Higher Gauge Theory, Division Algebras and Superstrings

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This research began as a puzzle. Explain this pattern:

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They have dimensions $k = 1, 2, 4$ and 8.
- The classical superstring makes sense only in dimensions $k + 2 = 3, 4, 6$ and 10.
- The classical super-2-brane makes sense only in dimensions $k + 3 = 4, 5, 7$ and 11.

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The explanation involves 'higher gauge theory'.

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:

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$$\bullet \xrightarrow{g} \bullet$$

since composition of paths then corresponds to multiplication:

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while reversing the direction corresponds to taking the inverse:

$$\bullet \xleftarrow{g^{-1}} \bullet$$

The associative law makes the holonomy along a triple composite unambiguous:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \arrow[r, bend left, "g''"] & \bullet
\end{tikzcd}
$$

So: *the topology dictates the algebra!*

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A '2-group' has objects:

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but also morphisms:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "f"] \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left, "g_1"{name=A}] \arrow[r, bend right, "g_1'"'{name=B}] & \bullet \arrow[r, bend left, "g_2"{name=C}] \arrow[r, bend right, "g_2'"'{name=D}] & \bullet \arrow[from=A, to=B, Rightarrow, "f_1"'] \arrow[from=C, to=D, Rightarrow, "f_2"'] \end{tikzcd}$$

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$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left, "g_2"] \arrow[r, bend right, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet
\end{tikzcd}
$$

and also compose morphisms:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"] \arrow[r, "g'"] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow f"] \arrow[r, phantom, shift right=3ex, "\Downarrow f'"] & \bullet
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But to get our hands on these, it's easiest to start with 'Lie 2-superalgebras'.

An $\boldsymbol{L_\infty}$**-algebra** is a chain complex

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Thus a Lie 2-algebra is a 'categorified' Lie algebra.

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- an L_0-valued 1-form A
- an L_1-valued 2-form B
- an L_2-valued 3-form C
- etc...

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(Don’t worry, soon I’ll tell you what $\mathfrak{siso}(T)$ actually *is!*)

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Let's see how it works in detail.

If V is a finite-dimensional real vector space with a quadratic form Q, the **Clifford algebra** $\mathrm{Cliff}(V)$ is the real associative algebra generated by V with relations

$$v^2 = -Q(v)$$

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This fact lets us define either two real representations of $\mathrm{Spin}(V)$, say S_+ and S_-, or one, say S.

In the first case $S_+ \not\cong S_-$. In the second, set $S_+ = S_- = S$. In either case, let's call S_+ and S_- **left- and right-handed spinors**.

$\mathrm{Cliff}_0(V)$ acts on S_+ and S_-. But the whole Clifford algebra acts on $S_+ \oplus S_-$, with odd elements interchanging the two parts. So, we can 'multiply' a spinor by a vector and get a spinor of the other handedness:

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So, let's see when $\dim(V) = \dim(S_+) = \dim(S_-)$.

Consider Euclidean space: $V = \mathbb{R}^k$ with

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V	$\mathrm{Cliff}(V)$	$\mathrm{Cliff}_0(V)$	$S_\pm$	
$\mathbb{R}^1$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	
$\mathbb{R}^2$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{C}$	
$\mathbb{R}^3$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	
$\mathbb{R}^4$	$\mathbb{H}[2]$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$(S_+ \ncong S_-)$
$\mathbb{R}^5$	$\mathbb{C}[4]$	$\mathbb{H}[2]$	$\mathbb{H}^2$	
$\mathbb{R}^6$	$\mathbb{R}[8]$	$\mathbb{C}[4]$	$\mathbb{C}^4$	
$\mathbb{R}^7$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}[8]$	$\mathbb{R}^8$	
$\mathbb{R}^8$	$\mathbb{R}[16]$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}^8$	$(S_+ \ncong S_-)$

Here $\mathbb{K}[n]$ is the algebra of $n \times n$ matrices with entries in $\mathbb{K}$.

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V	$S_\pm$	normed division algebra?
$\mathbb{R}^1$	$\mathbb{R}$	YES: $\mathbb{R}$
$\mathbb{R}^2$	$\mathbb{C}$	YES: $\mathbb{C}$
$\mathbb{R}^3$	$\mathbb{H}$	NO
$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

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$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

Increasing k by 8 multiplies $\dim(S_\pm)$ by 16, so these are the *only* normed division algebras!

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Now vectors V are the 2×2 Hermitian matrices with entries in $\mathbb{K}$:

$$V = \left\{ \begin{pmatrix} t+x & \overline{y} \\ y & t-x \end{pmatrix} : t, x \in \mathbb{R}, \quad y \in \mathbb{K} \right\}.$$

Now our quadratic form Q comes from the determinant:

$$\det \begin{pmatrix} t+x & \overline{y} \\ y & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

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Concretely:

$$[\psi, \phi] = \psi\phi^\dagger + \phi\psi^\dagger$$

It's symmetric!

So, we can define the **translation Lie superalgebra**

$$T = V \oplus S_+$$

with V as its even part and S_+ as its odd part. We define the bracket to be zero except for $[-,-]\colon S_+ \otimes S_+ \to V$. The Jacobi identity holds trivially.

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$\mathrm{Spin}(V)$ acts on everything, and its Lie algebra is $\mathfrak{so}(V)$, so we can form the semidirect product

$$\mathfrak{siso}(T) = \mathfrak{so}(V) \ltimes T$$

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The corresponding Lie supergroup acts as symmetries of 'Minkowski superspace'.

We get a Poincaré Lie superalgebra whenever we have an invariant symmetric bracket that takes two spinors and gives a vector. What's so special about the dimensions 3, 4, 6 and 10?

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obey the identity

$$[\psi,\psi]\cdot\psi = 0$$

Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using $\mathbb{K}$-valued matrices.

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$$[\psi, \psi] \cdot \psi = 0$$

Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using $\mathbb{K}$-valued matrices.

In fact, *only* for Minkowski spacetimes of dimension 3, 4, 6, and 10 does this identity hold!

The identity $[\psi, \psi] \cdot \psi = 0$ lets us extend the Poincaré Lie superalgebra $\mathfrak{siso}(T)$ to a Lie 2-superalgebra

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The idea:

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- $[-,-,-]$ is zero unless two arguments are spinors and one is a vector in $\mathfrak{siso}(T)$, and

$$[\psi, \phi, v] = g([\psi, \phi], v) \in \mathbb{R}$$

  where $\psi, \phi \in S_+$, $v \in V$, and $g \colon V \otimes V \to R$ is the **Minkowski metric**: the bilinear form corresponding to Q.

To get a Lie 2-superalgebra this way, the ternary bracket must obey an equation. This says that

$$[-,-,-]\colon \mathfrak{siso}(T)^{\otimes 3} \to \mathbb{R}$$

is a *3-cocycle* in Lie superalgebra cohomology.

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The equation

$$[\psi,\psi]\cdot\psi = 0$$

is this cocycle condition in disguise.

Let us call the resulting Lie 2-superalgebra $\mathfrak{superstring}(T)$.

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Theorem (John Huerta)

There is a 2-group in the category of supermanifolds, $\mathrm{Superstring}(T)$*, whose Lie 2-superalgebra is* $\mathfrak{superstring}(T)$*.*

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Theorem (John Huerta)

There is a 2-group in the category of supermanifolds, $\mathrm{Superstring}(T)$, whose Lie 2-superalgebra is $\mathfrak{superstring}(T)$.

This theorem takes real work to prove. Not every Lie 2-superalgebra has a corresponding 'Lie 2-supergroup' in such a simple-minded sense! There are important finite-dimensional Lie 2-algebras that don't come from 2-groups in the category of manifolds—instead, they come from 'stacky' Lie 2-groups.

He also went further:

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In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.

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In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.

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Moreover, there is a 3-group in the category of supermanifolds, $2\text{-}\mathrm{Brane}(T)$*, whose Lie 3-superalgebra is* $2\text{-}\mathfrak{brane}(T)$.

2-$\mathrm{Brane}(T)$ is relevant to the theory of supersymmetric 2-branes in dimension 4, 5, 7 and 11.

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The 11-dimensional — octonionic! — case also shows up in 11-dimensional supergravity, and thus presumably ‘*M*-theory’ (whatever that is).

The buck stops here: for 12-dimensional Minkowski spacetime, it seems all 5-cocycles on $\mathfrak{siso}(T)$ are trivial. Apparently the nonassociativity of the octonions spoils the calculation!