A NOTE ON STRONGLY SEPARABLE ALGEBRAS

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Abstract. Let A be an algebra over a field k. If M is an A–bimodule, we let M^A and M_A denote respectively the k–spaces of invariants and coinvariants of M, and $\varphi_M : M^A \to M_A$ be the natural map. In this note we characterize those algebras A for which φ_M is a natural isomorphism as those separable algebras for which the separability idempotent can be chosen to be symmetric, or as those finite dimensional algebras for which the trace form is non-degenerate. These algebras are called *strongly separable.* We also prove that the cotensor product of C–bicomodules has a right adjoint if and only if the coalgebra C is cosemisimple, and show that if C^* is strongly separable then the adjoint is given by maps of C–bicomodules.

1. Introduction

Let k be a field, G a group such that $|G| \neq 0$ in k and M a G–space. Then the natural map between the spaces of G–invariants and G–coinvariants

$$M^G = \{m \in M \ / \ gm = m \ \forall \ g \in G\} \hookrightarrow M \twoheadrightarrow M_G = M/\langle gm - m \ / \ g \in G, \ m \in M\rangle$$

is an isomorphism, since the map

$$M_G \to M^G, \ \bar{m} \mapsto \frac{1}{|G|}\sum_{g\in G} gm$$

is clearly a well-defined inverse.

In this note we consider the generalization of this question to arbitrary algebras A. In order to formulate the problem, one must consider A–bimodules M (there is no essential distinction between modules and bimodules for group algebras). The space of A–invariants of an A–bimodule M is

$$M^A = \{m \in M \ / \ am = ma \ \forall \ a \in A\}$$

and the space of A–coinvariants is

$$M_A = M/[A, M] \ , \text{ where } \ [A, M] = \langle am - ma \ / \ a \in A, \ m \in M\rangle \ .$$

Here, and elsewhere, $\langle S\rangle$ denotes the k–subspace generated by the set S. Whenever convenient, A–bimodules will be viewed as left A^e–modules, where $A^e = A \otimes A^{op}$.

Let $\varphi_M : M^A \hookrightarrow M \twoheadrightarrow M_A$ be the obvious natural map. The purpose of this note is to characterize those algebras A for which φ is an isomorphism for every M.

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A necessary condition is that A be projective as A^e–module. For, $M^A \cong \mathsf{Hom}_{A^e}(A, M)$ is a left exact functor of M, while $M_A \cong A \otimes_{A^e} M$ is a right exact one; hence, if these functors are naturally isomorphic, they are both exact, so

$$HH^*(M) = \mathsf{Ext}^*_{A^e}(A, M) \equiv 0 \quad \text{and} \quad HH_*(M) = \mathsf{Tor}^{A^e}_*(A, M) \equiv 0 \ ,$$

that is, A is projective as left A^e–module and flat as right A^e–module. A result of Villamayor [V, corollary 2] establishes that either of these conditions already implies the other. When they hold, the k–algebra A is said to be *separable.*

We will show, however, that separability alone is not sufficient to guarantee that φ be an isomorphism. For instance, we will show (corollary 3.1) that $A = M_n(k)$ does not have this property when $\mathsf{char} k$ divides n, even though it is separable. The additional condition that is needed is that the separability idempotent can be chosen to be symmetric (theorem 4.1). Such algebras have been studied before by Hattori [H], and called *strongly separable.* We further show that these algebras are characterized by the property that the trace form is non-degenerate (theorem 3.1). In the last section we show that for coalgebras that are dual to strongly separable algebras, the cotensor product of comodules has a right adjoint given by maps of comodules (corollary 5.1). A more comprehensive study of separability and strong separability, specially in connection to Hopf algebras, can be found in [KS].

2. Strongly separable algebras. Definition and examples

As already mentioned, a k–algebra A is called separable if A is projective as A^e–module. This is equivalent to the existence of an element $e \in A^e$ with the properties that

$$(a \otimes 1)e = (1 \otimes a^{op})e \ \forall a \in A \tag{1}$$

$$\mu_A(e) = 1 \tag{2}$$

where $\mu_A : A^e \to A$ is $\mu_A(a \otimes b^{op}) = ab$ (e arises as the image of $1 \in A$ under a splitting of μ_A). Such an element e is necessarily idempotent and is called a separability idempotent for A. For this basic material the reader is referred to [P], chapter 10. In general e is not unique. For instance if $A = M_n(k)$ then the elements $e_j = \sum_{i=1}^n e_{ij} \otimes e_{ji}$ are separability idempotents for each $j = 1, \ldots, n$ (e_{ij} denote the elementary matrices). Moreover, if $\mathsf{char} k$ does not divide n then $e = \frac{1}{n}\sum_{j=1}^n e_j$ is another separability idempotent. Unlike the others, this one has the additional property of being symmetric:

$$\tau(e) = e \tag{3}$$

where $\tau : A^e \to A^e$ is the antimorphism $\tau(a \otimes b^{op}) = b \otimes a^{op}$.

Definition 2.1. A k–algebra A is said to be strongly separable if it possesses a symmetric separability idempotent $e \in A^e$.

In particular such an algebra is separable. Explicitly, $e = \sum_i u_i \otimes v_i$ must satisfy

$$\sum_i au_i \otimes v_i = \sum_i u_i \otimes v_i a \ \forall a \in A \ , \tag{1}$$

$$\sum_i u_i v_i = 1 \quad \text{and} \tag{2}$$

$$\sum_i u_i \otimes v_i = \sum_i v_i \otimes u_i \ . \tag{3}$$

Remark 2.1. An obvious consequence of the last two equations is

$$\sum_i v_i u_i = 1 \ . \tag{4}$$

It turns out that conditions (1) and (4) suffice to imply the others. To prove this we only need to show that (3) holds, for then (2) follows by symmetry. This is done as follows

$$\sum_i u_i \otimes v_i \overset{(4)}{=} \sum_{i,j} u_i \otimes v_i v_j u_j \overset{(1)}{=} \sum_{i,j} v_j u_i \otimes v_i u_j \overset{(1)}{=} \sum_{i,j} v_j \otimes v_i u_i u_j \overset{(4)}{=} \sum_j v_j \otimes u_j \ .$$

This shows that definition 2.1 coincides with Hattori's [H]. Perhaps the terminology *symmetrically separable* would be more descriptive.

Examples 2.1.

1. The discussion above shows that if $A = M_n(k)$ and $\mathsf{char}k$ does not divide n then A is strongly separable. We will show (corollary 3.1) that if $\mathsf{char}k$ divides n then A is not strongly separable, even though it is separable.
2. If G is a finite group, $A = kG$ and $\mathsf{char}k$ does not divide $|G|$, then A is strongly separable with
$$e = \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1} \ .$$
If $\mathsf{char}k$ divides $|G|$ then A is not even separable, since it is not semisimple.
3. If X is a finite set and $A = k^X$, then A is strongly separable with
$$e = \sum_{x \in X} e_x \otimes e_x \ ,$$
where $e_x(y) = \delta_{x,y} \ \forall x, y \in X$. More generally, according to remark 2.1, any commutative separable k–algebra A is strongly separable, for in this case (4) is just (2). Moreover, this shows that any separability idempotent for A is symmetric. This, together with the uniqueness result in theorem 3.1 below, prove that in a commutative separable algebra the separability idempotent is unique (and symmetric).
4. If $\mathsf{char}k = 0$, then a k–algebra A is strongly separable if and only if it is finite dimensional and semisimple (see corollary 3.1).

We provide one more example of strongly separable algebras; it generalizes that of semisimple group algebras. For background on the relevant material on Hopf algebras the reader is referred to [S1, chapter V], [M, chapter 2] or [Sc, chapter 3]. Part 1 of the next proposition is well-known (cf. Maschke's theorem for Hopf algebras, as in [Sc, thm. 3.2]), except perhaps for the fact that finite dimensionality is not a necessary hypothesis but rather a consequence. We have learned from W. Nichols that part 2 of the next proposition was obtained earlier by F. Kreimer (unpublished). Similar results appear in [OS, section 3].

Proposition 2.1. *Let H be a Hopf algebra.*

1. *H is separable if and only if H is semisimple. In this case, H is finite dimensional.*
2. *If H is semisimple and involutory (i.e. if the antipode S satisfies $S^2 = \mathsf{id}_H$), then H is strongly separable.*

Proof. 1. If H is separable, then it is semisimple and finite dimensional by general results: [P], corollaries 10.4.b and 10.3.

Conversely, if H is semisimple then, first of all, it is finite dimensional by a result of Sweedler [S2, p330]. Now, by Maschke's theorem for Hopf algebras [M, theorem 2.2.1], there is a *left integral* $t \in H$ such that $\epsilon(t) \neq 0$. Also, since H is finite dimensional, S is bijective [M, theorem 2.1.3].

Let $e = \frac{1}{\epsilon(t)} \sum t_2 \otimes S^{-1}(t_1) \in H \otimes H$, where we have written $\Delta(t) = \sum t_1 \otimes t_2$ as usual. Let us show that e is a separability idempotent for H.

First, since S^{-1} is an antipode for H^{cop}, condition (2) for e holds.

Second, for $h \in H$ we have

$$\begin{aligned} S(h) \otimes \Delta(t) &= \sum S(h_1) \otimes \Delta(\epsilon(h_2)t) \\ &= \sum S(h_1) \otimes \Delta(h_2 t) = \sum S(h_1) \otimes h_2 t_1 \otimes h_3 t_2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \sum S(h)t_1 \otimes t_2 = \sum S(h_1)h_2 t_1 \otimes h_3 t_2 = t_1 \otimes h t_2 \\ &\Rightarrow \sum S^{-1}(t_1)h \otimes t_2 = \sum S^{-1}(t_1) \otimes h t_2 \\ &\Rightarrow \sum t_2 \otimes S^{-1}(t_1)h = \sum h t_2 \otimes S^{-1}(t_1)\ , \end{aligned}$$

which gives condition (1) for e. Thus e is a separability idempotent for H. This completes the proof of 1.

2. If in addition $S^2 = \mathsf{id}_H$, then condition (4) holds, so by remark 2.1 e is a symmetric separability idempotent.

□

Corollary 2.1. *Let H be a finite dimensional Hopf algebra H over a field of characteristic zero. The following conditions are equivalent:*

1. *H is involutory,*
2. *H is semisimple,*
3. *H is cosemisimple,*
4. *H is separable,*
5. *H is strongly separable.*

Proof. Conditions 1,2 and 3 are equivalent by theorems of Larson and Radford [LR1, theorem 4] and [LR2, theorem 4.4]. Conditions 1 and 2 imply 5 by proposition 2.1 (or simply by corollary 3.1 below). The implications $5 \Rightarrow 4 \Rightarrow 2$ are trivial. □

Remark 2.2. Recently it has been shown by Etingof and Gelaki [EK] that any (finite dimensional) semisimple and cosemisimple Hopf algebra is involutory, regardless of the characteristic of the field. It follows from proposition 2.1 that any such algebra is strongly separable.

It is conjectured [K] that any (finite dimensional) semisimple Hopf algebra is involutory. This would imply that any semisimple Hopf algebra is strongly separable.

3. Strongly separable algebras and the non-degeneracy of the trace

In this section we characterize strongly separable algebras in terms of the trace form.

Recall that for any finite dimensional k–algebra A the trace form $\mathsf{Tr} : A \to k$ is defined as $\mathsf{Tr}(a) = \mathsf{tr}(l(a))$, where $l : A \to \mathsf{End}_k(A)$ is the left regular representation and tr is the ordinary trace of a linear endomorphism. It gives rise to a symmetric bilinear form $\mathsf{T} : A \times A \to k$ via $\mathsf{T}(a, b) = \mathsf{Tr}(ab)$. When $A = M_n(k)$, $\mathsf{Tr}(a) = n\mathsf{tr}(a)$. In particular $\mathsf{Tr} \equiv 0$ if $\mathsf{char} k$ divides n. Similarly, when $A = kG$, $\mathsf{Tr}(\sum_{g\in G} a_g g) = |G| \cdot a_e$.

The connection between separability and the non-degeneracy of T is well-known in the commutative case (e.g. [J] lemma on page 621). There is a more general result as follows. First let us name one further condition

$$(1)' \qquad \sum_i u_i \otimes av_i = \sum_i u_i a \otimes v_i \ \forall a \in A \ ;$$

obviously $(1) + (3) \Rightarrow (1)'$.

Theorem 3.1. *Let k be a field and A a k–algebra. Then the following are equivalent:*

1. *A is strongly separable.*
2. *A is finite dimensional and $\mathsf{T} : A \times A \to k$ is non-degenerate.*

Moreover, if these conditions hold, then the symmetric separability idempotent is unique.

Proof.
$\underline{1 \Rightarrow 2}$. Let $e = \sum_i u_i \otimes v_i^{op} \in A^e$ be the symmetric separability idempotent. We will show that

$$x = \sum_i \mathsf{Tr}(xu_i)v_i \ \forall x \in A \ ,$$

from where it immediately follows that T is non-degenerate.

First recall that by a theorem of Villamayor and Zelinski ([P, corollary 10.3], or [VZ, proposition 1.1]), any separable algebra is finite dimensional. Thus, the trace form is defined.

We can assume that $\{u_i \ / \ i \in I\}$ is a k–basis for A. For each $j \in I$ write

$$xu_iu_j = \sum_h x^i_{jh} u_h, \text{ with } x^i_{jh} \in k \ .$$

Then

$$\sum_i u_i \otimes u_j v_i x \overset{(1),(1)'}{=} \sum_i x u_i u_j \otimes v_i = \sum_{i,h} x^i_{jh} u_h \otimes v_i = \sum_h u_h \otimes \sum_i x^i_{jh} v_i$$

Hence $u_j v_h x = \sum_i x^i_{jh} v_i$ for each $j, h \in I$. Thus

$$\sum_i \mathsf{Tr}(x u_i) v_i = \sum_i \sum_j x^i_{jj} v_i = \sum_j u_j v_j x \overset{(2)}{=} x$$

as announced.

<u>$2 \Rightarrow 1$</u>. Let $\{u_i \ / \ i \in I\}$ be a k–basis for A and $\{v_i \ / \ i \in I\}$ its dual basis with respect to T, so that

$$x = \sum_i \mathsf{Tr}(x u_i) v_i \ \forall x \in A. \tag{*}$$

We will show that $e = \sum u_i \otimes v_i^{op} \in A^e$ is a symmetric separability idempotent.

Since T is a symmetric form, e is symmetric:

$$\sum_j u_j \otimes v_j \overset{(*)}{=} \sum_{i,j} \mathsf{Tr}(u_j u_i) v_i \otimes v_j = \sum_{i,j} v_i \otimes \mathsf{Tr}(u_i u_j) v_j \overset{(*)}{=} \sum_{i,j} v_i \otimes u_i \ ,$$

proving (3). Now,

$$\sum_i u_i \otimes v_i a \overset{(*)}{=} \sum_{i,j} u_i \otimes \mathsf{Tr}(v_i a u_j) v_j = \sum_{i,j} \mathsf{Tr}(v_i a u_j) u_i \otimes v_j =$$

$$= \sum_{i,j} \mathsf{Tr}(a u_j v_i) u_i \otimes v_j \overset{(3)}{=} \sum_{i,j} \mathsf{Tr}(a u_j u_i) v_i \otimes v_j \overset{(*)}{=} \sum_j a u_j \otimes v_j \ ,$$

proving (1). Recall that $(1) + (3) \Rightarrow (1)'$.

Finally, for each $i, j \in I$ write

$$u_i u_j = \sum_h a^i_{jh} u_h \text{ with } a^i_{jh} \in k \ . \tag{**}$$

Then for each $j \in I$,

$$\sum_i u_i \otimes u_j v_i \overset{(1)'}{=} \sum_i u_i u_j \otimes v_i = \sum_{i,h} a^i_{jh} u_h \otimes v_i = \sum_h u_h \otimes \sum_i a^i_{jh} v_i \ ,$$

from where

$$u_j v_h = \sum_i a^i_{jh} v_i \ \forall j, h \in I \tag{***}$$

Hence

$$1 \overset{(*)}{=} \sum_i \mathsf{Tr}(u_i) v_i \overset{(**)}{=} \sum_{i,j} a^i_{jj} v_i \overset{(***)}{=} \sum_j u_j v_j \ ,$$

proving (2) and thus completing the proof of the implication.

Notice also that the uniqueness of the symmetric separability idempotent has been settled: it is the element of $A \otimes A$ corresponding to $\mathsf{id}_A \in \mathsf{End}_k(A)$ under the isomorphism $t : A \otimes A \to \mathsf{End}_k(A)$, $t(a \otimes b)(c) = \mathsf{Tr}(ca)b$. □

Corollary 3.1. *Let A be a k–algebra.*

1. *If* $\mathsf{char}k = 0$ *then A is strongly separable if and only if it is finite dimensional and semisimple.*
2. *If* $\mathsf{char}k$ *divides n and $A = M_n(k)$ then A is not strongly separable.*

Proof.

1. If A is separable then it is finite dimensional and semisimple by [P], cor.10.3 and 10.4.b. Conversely, assume that A is finite dimensional and semisimple. Let K be an algebraic closure of k. Then
$$A \otimes K \cong M_{n_1}(K) \times \ldots \times M_{n_r}(K)$$
by the Wedderburn-Artin theorem. It follows from the remarks preceding theorem 3.1 that the trace form for $A \otimes K$ over K is non-degenerate, since $\mathsf{char}K = 0$. But the trace form is invariant under extension of scalars, so the same conclusion holds for the trace form for A over k, and the theorem applies to conclude that A is strongly separable.
2. As mentioned before, $\mathsf{Tr} \equiv 0$ in this case, so the theorem applies.

□

Corollary 3.2. *If K is algebraically closed field and A a strongly separable K–algebra, then* $\mathsf{char}K$ *does not divide* $\mathsf{dim}_K S$ *for any simple A–module S.*

Proof. This follows from the proof of the previous corollary, since the dimensions of the simple A–modules are precisely the numbers n_i in that proof. □

Remark 3.1. The previous result generalizes corollary 8 in [R], which is the particular case when A is a Hopf algebra in the hypothesis of proposition 2.1.

4. Symmetrically separable algebras and the isomorphism between invariants and coinvariants

Consider now A–bimodules M. These can be regarded as left A^e–modules via $(a \otimes b^{op})m = amb$. We will adopt either point of view as convenient. Recall the definitions of M^A, M_A and φ_M from the introduction. For instance $M^A = \{m \in M \ / \ am = ma \ \forall \ a \in A\} = \{m \in M \ / \ (a \otimes 1)m = (1 \otimes a^{op})m\}$.

Lemma 4.1. *Let A be a separable k–algebra, $e \in A^e$ any separability idempotent and M an A–bimodule. Then*

(a) *$M^A = eM$, and*

(b) *$[A, M] = \{m \in M \ / \ \tau(e)m = 0\} =: \tau(e)^r$.*

Proof.

(a) $\subseteq$: If $m \in M^A$ then $em = \sum_i u_i m v_i = \sum_i u_i v_i m \overset{(2)}{=} m$ so $m \in eM$.

$\supseteq$: If $m \in M$ and $a \in A$ then $aem = (a \otimes 1)em \overset{(1)}{=} (1 \otimes a^{op})em = ema$ so $em \in M^A$.

(b) $\subseteq$: $\tau(e)(am - ma) = \tau(e)(a \otimes 1 - 1 \otimes a^{op})m = \tau((1 \otimes a^{op} - a \otimes 1)e)m \overset{(1)}{=} 0$ so $am - ma \in \tau(e)^r$.

$\supseteq$: If $\tau(e)m = 0$ then $m = (1 \otimes 1 - \tau(e))m \overset{(2)}{=} (\sum_i 1 \otimes v_i^{op} u_i^{op} - v_i \otimes u_i^{op})m = \sum_i (1 \otimes v_i^{op} - v_i \otimes 1)(1 \otimes u_i^{op})m = \sum_i (mu_i)v_i - v_i(mu_i)$ so $m \in [A, M]$.

$\square$

Theorem 4.1. *Let A be a k–algebra. Then the following are equivalent:*

1. *A is strongly separable.*
2. *The natural map $\varphi_M : M^A \to M_A$ is an isomorphism for every A–bimodule M.*

Moreover, if e is the symmetric separability idempotent, then the map

$$M_A \to M^A, \ \bar{m} \mapsto em$$

is the inverse of φ_M.

Proof.
<u>$1 \Rightarrow 2$</u>. Let e be the symmetric separability idempotent for A. Then, by the lemma,

$$M^A = eM \text{ and } M_A = M/e^r \ .$$

On the other hand, $M = eM \oplus (1-e)M = eM \oplus e^r$ obviously. Hence

$$\varphi_M : M^A = eM \hookrightarrow M \twoheadrightarrow M/e^r = M_A$$

is an isomorphism. Moreover, it is clear that its inverse is as stated.

<u>$2 \Rightarrow 1$</u>. Let $M = A \otimes A$, an A–bimodule under $a(x \otimes y)b = ax \otimes yb$ (so $M \cong A^e$). Then $M_A = A \otimes A / \langle ax \otimes y - x \otimes ya \rangle \cong A \otimes_A A \cong A$ (under the switch followed by the multiplication map of A). Let $e \in M^A$ be such that $\varphi_M(e) = \overline{1 \otimes 1} \in M_A$, say $e = \sum_i u_i \otimes v_i \in A \otimes A$. Since $e \in M^A$ we have $\sum au_i \otimes v_i = \sum u_i \otimes v_i a \ \forall a \in A$, that is (1). Since $e \in M^A$ maps to $1 \in A$ (under φ_M and the map described above), we have $\sum v_i u_i = 1$, that is (4).

According to remark 2.1, e is a symmetric separability idempotent for A. $\square$

Examples 4.1.

1. Let $A = kG$, where G is a finite group with $\mathsf{char} k$ does not divide $|G|$, so that A is strongly separable. Let M be a left G–space; view it as a kG–bimodule with trivial G–action. Then

$$M^G = \{m \in M \ / \ gm = m \ \forall \ g \in G\} \text{ and } M_G = M/\langle gm - m \ / \ g \in G, \ m \in M \rangle$$

are the usual spaces of G–invariants and coinvariants, and theorem 4.1 recovers the well-known fact that $M^G \cong M_G$ under the present hypothesis.

2. Let A be a strongly separable k–algebra and $M = A$ with its usual A–bimodule structure, $a \cdot m \cdot b = amb$. Then $M^A = \mathsf{Z}(A)$, the center of the algebra A, and $M_A = A/[A, A]$, where $[A, A]$ is the k–subspace of A generated by all commutators $ab - ba$. Therefore theorem 4.1 implies that if A is strongly separable then

$$\mathsf{Z}(A) \cong A/[A, A] \ .$$

This observation already appears in [H].

Now suppose that A is actually a strongly separable bialgebra (as for instance in proposition 2.1). Then $(A/[A, A])^* = \mathsf{Cocom}(A^*)$, the space of cocommutative elements of the coalgebra A^*. We deduce that in this case

$$\mathsf{Z}(A)^* \cong \mathsf{Cocom}(A^*), \text{ in particular } \mathsf{dim}\mathsf{Z}(A) = \mathsf{dim}\mathsf{Cocom}(A^*).$$

A similar result is [R, corollary 4].

5. Symmetrically separable algebras and the cotensor product of comodules

Let A be a strongly separable k–algebra, V a left A–module and W a right one. Let $M = V \otimes W$, with its obvious A–bimodule structure. Then

$$M_A = V \otimes W/\langle av \otimes w - v \otimes wa \ / \ v \in V, \ w \in W, \ a \in A\rangle \cong W \otimes_A V$$

under the switch map. On the other hand, let A^* be the dual coalgebra of A (recall that A is necessarily finite dimensional) and view V as a right A^*–comodule and W as a left A^*–comodule as follows:

$$s : V \to V \otimes A^* \quad v \mapsto \sum_i v_i \otimes f_i \qquad \text{iff } \ av = \sum_i f_i(a)v_i \qquad \forall a \in A$$

$$t : W \to A^* \otimes W \quad w \mapsto \sum_i f_i \otimes w_i \qquad \text{iff } \ wa = \sum_i f_i(a)w_i \qquad \forall a \in A$$

For background on this material see [S1]. Recall that if C is a k–coalgebra and V and W are right and left C–comodules via s and t as above, then the cotensor product of V and W over C is defined as

$$V \otimes^C W = \mathsf{Ker}(V \otimes W \xrightarrow{s \otimes \mathsf{id}_W - \mathsf{id}_V \otimes t} V \otimes C \otimes W) \ .$$

In the case when $C = A^*$ and V, W and $M = V \otimes W$ are as above, it follows immediately that

$$V \otimes^{A^*} W = M^A \ .$$

Thus theorem 4.1 above says that for a strongly separable algebra A with symmetric separability idempotent $e = \sum u_j \otimes v_j$, there is a natural isomorphism

$$V \otimes^{A^*} W \to W \otimes_A V \ , \ \sum v_i \otimes w_i \mapsto \sum w_i \otimes_A v_i \ ,$$

with inverse

$$W \otimes_A V \to V \otimes^{A^*} W \ , \ w \otimes_A v \mapsto \sum_j u_j v \otimes w v_j \ .$$

Elaborating on this observation we will now show that the cotensor product over "strongly coseparable" coalgebras has a right adjoint.

Pondering on when this was true is what started us into the considerations of this note.

Corollary 5.1. *Let C, D, and E be k–coalgebras. Consider bicomodules as follows: U a C–D–bicomodule, V an E–D–bicomodule and W an E–C–bicomodule. Assume that C and E are finite dimensional, so that $\mathsf{Hom}_D(U,V)$ carries a natural structure of E–C–bicomodule as follows:*

$$s : \mathsf{Hom}_D(U,V) \to \mathsf{Hom}_D(U,V) \otimes C \quad T \mapsto \sum_j T_j \otimes c_j$$

$$\textit{iff } (\mathsf{id}_C \otimes T)\alpha_U(u) = \sum c_j \otimes T_j(u) \ \forall \ u \in U \ ,$$

where $\alpha_U : U \to C \otimes U$ is the left C–comodule structure on U, and

$$t : \mathsf{Hom}_D(U,V) \to E \otimes \mathsf{Hom}_D(U,V) \quad T \mapsto \sum_i e_i \otimes T_i$$

$$\textit{iff } \alpha_V T(u) = \sum_i e_i \otimes T_i(u) \ \forall u \in U \ ,$$

where $\alpha_V : V \to E \otimes V$ is the left E–comodule structure on V.

Then, if C^* is strongly separable, there is a natural isomorphism

$$\mathsf{Hom}_{E\text{–}D}(W \otimes^C U, V) \cong \mathsf{Hom}_{E\text{–}C}(W, \mathsf{Hom}_D(U,V)) \ .$$

Proof. Since C is finite dimensional, there is an equivalence of categories

$$\{\text{right } C\text{–comodules}\} \cong \{\text{left } C^*\text{–modules}\}$$

([S1], chapter II). The comments preceding the corollary show that under the present hypothesis we have $W \otimes^C U \cong U \otimes_{C^*} W$ naturally. We combine these facts together with the well-known adjunction for tensor products of modules over algebras to obtain

$$\mathsf{Hom}_{E\text{–}C}(W, \mathsf{Hom}_D(U,V)) = \mathsf{Hom}_{C^*\text{–}E^*}(W, \mathsf{Hom}_{D^*}(U,V)) \cong \mathsf{Hom}_{D^*\text{–}E^*}(U \otimes_{C^*} W, V)$$

$$= \mathsf{Hom}_{E\text{–}D}(U \otimes_{C^*} W, V) \cong \mathsf{Hom}_{E\text{–}D}(W \otimes^C U, V) \ .$$

The proof is complete. □

Let U be a C–D–bicomodule. The functor

$$(-) \otimes^C U : \{E\text{–}C\text{–bicomodules}\} \to \{E\text{–}D\text{–bicomodules}\}$$

may have a right adjoint even if C^* is not strongly separable. In fact, we will close this note by characterizing these coalgebras. In particular, it follows from theorem 5.1 below that any group-like coalgebra $C = kX$ has this property, while $C^* = k^X$ is strongly separable only when X is finite. However, it is only under these extra assumptions that we are able to find an explicit description for the right adjoint functor.

Definition 5.1. A coalgebra C is called cosemisimple if every left C–comodule is a direct sum of simple subcomodules, or, equivalently, if every short exact sequence of left C–comodules splits.

Theorem 5.1. *The following conditions are equivalent for any coalgebra C:*

1. *C is cosemisimple.*
2. *For any coalgebras D and E and C–D–bicomodule U, the functor*
$$(-) \otimes^C U : \{E\text{–}C\text{–}bicomodules\} \to \{E\text{–}D\text{–}bicomodules\}$$
has a right adjoint.
3. *For any left C–comodule U, the functor*
$$(-) \otimes^C U : \{right\ C\text{–}comodules\} \to \{k\text{–}spaces\}$$
has a right adjoint.

Proof.
$\underline{1 \Rightarrow 2}$. Since every short exact sequence of left C–comodules splits, the functor
$$\mathsf{Hom}_C(-, U) : \{\text{left } C\text{–comodules}\} \to \{k\text{–spaces}\}$$
is right exact. By a result of Takeuchi [T1, proposition A.2.1], this functor is right exact if and only if so is the functor
$$(-) \otimes^C U : \{\text{right } C\text{–comodules}\} \to \{k\text{–spaces}\}\ .$$
It follows that the functor
$$(-) \otimes^C U : \{E\text{–}C\text{–bicomodules}\} \to \{E\text{–}D\text{–bicomodules}\}$$
is also right exact. Since it clearly preserves direct sums, it is cocontinuous (i.e. preserves coequalizers and small coproducts). The result now follows from the special adjoint functor theorem [ML, corollary V.8], since the category $\{E\text{–}C\text{–bicomodules}\}$ is well-powered, small cocomplete and skeletally small.

$\underline{2 \Rightarrow 3}$. Obvious.

$\underline{3 \Rightarrow 1}$. Since $(-) \otimes^C U$ has a right adjoint, it is right exact. This implies, as in the proof $1 \Rightarrow 2$, that C is cosemisimple. ☐

Remark 5.1. On the other hand, it follows from another result of Takeuchi [T2, proposition 1.10] that the functor $(-) \otimes^C U : \{\text{right } C\text{–comodules}\} \to \{k\text{–spaces}\}$ has a *left* adjoint if and only if U is finite dimensional.

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References

[EK]: P.Etingof and S.Gelaki *On finite dimensional semisimple and cosemisimple Hopf algebras in positive characteristic,* to appear in IMRN.

[H]: A.Hattori, *On strongly separable algebras,* Osaka J. Math. 2 (1965), 369-372.

[J]: N.Jacobson, *Basic Algebra 2,* W.H. Freeman and Co (1989).

[K]: L.Kadison and A.A.Stolin, *Separability and Hopf Algebras,* Göteborg University Preprint (1999).

[KS]: I.Kaplansky, *Bialgebras,* University of Chicago Lecture Notes (1975).

[LR1]: R.G.Larson and D.E.Radford, *Semisimple cosemisimple Hopf algebras,* Amer. J. Math 110 (1988), 187-195.

[LR2]: R.G.Larson and D.E.Radford, *Finite Dimensional Cosemisimple Hopf Algebras in Characteristic 0 Are Semisimple* , J. Algebra 117 (1988), 267-289.

[ML]: S.Mac Lane, *Categories for the Working Mathematician,* GTM 5 (1971).

[M]: S.Montgomery, *Hopf algebras and their actions on rings,* CBMS n82 (1993).

[0S]: U.Oberst and H.-J.Schneider, *Über untergruppen endlicher algebraischer gruppen,* Manuscripta math. 8, (1973), 217-241.

[P]: R.S.Pierce, *Associative algebras,* GTM 88 Springer Verlag (1982).

[R]: D.E.Radford, *The trace function and Hopf algebras,* J. of Algebra 163 (1994), 583-622. (1988), Vol 2.

[Sc]: H.-J.Schneider, *Lectures on Hopf Algebras,* Universidad de Córdoba Trabajos de Matemática, Serie “B”, No.31/95 (1995).

[S1]: M.Sweedler, *Hopf Algebras,* W.A. Benjamin (1969).

[S2]: M.Sweedler, *Integrals for Hopf algebras,* Annals of Math 89 (1969), 323-335.

[T1]: M.Takeuchi, *Formal Schemes Over Fields,* Comm. Algebra 5 (1977) 1483-1528.

[T2]: M.Takeuchi, *Morita theorems for categories of comodules,* J. Fac. Sci. Univ. Tokyo 24 (1977) 629-644.

[V]: O.Villamayor, *On weak dimension of algebras,* Pacific J. Math. Vol 9 n3 (1959), 941-951.

[VZ]: O.Villamayor and D.Zelinsky *Galois theory for rings with finitely many idempotents,* Nagoya Math. J. Vol 27 n2 (1966), 721-731.

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