lie 2-group** is a small category $\mathcal{G}$ such that

- the set of objects $\mathcal{G}_0$ and
- the set of morphisms $\mathcal{G}_1$

are Lie groups;

- source and target $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$,
- the identity assigning function $i\colon \mathcal{G}_0 \to \mathcal{G}_1$,
- composition $\circ\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$

are all homomorphisms of Lie groups.

By **Lie group** we mean a group G which is also a manifold such that the map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh^{-1} \end{aligned}$$

is smooth, but by **manifold** we could mean an infinite dimensional manifold modelled on a locally convex topological vector space, for example a **Fréchet space**. In particular, we will consider **Fréchet Lie groups** and hence **Fréchet Lie 2-groups**.

Remarks

1. There is a notion of **weak** Lie 2-group.
2. Lie 2-groups are the same as **Lie crossed modules**

$$\begin{aligned} \partial&\colon H \to G \\ \alpha&\colon G \to \mathrm{Aut}(H) \end{aligned}$$

3. Every Lie 2-group $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ has an associated **Lie 2-algebra** $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1)$

$$\mathfrak{g}_0 = \mathrm{Lie}(\mathcal{G}_0)$$
$$\mathfrak{g}_1 = \mathrm{Lie}(\mathcal{G}_1)$$

The differentials ds, dt, di and $d \circ \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$ are all Lie algebra homomorphisms.

Let G be a compact, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$. For any $k \in \mathbb{R}$ Baez and Crans construct a (weak) Lie 2-algebra $\mathfrak{g}_k$ with

$$\mathrm{Ob}(\mathfrak{g}_k) = \mathfrak{g}$$
$$\mathrm{Mor}(\mathfrak{g}_k) = \mathfrak{g} \oplus i\mathbb{R}$$

but where the 'Jacobiator' is given by the **basic** 3-form $k\langle x, [y, z]\rangle$ on G.

Question: Is $\mathfrak{g}_k$ the Lie 2-algebra of some Lie 2-group**??**

For any $k \in \mathbb{Z}$ Baez and Lauda construct a (weak) 2-group G_k but this is **not** a **Lie** 2-group.

We will explain how to construct a **Fréchet** Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra is equivalent to $\mathfrak{g}_k$. $\mathcal{P}_k G$ is closely related to central extensions of **loop groups** and to the **basic gerbe** on G.

Loop Groups

Let G be a compact, simple, simply connected Lie group. We define

$$LG = \{f\colon [0, 2\pi] \to G |\ f \text{ is } C^\infty, f(0) = f(2\pi)\} \ - \ \text{the **loop group**}$$

and

$$\Omega G = \{f \in LG |\ f(0) = 1\} \ - \ \text{the **based** loopgroup}$$

LG and ΩG are Fréchet Lie groups. ΩG behaves like a compact Lie group in many ways except that there exist **topologically non-trivial central extensions**

$$1 \to \mathbb{T} \to \widehat{\Omega_k G} \to \Omega G \to 1 \ - \ \text{the **Kac-Moody** group}$$

where $\widehat{\Omega_k G}$ is the **Kac-Moody** group. There is a corresponding central extension of Lie algebras

$$0 \to i\mathbb{R} \to \Omega_k \mathfrak{g} \to \Omega g \to 0$$

This is easier to understand: it is determined up to isomorphism by the **Kac-Moody 2-cocycle**

$$\omega(f,g) = 2\int_0^{2\pi} \langle f(\theta), g'(\theta)\rangle d\theta$$

where $f, g \in \Omega g = \{f\colon [0,2\pi] \to \mathfrak{g} | \ f \text{ is } C^\infty, f(0) = f(2\pi) = 0\}$ and $\langle\, ,\, \rangle$ denotes the Killing form normalised so that $|h_\theta| = \frac{1}{\sqrt{2\pi}}$ where h_θ is the co-root associated to the longest root θ.

Kac-Moody Central Extension

Let $\mathcal{G}$ be a simply connected Lie group forming part of a central extension

$$1 \to \mathbb{T} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

so that $\mathcal{G}$ is the total space of a principal $\mathbb{T}$-bundle over $\mathcal{G}$ which is also a group containing $\mathbb{T}$ as a central subgroup. Suppose also that $\mathcal{G}$ is equipped with a connection ∇ whose curvature 2-form is F_∇. Denote by

$$P_0\mathcal{G} = \{f\colon [0,2\pi] \to G| \ f \text{ is } C^\infty, f(0) = 1\}$$

the group of **based paths** in $\mathcal{G}$. $P_0\mathcal{G}$ is a group under pointwise multiplication of paths in $\mathcal{G}$. Note that there is a homomorphism $\pi\colon P_0\mathcal{G} \to \mathcal{G}$ which evaluates a path at its endpoint: $\pi(f) = f(2\pi)$. The kernel of π is just the based loop group ΩG.

We can use the homomorphism π to pullback the central extension $\hat{\mathcal{G}}$ to obtain a new group $\pi^*\hat{\mathcal{G}}$ which is a central extension of $P_0\mathcal{G}$

$$\begin{tikzcd} \pi^*\hat{\mathcal{G}} \arrow[r] \arrow[d] & \hat{\mathcal{G}} \arrow[d] \\ P_0\mathcal{G} \arrow[r, "\pi"] & \mathcal{G} \end{tikzcd}$$

$P_0\mathcal{G}$ is **contractible** and hence the central extension $1 \to \mathbb{T} \to \pi^*\hat{\mathcal{G}} \to P_0\mathcal{G} \to 1$ is **split**. In fact a splitting can be constructed explicitly as follows using the connection ∇: if f is a based path in $P_0\mathcal{G}$ denote by $\hat{f}$ the unique horizontal lift of f to a path in $\hat{\mathcal{G}}$ starting at 1. Then $\sigma\colon P_0\mathcal{G} \to \pi^*\hat{\mathcal{G}}$ defined by $\sigma(f) = (f, \hat{f})$ provides such a splitting. In particular we obtain an isomorphism $\pi^*\hat{\mathcal{G}} \cong P_0\mathcal{G} \times \mathbb{T}$. The product on the group $P_0\mathcal{G} \times \mathbb{T}$ is determined however by a $\mathbb{T}$-valued 2-cocycle

$$c\colon P_0\mathcal{G} \times P_0\mathcal{G} \to \mathbb{T}$$

If $\mathcal{G} = \Omega G$ then

$$c(f,g) = \exp\left(2ik\int_0^{2\pi}\int_0^{2\pi} \langle f(t)^{-1}f'(t), g'(\theta)g(\theta)^{-1}\rangle d\theta\, dt\right)$$

where $f = f(t,\theta), g = g(t,\theta) \in P_0\Omega G$.

So we have the commutative diagram of groups and group homomorphisms

$$\begin{array}{ccc} P_0\mathcal{G}\times\mathbb{T} & \xrightarrow{\hat{\pi}} & \hat{\mathcal{G}} \\ \downarrow & & \downarrow \\ P_0\mathcal{G} & \xrightarrow{\pi} & \mathcal{G} \end{array}$$

where $\hat{\pi}(f,z) = \hat{f}(2\pi)z$. The kernel $\ker\hat{\pi}$ is the normal subgroup

$$\begin{aligned} \ker\hat{\pi} &= \{(f,z)|\ \hat{f}(2\pi) = z^{-1}, f(2\pi) = 1\} \\ &= \{(f,z)|z^{-1} = \mathrm{Hol}_f(\nabla)\} \end{aligned}$$

where $\mathrm{Hol}_f(\nabla)$ denotes the **holonomy** of ∇ around the loop f. Since $\mathcal{G}$ is simply connected,

$$\mathrm{Hol}_f(\nabla) = \exp\left(2\pi i\int_{D_f} F_\nabla\right)$$

where D_f is any disc with boundary the loop f. If $\mathcal{G} = \Omega G$ then F_∇ is the left invariant 2-form whose value at the identity is just the Kac-Moody 2-cocycle:

$$F_\nabla(\xi,\eta) = 2\int_0^{2\pi}\langle\xi(\theta),\eta'(\theta)\rangle d\theta$$

The point of this construction is that we can recover $\hat{\mathcal{G}}$ by

1. equipping $P_0\mathcal{G}\times\mathbb{T}$ with the product coming from the 2-cocycle c, and
2. letting $\hat{\mathcal{G}}$ be the **quotient**

$$\hat{\mathcal{G}} = \frac{P_0\mathcal{G}\times\mathbb{T}}{N}$$

where N is the **normal subgroup**

$$N = \{(\gamma,z)|\ \gamma\in\Omega G, z^{-1} = \exp\left(2\pi i\int_{D_\gamma} F_\nabla\right)\}$$

Construction of $\mathcal{P}_k G$

As above, let $P_0 G = \{f\colon [0,2\pi] \to G |\ f \text{ is} C^\infty, f(0) = 1\}$. $P_0 G$ acts on ΩG by conjugation and induces an action on $P_0\Omega G$. Define an action of $P_0 G$ on $P_0\Omega G \times \mathbb{T}$ by

$$p \cdot (f, z) = (p^{-1} f p, z \exp\left(ik \int_0^{2\pi} \int_0^{2\pi} \langle f(t)^{-1} f'(t), p'(\theta) p(\theta)^{-1} \rangle d\theta\, dt \right))$$

where $p = p(\theta) \in P_0 G$ and $f = f(t, \theta) \in P_0\Omega G$. This action of $P_0 G$ preserves the normal subgroup N and induces an action of $P_0 G$ on $\widehat{\Omega_k G}$ by automorphisms, so that we have a Fréchet Lie crossed module

$$\begin{aligned} &\alpha\colon P_0 G \to \mathrm{Aut}(\widehat{\Omega_k G}) \\ &\partial\colon \widehat{\Omega_k G} \to P_0 G \end{aligned}$$

where ∂ is defined as the composite $\widehat{\Omega_k G} \xrightarrow{p} \Omega G \xhookrightarrow{i} P_0 G$.

Let $\mathcal{P}_k G$ have Fréchet Lie group $\mathrm{Ob}(\mathcal{P}_k G)$ of **objects** given by

$$\mathrm{Ob}(\mathcal{P}_k G) = P_0 G$$

and Fréchet Lie group $\mathrm{Mor}(\mathcal{P}_k G)$ of **morphisms** given by the semi-direct product

$$\mathrm{Mor}(\mathcal{P}_k G) = P_0 G \ltimes \widehat{\Omega_k G}$$

Then $\mathcal{P}_k G = (P_0 G, P_0 G \ltimes \widehat{\Omega_k G})$ is a Fréchet Lie 2-group when source, target, composition etc are defined as follows:

source $s(p, \hat\gamma) = p$

target $t(p, \hat\gamma) = p\partial(\hat\gamma)$

composition $(p_1, \hat\gamma_1) \circ (p_2, \hat\gamma_2) = (p_1, \hat\gamma_1\hat\gamma_2)$ when $t(p_1, \hat\gamma_1) = s(p_2, \hat\gamma_2)$, i.e. $p_2 = p_1\partial(\hat\gamma_1)$.

identities $i(p) = (p, 1)$.

where $p \in P_0 G$ and $\hat\gamma \in \widehat{\Omega_k G}$.

Theorem 1. *The Lie 2-algebra of $\mathcal{P}_k G$ is equivalent to $\mathfrak{g}_k$.*

Topology of $\mathcal{P}_k G$

The **nerve** of any topological 2-group $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a **simplicial** topological group and we therefore obtain on passing to **geometric realisations** a topological group

$$|\mathcal{P}_k G|$$

In fact more is true: there is an **exact sequence** of topological 2-groups

$$1 \to \mathcal{L}_k G \to \mathcal{P}_k G \to G \to 1$$

where $\mathcal{L}_k G$ is the topological 2-group with

$$\mathrm{Ob}(\mathcal{L}_k G) = \Omega G$$
$$\mathrm{Mor}(\mathcal{L}_k G) = \Omega G \ltimes \widehat{\Omega_k G}$$

and where G is considered as a **discrete**2-group (i.e. there only exist identity morphisms). Here by exact sequence we mean only that the sequences of groups $1 \to \mathrm{Ob}(\mathcal{L}_k G) \to \mathrm{Ob}(\mathcal{P}_k G) \to \mathrm{Ob}(G) \to 1$ and $1 \to \mathrm{Mor}(\mathcal{L}_k G) \to \mathrm{Mor}(\mathcal{P}_k G) \to \mathrm{Mor}(G) \to 1$ are both exact.

Applying the geometric realisation functor $|\cdot|$ we get a short exact sequence of topological groups

$$1 \to |\mathcal{L}_k G| \to |\mathcal{P}_k G| \to G \to 1$$

We note the following two facts:

- $|\mathcal{L}_k G|$ has the homotopy type of a $K(\mathbb{Z}, 2)$
- $|\mathcal{P}_k G| \to G$ is a locally trivial fibre bundle with fibre $K(\mathbb{Z}, 2)$.

Recall that $K(\mathbb{Z}, 2)$ bundles on a space X are classified up to isomorphism by their **Dixmier-Douady** invariants in $H^3(X; \mathbb{Z})$. We have

Theorem 2. *The Dixmier-Douady class of the $K(\mathbb{Z}, 2)$-bundle $|\mathcal{P}_k G| \to G$ is k times the generator of $H^3(G; \mathbb{Z}) = \mathbb{Z}$. When $k = \pm 1$, $|\mathcal{P}_k G|$ is $\hat{G}$ — the topological group obtained by killing the third homotopy group of G.*

Fundamental particles (for example electrons) have extra degrees of freedom (**spin**) — so we need to enlarge the group of symmetries to $\mathrm{Spin}(n)$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

For string theory we need to enlarge the group of symmetries to an even bigger group $\mathrm{String}(n)$

$$1 \to K(\mathbb{Z}, 2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1$$

When $G = \mathrm{Spin}(n)$ and $k = \pm 1$, we obtain the important result that $|\mathcal{P}_k G| = \mathrm{String}(n)$.

$\mathcal{P}_k G$ and gerbes

Note that the Lie 2-group $\mathcal{P}_k G = (P_0 G, P_0 G \ltimes \widehat{\Omega_k G})$ fits into a short exact sequence of groupoids

$$\begin{tikzcd}
\mathbb{T} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P_0 G \ltimes \widehat{\Omega_k G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P_0 G \ltimes \Omega G \arrow[d, shift left] \arrow[d, shift right] \\
* \arrow[r] & P_0 G \arrow[r] & P_0 G
\end{tikzcd}$$

This exhibits $\mathcal{P}_k G$ as a $\mathbb{T}$-**central extension** of groupoids. So $\mathcal{P}_k G$ is a $\mathbb{T}$-**bundle gerbe** in the sense of **Murray**. In this way $\mathcal{P}_k G$ provides a realisation of the **basic gerbe** on G.

In fact $\mathcal{P}_k G$ is a **multiplicative** $\mathbb{T}$-bundle gerbe. Recall that a multiplicative gerbe on G consists of the following data:

- a $\mathbb{T}$-gerbe $\mathcal{G}$ on G
- a morphism $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$
- a coherent natural isomorphism

$$\begin{tikzcd}
\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & \mathcal{G} \otimes \mathcal{G} \arrow[d, "m"] \\
\mathcal{G} \otimes \mathcal{G} \arrow[r, "m"'] & \mathcal{G}
\end{tikzcd}$$

Multiplicative gerbes play a role in Chern-Simons theory and twisted K-theory. It is interesting to note that $\mathcal{P}_k G$ is a **strictly** multiplicative gerbe.

String structures

Suppose that $P \xrightarrow{G} M$ is a principal G-bundle where G is as above. Let $[\nu]$ denote the generator of $H^3(G; \mathbb{Z}) = \mathbb{Z}$. $[\nu]$ is universally transgressive: let $[c] \in H^4(M; \mathbb{Z})$ denote the transgression of $[\nu]$ in the fibre bundle $P \to M$. If $G = \mathrm{Spin}(n)$ then $2[c] = p_1$.

A **string structure** for P is a lift of the structure group of the bundle ΩP to $\widehat{\Omega_k G}$:

$$\begin{tikzcd}
\widehat{\Omega P} \arrow[rr] \arrow[dr, dotted, "\widehat{\Omega_k G}"'] & & \Omega P \arrow[dl, "\Omega G"] \\
& \Omega M &
\end{tikzcd}$$

where $\Omega P \to \Omega M$ is the principal ΩG-bundle one gets by applying the based loops functor Ω to the pointed spaces P and M.

Assume M is 2-connected. Then a string structure for P exists iff $[c] = 0$. In this case, one can construct explicitly a **non-abelian gerbe** for the crossed module $\widehat{\Omega_k G} \xrightarrow{\partial} P_0 G$ in the sense of Breen. The existence of a string structure can also be interpreted, when $G = \mathrm{Spin}(n)$, as a solution to the obstruction problem

$$\begin{tikzcd} & B\mathrm{String}(n) \arrow[d] \\ M \arrow[r] \arrow[ur, dotted] & B\mathrm{Spin}(n) \end{tikzcd}$$

In my opinion, the gerbe referred to above is analogous to an extension of the structure group of P to $\mathrm{String}(n)$.