### LOOP GROUPS AND CATEGORIFIED GEOMETRY

Notes for talk at Streetfest

(joint work with John Baez, Alissa Crans and Urs Schreiber)

## Lie 2-groups

A (strict) Lie 2-group is a small category  $\mathcal{G}$  such that

- the set of objects  $\mathcal{G}_0$  and
- the set of morphisms  $\mathcal{G}_1$

are Lie groups;

- source and target  $s, t: \mathcal{G}_1 \to \mathcal{G}_0$ ,
- the identity assigning function  $i: \mathcal{G}_0 \to \mathcal{G}_1$ ,
- composition  $\circ: \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$

are all homomorphisms of Lie groups.

By **Lie group** we mean a group G which is also a manifold such that the map

$$G \times G \to G$$
$$(g,h) \mapsto gh^{-1}$$

is smooth, but by **manifold** we could mean an infinite dimensional manifold modelled on a locally convex topological vector space, for example a **Fréchet space**. In particular, we will consider **Fréchet Lie groups** and hence **Fréchet Lie 2-groups**.

#### Remarks

- 1. There is a notion of **weak** Lie 2-group.
- 2. Lie 2-groups are the same as Lie crossed modules

$$\partial \colon H \to G$$
$$\alpha \colon G \to \operatorname{Aut}(H)$$

3. Every Lie 2-group  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$  has an associated Lie 2-algebra  $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1)$ 

$$\mathfrak{g}_0 = \operatorname{Lie}(\mathcal{G}_0)$$
  
 $\mathfrak{g}_1 = \operatorname{Lie}(\mathcal{G}_1)$ 

The differentials ds, dt, di and  $d \circ \mathfrak{g}_1 \times_{\mathfrak{g}_0} \mathfrak{g}_1 \to \mathfrak{g}_1$  are all Lie algebra homomorphisms.

Let G be a compact, simple, simply connected Lie group with Lie algebra  $\mathfrak{g}$ . For any  $k \in \mathbb{R}$  Baez and Crans construct a (weak) Lie 2-algebra  $\mathfrak{g}_k$  with

$$Ob(\mathfrak{g}_k) = \mathfrak{g}$$
$$Mor(\mathfrak{g}_k) = \mathfrak{g} \oplus i\mathbb{R}$$

but where the 'Jacobiator' is given by the **basic** 3-form  $k\langle x, [y, z] \rangle$  on G.

**Question**: Is  $\mathfrak{g}_k$  the Lie 2-algebra of some Lie 2-group??

For any  $k \in \mathbb{Z}$  Baez and Lauda construct a (weak) 2-group  $G_k$  but this is **not** a Lie 2-group.

We will explain how to construct a **Fréchet** Lie 2-group  $\mathcal{P}_k G$  whose Lie 2-algebra is equivalent to  $\mathfrak{g}_k$ .  $\mathcal{P}_k G$  is closely related to central extensions of **loop groups** and to the **basic gerbe** on G.

### Loop Groups

Let G be a compact, simple, simply connected Lie group. We define

$$LG = \{f \colon [0, 2\pi] \to G | f \text{ is } C^{\infty}, f(0) = f(2\pi)\}$$
 – the **loop group**

and

 $\Omega G = \{ f \in LG | f(0) = 1 \}$  – the **based** loop group

LG and  $\Omega G$  are Fréchet Lie groups.  $\Omega G$  behaves like a compact Lie group in many ways except that there exist **topologically non-trivial central extensions** 

 $1 \to \mathbb{T} \to \widehat{\Omega_k G} \to \Omega G \to 1$  – the **Kac-Moody** group

where  $\widehat{\Omega_k G}$  is the **Kac-Moody** group. There is a corresponding central extension of Lie algebras

$$0 \to i\mathbb{R} \to \Omega_k \mathfrak{g} \to \Omega g \to 0$$

This is easier to understand: it is determined up to isomorphism by the **Kac-Moody** 2-cocycle

$$\omega(f,g) = 2 \int_0^{2\pi} \langle f(\theta), g'(\theta) \rangle d\theta$$

where  $f, g \in \Omega g = \{f : [0, 2\pi] \to \mathfrak{g} | f \text{ is } C^{\infty}, f(0) = f(2\pi) = 0\}$  and  $\langle , \rangle$  denotes the Killing form normalised so that  $|h_{\theta}| = \frac{1}{\sqrt{2\pi}}$  where  $h_{\theta}$  is the co-root associated to the longest root  $\theta$ .

#### Kac-Moody Central Extension

Let  $\mathcal{G}$  be a simply connected Lie group forming part of a central extension

$$1 \to \mathbb{T} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

so that  $\mathcal{G}$  is the total space of a principal  $\mathbb{T}$ -bundle over  $\mathcal{G}$  which is also a group containing  $\mathbb{T}$  as a central subgroup. Suppose also that  $\mathcal{G}$  is equipped with a connection  $\nabla$  whose curvature 2-form is  $F_{\nabla}$ . Denote by

$$P_0 \mathcal{G} = \{ f : [0, 2\pi] \to G | f \text{ is } C^\infty, f(0) = 1 \}$$

the group of **based paths** in  $\mathcal{G}$ .  $P_0\mathcal{G}$  is a group under pointwise multiplication of paths in  $\mathcal{G}$ . Note that there is a homomorphism  $\pi: P_0\mathcal{G} \to \mathcal{G}$  which evaluates a path at its endpoint:  $\pi(f) = f(2\pi)$ . The kernel of  $\pi$  is just the based loop group  $\Omega G$ .

We can use the homomorphism  $\pi$  to pullback the central extension  $\mathcal{G}$  to obtain a new group  $\pi^* \hat{\mathcal{G}}$  which is a central extension of  $P_0 \mathcal{G}$ 

$$\begin{array}{c} \pi^* \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}} \\ \downarrow & \downarrow \\ P_0 \mathcal{G} \xrightarrow{\pi} \mathcal{G} \end{array}$$

 $P_0\mathcal{G}$  is **contractible** and hence the central extension  $1 \to \mathbb{T} \to \pi^* \hat{\mathcal{G}} \to P_0\mathcal{G} \to 1$  is **split**. In fact a splitting can be constructed explicitly as follows using the connection  $\nabla$ : if f is a based path in  $P_0\mathcal{G}$  denote by  $\hat{f}$  the unique horizontal lift of f to a path in  $\hat{\mathcal{G}}$  starting at 1. Then  $\sigma: P_0\mathcal{G} \to \pi^*\hat{\mathcal{G}}$  defined by  $\sigma(f) = (f, \hat{f})$  provides such a splitting. In particular we obtain an isomorphism  $\pi^*\hat{\mathcal{G}} \cong P_0\mathcal{G} \times \mathbb{T}$ . The product on the group  $P_0\mathcal{G} \times \mathbb{T}$  is determined however by a  $\mathbb{T}$ -valued 2-cocycle

$$c\colon P_0\mathcal{G}\times P_0\mathcal{G}\to\mathbb{T}$$

If  $\mathcal{G} = \Omega G$  then

$$c(f,g) = \exp\left(2ik\int_0^{2\pi}\int_0^{2\pi} \langle f(t)^{-1}f'(t), g'(\theta)g(\theta)^{-1}\rangle d\theta \, dt\right)$$

where  $f = f(t, \theta), g = g(t, \theta) \in P_0\Omega G$ .

So we have the commutative diagram of groups and group homomorphisms

$$\begin{array}{c} P_0 \mathcal{G} \times \mathbb{T} \xrightarrow{\hat{\pi}} \hat{\mathcal{G}} \\ \downarrow & \downarrow \\ P_0 \mathcal{G} \xrightarrow{\pi} \mathcal{G} \end{array}$$

where  $\hat{\pi}(f, z) = \hat{f}(2\pi)z$ . The kernel ker  $\hat{\pi}$  is the normal subgroup

$$\ker \hat{\pi} = \{ (f, z) | \hat{f}(2\pi) = z^{-1}, f(2\pi) = 1 \}$$
$$= \{ (f, z) | z^{-1} = \operatorname{Hol}_{f}(\nabla) \}$$

where  $\operatorname{Hol}_f(\nabla)$  denotes the **holonomy** of  $\nabla$  around the loop f. Since  $\mathcal{G}$  is simply connected,

$$\operatorname{Hol}_{f}(\nabla) = \exp\left(2\pi i \int_{D_{f}} F_{\nabla}\right)$$

where  $D_f$  is any disc with boundary the loop f. If  $\mathcal{G} = \Omega G$  then  $F_{\nabla}$  is the left invariant 2-form whose value at the identity is just the Kac-Moody 2-cocycle:

$$F_{\nabla}(\xi,\eta) = 2 \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle d\theta$$

The point of this construction is that we can recover  $\hat{\mathcal{G}}$  by

- 1. equipping  $P_0 \mathcal{G} \times \mathbb{T}$  with the product coming from the 2-cocycle c, and
- 2. letting  $\hat{\mathcal{G}}$  be the **quotient**

$$\hat{\mathcal{G}} = \frac{P_0 \mathcal{G} \times \mathbb{T}}{N}$$

where N is the **normal subgroup** 

$$N = \{(\gamma, z) \mid \gamma \in \Omega G, z^{-1} = \exp\left(2\pi i \int_{D_{\gamma}} F_{\nabla}\right)\}$$

### Construction of $\mathcal{P}_k G$

As above, let  $P_0G = \{f : [0, 2\pi] \to G | f \text{ is} C^{\infty}, f(0) = 1\}$ .  $P_0G$  acts on  $\Omega G$  by conjugation and induces an action on  $P_0\Omega G$ . Define an action of  $P_0G$  on  $P_0\Omega G \times \mathbb{T}$  by

$$p \cdot (f, z) = (p^{-1}fp, z \exp\left(ik \int_0^{2\pi} \int_0^{2\pi} \langle f(t)^{-1}f'(t), p'(\theta)p(\theta)^{-1} \rangle d\theta \, dt\right))$$

where  $p = p(\theta) \in P_0G$  and  $f = f(t, \theta) \in P_0\Omega G$ . This action of  $P_0G$  preserves the normal subgroup N and induces an action of  $P_0G$  on  $\widehat{\Omega_k G}$  by automorphisms, so that we have a Fréchet Lie crossed module

$$\alpha \colon P_0 G \to \operatorname{Aut}(\widehat{\Omega}_k \widehat{G})$$
$$\partial \colon \widehat{\Omega_k G} \to P_0 G$$

where  $\partial$  is defined as the composite  $\widehat{\Omega_k G} \xrightarrow{p} \Omega G \xrightarrow{i} P_0 G$ .

Let  $\mathcal{P}_k G$  have Fréchet Lie group  $Ob(\mathcal{P}_k G)$  of **objects** given by

$$\operatorname{Ob}(\mathcal{P}_k G) = P_0 G$$

and Fréchet Lie group  $Mor(\mathcal{P}_k G)$  of **morphisms** given by the semi-direct product

$$Mor(\mathcal{P}_k G) = P_0 G \ltimes \widehat{\Omega}_k \widehat{G}$$

Then  $\mathcal{P}_k G = (P_0 G, P_0 G \ltimes \widehat{\Omega_k G})$  is a Fréchet Lie 2-group when source, target, composition etc are defined as follows:

- source  $s(p, \hat{\gamma}) = p$
- target  $t(p, \hat{\gamma}) = p\partial(\hat{\gamma})$
- **composition**  $(p_1, \hat{\gamma}_1) \circ (p_2, \hat{\gamma}_2) = (p_1, \hat{\gamma}_1 \hat{\gamma}_2)$  when  $t(p_1, \hat{\gamma}_1) = s(p_2, \hat{\gamma}_2)$ , i.e.  $p_2 = p_1 \partial(\hat{\gamma}_1)$ .

identities i(p) = (p, 1).

where  $p \in P_0 G$  and  $\hat{\gamma} \in \widehat{\Omega_k G}$ .

**Theorem 1.** The Lie 2-algebra of  $\mathcal{P}_k G$  is equivalent to  $\mathfrak{g}_k$ .

# **Topology of** $\mathcal{P}_k G$

The **nerve** of any topological 2-group  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$  is a **simplicial** topological group and we therefore obtain on passing to **geometric realisations** a topological group

$$|\mathcal{P}_k G|$$

In fact more is true: there is an **exact sequence** of topological 2-groups

$$1 \to \mathcal{L}_k G \to \mathcal{P}_k G \to G \to 1$$

where  $\mathcal{L}_k G$  is the topological 2-group with

$$Ob(\mathcal{L}_k G) = \Omega G$$
$$Mor(\mathcal{L}_k G) = \Omega G \ltimes \widehat{\Omega_k G}$$

and where G is considered as a **discrete**2-group (i.e. there only exist identity morphisms). Here by exact sequence we mean only that the sequences of groups  $1 \to \operatorname{Ob}(\mathcal{L}_k G) \to \operatorname{Ob}(\mathcal{P}_k G) \to \operatorname{Ob}(G) \to 1$  and  $1 \to \operatorname{Mor}(\mathcal{L}_k G) \to \operatorname{Mor}(\mathcal{P}_k G) \to \operatorname{Mor}(G) \to 1$  are both exact.

Applying the geometric realisation functor  $|\cdot|$  we get a short exact sequence of topological groups

$$1 \to |\mathcal{L}_k G| \to |\mathcal{P}_k G| \to G \to 1$$

We note the following two facts:

- $|\mathcal{L}_k G|$  has the homotopy type of a  $K(\mathbb{Z}, 2)$
- $|\mathcal{P}_k G| \to G$  is a locally trivial fibre bundle with fibre  $K(\mathbb{Z}, 2)$ .

Recall that  $K(\mathbb{Z}, 2)$  bundles on a space X are classified up to isomorphism by their **Dixmier-Douady** invariants in  $H^3(X; \mathbb{Z})$ . We have

**Theorem 2.** The Dixmier-Douady class of the  $K(\mathbb{Z}, 2)$ -bundle  $|\mathcal{P}_k G| \to G$ is k times the generator of  $H^3(G; \mathbb{Z}) = \mathbb{Z}$ . When  $k = \pm 1$ ,  $|\mathcal{P}_k G|$  is  $\hat{G}$  — the topological group obtained by killing the third homotopy group of G.

Fundamental particles (for example electrons) have extra degrees of freedom (spin) — so we need to enlarge the group of symmetries to Spin(n)

$$1 \to \mathbb{Z}_2 \to \operatorname{Spin}(n) \to \operatorname{SO}(n) \to 1$$

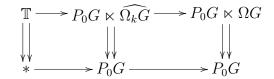
For string theory we need to enlarge the group of symmetries to an even bigger group String(n)

$$1 \to K(\mathbb{Z}, 2) \to \operatorname{String}(n) \to \operatorname{Spin}(n) \to 1$$

When G = Spin(n) and  $k = \pm 1$ , we obtain the important result that  $|\mathcal{P}_k G| = \text{String}(n)$ .

## $\mathcal{P}_k G$ and gerbes

Note that the Lie 2-group  $\mathcal{P}_k G = (P_0 G, P_0 G \ltimes \widehat{\Omega_k G})$  fits into a short exact sequence of groupoids



This exhibits  $\mathcal{P}_k G$  as a **T-central extension** of groupoids. So  $\mathcal{P}_k G$  is a **T-bundle gerbe** in the sense of **Murray**. In this way  $\mathcal{P}_k G$  provides a realisation of the **basic gerbe** on G.

In fact  $\mathcal{P}_k G$  is a **multiplicative**  $\mathbb{T}$ -bundle gerbe. Recall that a multiplicative gerbe on G consists of the following data:

- a  $\mathbb{T}$ -gerbe  $\mathcal{G}$  on G
- a morphism  $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$
- a coherent natural isomorphism

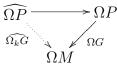
$$\begin{array}{c|c} \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} \xrightarrow{m \otimes 1} \mathcal{G} \otimes \mathcal{G} \\ \downarrow^{1 \otimes m} & & \downarrow^{m} \\ \mathcal{G} \otimes \mathcal{G} \xrightarrow{m} \mathcal{G} \end{array}$$

Multiplicative gerbes play a role in Chern-Simons theory and twisted K-theory. It is interesting to note that  $\mathcal{P}_k G$  is a **strictly** multiplicative gerbe.

### String structures

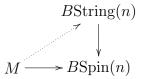
Suppose that  $P \xrightarrow{G} M$  is a principal *G*-bundle where *G* is as above. Let  $[\nu]$  denote the generator of  $H^3(G; \mathbb{Z}) = \mathbb{Z}$ .  $[\nu]$  is universally transgressive: let  $[c] \in H^4(M; \mathbb{Z})$  denote the transgression of  $[\nu]$  in the fibre bundle  $P \to M$ . If G = Spin(n) then  $2[c] = p_1$ .

A string structure for P is a lift of the structure group of the bundle  $\Omega P$  to  $\widehat{\Omega_k G}$ :



where  $\Omega P \to \Omega M$  is the principal  $\Omega G$ -bundle one gets by applying the based loops functor  $\Omega$  to the pointed spaces P and M.

Assume M is 2-connected. Then a string structure for P exists iff [c] = 0. In this case, one can construct explicitly a **non-abelian gerbe** for the crossed module  $\widehat{\Omega_k G} \xrightarrow{\partial} P_0 G$  in the sense of Breen. The existence of a string structure can also be interpreted, when G = Spin(n), as a solution to the obstruction problem



In my opinion, the gerbe referred to above is analogous to an extension of the structure group of P to String(n).