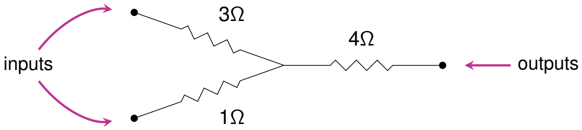
STRUCTURED vs DECORATED COSPANS



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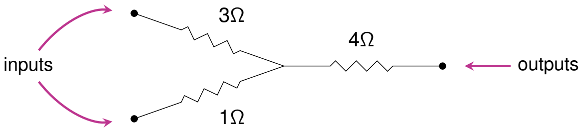
Around 2012, I asked Brendan Fong to create and study a category having “open electrical circuits” as morphisms:

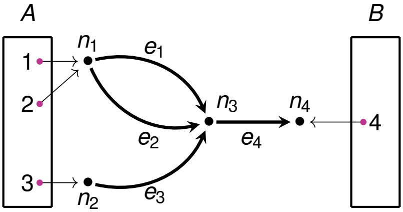


He invented the theory of “decorated cospans” to do this. He published it in 2015, and we finished our work on electrical circuits in 2018.

Decorated cospans have now been used to study open systems of many kinds. But they’re subtle, and I feel they were only fully understood in 2021. (Or later?)

The simplest example: it would be nice to have a category with "open graphs" as morphisms.

Here is an open graph with inputs A and outputs B:



We compose two open graphs by gluing the outputs of one to the inputs of the other.

So, an **open graph** is a cospan of finite sets:

$$
\begin{tikzcd}
 & N & \\
A \arrow[ur, "i"] & & B \arrow[ul, "o"']
\end{tikzcd}
$$

where the apex N is equipped with extra data: a finite set E and two functions giving a **graph**:

$$
\begin{tikzcd}
E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N
\end{tikzcd}
$$

Brendan treated the apex N as "decorated" with an element of the set

$$F(N) = \left\{ \text{graphs} \quad \begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N \end{tikzcd} \right\}$$

Theorem (Fong)

Let A *be a category with finite colimits and* $F \colon (\mathsf{A}, +) \to (\mathsf{Set}, \times)$ *a symmetric lax monoidal functor. Then there is a symmetric monoidal category* **FCsp** *where:*

- *an object is an object of* A
- *a morphism is an isomorphism class of* **decorated cospans***:*

$$a \xrightarrow{i} m \xleftarrow{o} b \qquad d \in F(m)$$

Here d *is called the* **decoration***, and two decorated cospans are* **isomorphic** *if there's an isomorphism* $h \colon m \xrightarrow{\sim} m'$ *such that this commutes:*

$$\begin{tikzcd} & m \arrow[dd, "h"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & m' & \end{tikzcd} \qquad \begin{array}{c} d \in F(m) \\ \\ d' \in F(m') \end{array}$$

and $F(h)(d) = d'$.

Given decorated cospans

$$M = (a \to m \leftarrow b, d \in F(m)) \qquad N = (b \to n \leftarrow c, e \in F(n))$$

we compose their underlying cospans by pushout:

$$\begin{tikzcd}
 & & m +_b n & & \\
 & & m + n \arrow[u, "\psi"] & & \\
 & m \arrow[ur] & & n \arrow[ul] & \\
a \arrow[ur] \arrow[uuurr, bend left] & & b \arrow[ul] \arrow[ur] & & c \arrow[ul] \arrow[uuull, bend right]
\end{tikzcd}$$

and give it the decoration that's the image of (d, e) under this composite:

$$(d, e) \in F(m) \times F(n) \xrightarrow{\phi_{m,n}} F(m + n) \xrightarrow{F(\psi)} F(m +_b n)$$

where $\phi_{m,n}$ comes from F being lax monoidal.

There are problems with applying this theorem to open graphs! In historical order — and order of increasing seriousness:

1) The “set”

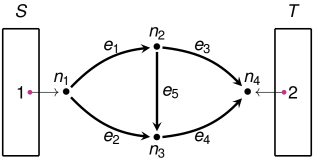
$$F(N) = \left\{ \text{graphs} \quad E \underset{t}{\overset{s}{\rightrightarrows}} N \quad \right\}$$

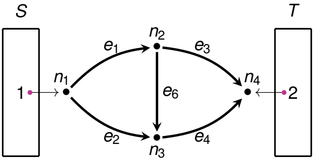
is a proper class, so we do not have a functor $F\colon \mathrm{FinSet} \to \mathrm{Set}$.

Solution: replace FinSet by an equivalent small category, which we again call FinSet. Then we get $F\colon \mathrm{FinSet} \to \mathrm{Set}$.

2) Open graphs that “look isomorphic” may not be isomorphic as decorated cospans!

These are not isomorphic as decorated cospans:





Here's why: given a graph

$$E \underset{t}{\overset{s}{\rightrightarrows}} N \quad = d \in F(N)$$

and a function $h \colon N \to N'$, we have

$$E \underset{h \circ t}{\overset{h \circ s}{\rightrightarrows}} N' \quad = F(h)(d) \in F(N')$$

So, the map $F(h) \colon F(N) \to F(N')$ can relabel the *nodes* of a graph, but *not its edges!*

Solution: get used to it. Learn to live with a vast number of open graphs that look isomorphic but technically aren't.

Or: let $F(N)$ be a set of *equivalence classes* of graphs with N as the set of nodes. Alas, you can't point to a specific edge in such an equivalence class.

3) If we take

$$F(N) = \left\{ \text{graphs} \quad E \underset{t}{\overset{s}{\rightrightarrows}} N \quad \right\}$$

then $F\colon (\mathrm{FinSet}, +) \to (\mathrm{Set}, \times)$ is *not* lax symmetric monoidal!

This diagram fails to commute:

$$\begin{tikzcd}
F(N) \times F(N') \arrow[r, "\beta_{F(N),F(N')}"] \arrow[d, "\phi_{N,N'}"'] & F(N') \times F(N) \arrow[d, "\phi_{N',N}"] \\
F(N + N') \arrow[r, "F(\beta_{N,N'})"'] & F(N' + N)
\end{tikzcd}$$

where the horizontal arrows are braidings.

Solution: Let $F(N)$ be a set of *equivalence classes* of graphs with N as the set of nodes.

Or better: go back to the drawing board and improve the whole formalism.

There are two ways to solve problems 2) and 3):

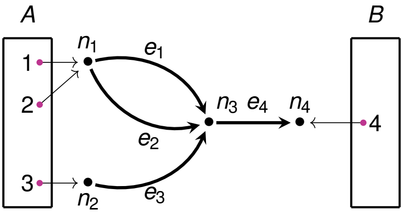
- “structured cospans”
- the new improved “decorated cospans”.

They are equivalent when they both apply. Structured cospans are easier, while decorated cospans are more general.

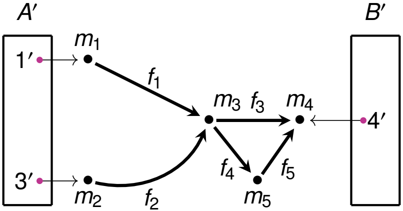
Everything is explained here, with lots of examples:

- John Baez and Kenny Courser, Structured cospans.
- John Baez, Kenny Courser and Christina Vasilakopoluou, Structured versus decorated cospans.

Both structured and decorated cospans give symmetric monoidal *double* categories. For example there is a 2-morphism from this open graph:



to this one:



We explain how you can water down a double category of structured or decorated cospans to get a symmetric monoidal *bicategory*, or just a symmetric monoidal *category*.

But let’s see how the double categories work!

There are two ways to equip an object of a category A with extra data:

- **"Structuring."** Given a right adjoint $R\colon \mathsf{X} \to \mathsf{A}$, we can give $a \in \mathsf{A}$ extra structure by choosing $x \in \mathsf{X}$ with $R(x) = a$.

- **"Decorating."** Given a pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$, we can decorate $a \in A$ with an object $d \in F(a)$.

The first is more convenient. The second is more general. So, we want to relate the two approaches!

The key is the Grothendieck construction: any pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ gives a functor $R\colon \int F \to \mathsf{A}$ which *under certain conditions* is a right adjoint.

Given a right adjoint

$$R\colon \mathsf{X} \to \mathsf{A}$$

a structured cospan is a diagram in A of this form:

$$\begin{tikzcd} & R(x) & \\ a \arrow[ur] & & b \arrow[ul] \end{tikzcd}$$

Think of A as a category of objects with “less structure”, and X as a category of objects with “more structure”.

Given a left adjoint

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram in X of this form:

$$\begin{tikzcd} & x & \\ L(a) \arrow[ur] & & L(b) \arrow[ul] \end{tikzcd}$$

Now we can compose structured cospans by doing pushouts in X.

Theorem (Baez–Courser)

Suppose A *and* X *have finite colimits and* $L\colon \mathsf{A} \to \mathsf{X}$ *preserves finite colimits. Then there is a symmetric monoidal double category* ${}_L\mathbb{C}\mathbf{sp}(\mathbf{X})$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan*

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

- *a 2-morphism is a commutative diagram*

$$\begin{tikzcd}
L(a) \arrow[r, "i"] \arrow[d, "L(f)"'] & x \arrow[d, "h"'] & L(b) \arrow[l, "o"'] \arrow[d, "L(g)"] \\
L(a') \arrow[r, "i'"'] & x' & L(b') \arrow[l, "o'"]
\end{tikzcd}$$

Horizontal composition is defined using pushouts in X. Composing these:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x & \longleftarrow & L(b) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' & \longleftarrow & L(b') \end{array} \qquad \begin{array}{ccccc} L(b) & \longrightarrow & y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(b') & \longrightarrow & y' & \longleftarrow & L(c') \end{array}$$

gives this:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c') \end{array}$$

Vertical composition is straightforward.

The symmetric monoidal structure uses binary coproducts in both A and X, and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves these:

$$
\begin{array}{ccccc}
L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2)
\end{array}
\quad \otimes \quad
\begin{array}{ccccc}
L(a_1') & \longrightarrow & x_1' & \longleftarrow & L(b_1') \\
\downarrow & & \downarrow & & \downarrow \\
L(a_2') & \longrightarrow & x_2' & \longleftarrow & L(b_2')
\end{array}
$$

$$
= \quad
\begin{array}{ccccc}
L(a_1 + a_1') & \longrightarrow & x_1 + x_1' & \longleftarrow & L(b_1 + b_1') \\
\downarrow & & \downarrow & & \downarrow \\
L(a_2 + a_2') & \longrightarrow & x_2 + x_2' & \longleftarrow & L(b_2 + b_2')
\end{array}
$$

Example. There is a category **Graph** where objects are graphs:

$$E \underset{t}{\overset{s}{\rightrightarrows}} N$$

There is a functor $R\colon \mathrm{Graph} \to \mathrm{FinSet}$ sending any graph to its set of nodes, N.

This has a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Graph}$ sending any finite set N to the graph with that set of nodes and no edges.

We obtain a symmetric monoidal double category

$$_L\mathbb{C}\mathbf{sp}(\mathrm{Graph})$$

where the horizontal 1-cells are exactly open graphs.

In the new improved decorated cospans, we use a *category* of decorations instead of a mere *set*.

Given a lax monoidal pseudofunctor $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$, a **decorated cospan** is a diagram in A of this form:

$$\begin{tikzcd} & m & \\ a \arrow[ur] & & b \arrow[ul] \end{tikzcd}$$

together with a **decoration** $d \in F(m)$.

Theorem (Baez–Courser–Vasilakopoulou)

Let A *be a category with finite colimits and* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *a symmetric lax monoidal pseudofunctor. Then there is a symmetric monoidal double category* $\boldsymbol{F\mathbb{C}\mathbf{sp}}$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a decorated cospan:*

$$a \xrightarrow{i} m \xleftarrow{o} b \qquad d \in F(m)$$

- *a 2-morphism is a commuting diagram*

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "f"'] & m \arrow[d, "h"'] & b \arrow[l, "o"'] \arrow[d, "g"] & d \in F(m) \\ a' \arrow[r, "i'"] & m' & b' \arrow[l, "o'"'] & d' \in F(m') \end{tikzcd}$$

together with a **decoration morphism** $\tau\colon F(h)(d) \to d'$.

When are decorated cospans also structured cospans?

Theorem (Baez–Courser–Vasilakopoulou)

Suppose A *has finite colimits and* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *is a symmetric lax monoidal pseudofunctor. Suppose the corresponding pseudofunctor* $F\colon \mathsf{A} \to \mathbf{SymMonCat}$ *factors through* **Rex***, the 2-category of categories with finite colimits. Then the symmetric monoidal double categories:*

- $F\mathbb{C}\mathbf{sp}$ *of decorated cospans*

and

- ${}_L\mathbb{C}\mathbf{sp}(\int F)$ *of structured cospans*

are isomorphic, where $L\colon \mathsf{A} \to \int F$ *is a left adjoint of the functor* $R\colon \int F \to \mathsf{A}$ *given by the Grothendieck construction.*

What's "the corresponding pseudofunctor $F\colon \mathsf{A} \to \mathbf{SymMonCat}$"?

Theorem (Shulman and Moeller–Vasilakopoulou)

If $(\mathsf{A}, +)$ *has finite coproducts, these three things correspond to each other:*

- *symmetric lax monoidal pseudofunctors* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$
- *pseudofunctors* $F\colon \mathsf{A} \to \mathbf{SymMonCat}$
- *symmetric monoidal opfibrations* $R\colon (\mathsf{X}, \otimes) \to (\mathsf{A}, +)$.

We build X from F using the Grothendieck construction:

$$X = \int F$$

- objects of $\int F$ are pairs $(a \in A, d \in F(a))$
- morphisms are pairs $(f\colon a \to a', \tau\colon F(f)(d) \to d')$.

Example. There is a pseudofunctor $F\colon \mathrm{FinSet} \to \mathbf{Cat}$ sending each finite set N to the category where objects are graphs

$$\begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N \end{tikzcd}$$

and the morphisms are diagrams

$$\begin{tikzcd} E \arrow[dd, "f"'] \arrow[dr, shift left, "s"] \arrow[dr, shift right, "t"'] & \\ & N \\ E' \arrow[ur, shift left, "s'"] \arrow[ur, shift right, "t'"'] & \end{tikzcd}$$

where $s' \circ f = s, t' \circ f = t$.

$F\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$ is lax symmetric monoidal, using the obvious map

$$\phi_{M,N}\colon F(M) \times F(N) \to F(M+N)$$

The Grothendieck construction gives

$$\int F \cong \mathrm{Graph}$$

and it gives a functor

$$R\colon \mathrm{Graph} \to \mathrm{FinSet}$$

which sends each graph to its set of nodes. The left adjoint

$$L\colon \mathrm{FinSet} \to \mathrm{Graph}$$

sends each finite set to the graph with that set of nodes and no edges.

The general theory assures us that

- the symmetric monoidal double category of open graphs built using *decorated* cospans

is isomorphic to

- the symmetric monoidal double category of open graphs built using *structured* cospans.

But there are also decorated cospans that *cannot* be described as structured cospans. So, both approaches are useful!