

# Higher Gauge Theory, Division Algebras and Superstrings

John Baez

March 25, 2010  
Hong Kong University

This research began as a puzzle. Explain this pattern:

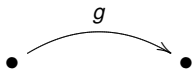
- The only normed division algebras are  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$ .  
They have dimensions  $k = 1, 2, 4$  and  $8$ .
- The classical superstring makes sense only in dimensions  $k + 2 = 3, 4, 6$  and  $10$ .
- The classical super-2-brane makes sense only in dimensions  $k + 3 = 4, 5, 7$  and  $11$ .

This research began as a puzzle. Explain this pattern:

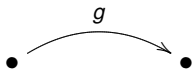
- The only normed division algebras are  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$ .  
They have dimensions  $k = 1, 2, 4$  and  $8$ .
- The classical superstring makes sense only in dimensions  $k + 2 = 3, 4, 6$  and  $10$ .
- The classical super-2-brane makes sense only in dimensions  $k + 3 = 4, 5, 7$  and  $11$ .

The explanation involves 'higher gauge theory'.

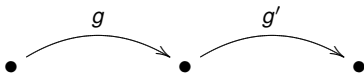
Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:



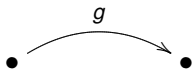
Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:



since composition of paths then corresponds to multiplication:



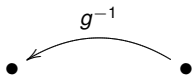
Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:



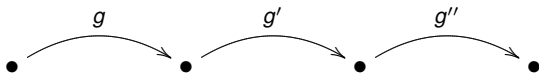
since composition of paths then corresponds to multiplication:



while reversing the direction corresponds to taking the inverse:



The associative law makes the holonomy along a triple composite unambiguous:



*So: the topology dictates the algebra!*

Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings.



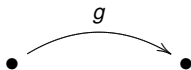
Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings.

For this we must 'categorify' the notion of a group!

Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings.

For this we must 'categorify' the notion of a group!

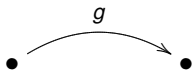
A '2-group' has objects:



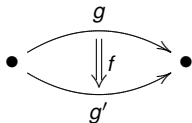
Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings.

For this we must 'categorify' the notion of a group!

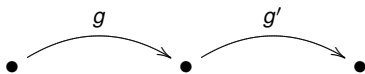
A '2-group' has objects:



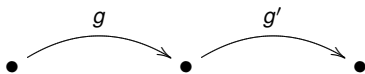
but also morphisms:



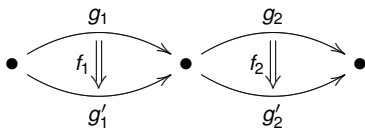
We can multiply objects:



We can multiply objects:



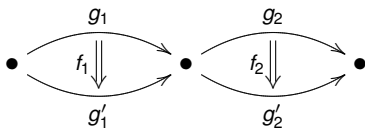
multiply morphisms:



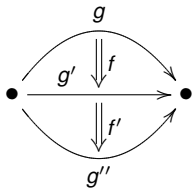
We can multiply objects:



multiply morphisms:



and also compose morphisms:



Various laws should hold...

Various laws should hold...  
again, *the topology dictates the algebra.*



Various laws should hold...  
again, *the topology dictates the algebra.*

Just as a group is a monoid where every element has an inverse, a **2-group** is a monoidal category where every object and every morphism has an inverse.

Various laws should hold...  
again, *the topology dictates the algebra*.

Just as a group is a monoid where every element has an inverse, a **2-group** is a monoidal category where every object and every morphism has an inverse.

For higher gauge theory, we really want ‘Lie 2-groups’.

Various laws should hold...  
again, *the topology dictates the algebra*.

Just as a group is a monoid where every element has an inverse, a **2-group** is a monoidal category where every object and every morphism has an inverse.

For higher gauge theory, we really want ‘Lie 2-groups’.

To study *superstrings* using higher gauge theory, we really want ‘Lie 2-supergroups’.

Various laws should hold...  
again, *the topology dictates the algebra*.

Just as a group is a monoid where every element has an inverse, a **2-group** is a monoidal category where every object and every morphism has an inverse.

For higher gauge theory, we really want ‘Lie 2-groups’.

To study *superstrings* using higher gauge theory, we really want ‘Lie 2-supergroups’.

But to get our hands on these, it’s easiest to start with ‘Lie 2-superalgebras’.

An  $L_\infty$ -**algebra** is a chain complex

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \cdots \xleftarrow{d} L_n \xleftarrow{d} \cdots$$

equipped with the structure of a Lie algebra ‘up to coherent chain homotopy’.

An  $L_\infty$ -**algebra** is a chain complex

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \cdots \xleftarrow{d} L_n \xleftarrow{d} \cdots$$

equipped with the structure of a Lie algebra 'up to coherent chain homotopy'.

So,  $L$  has:

- a map  $d: L \rightarrow L$  of grade -1 with  $d^2 = 0$

An  $L_\infty$ -**algebra** is a chain complex

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \cdots \xleftarrow{d} L_n \xleftarrow{d} \cdots$$

equipped with the structure of a Lie algebra 'up to coherent chain homotopy'.

So,  $L$  has:

- a map  $d: L \rightarrow L$  of grade -1 with  $d^2 = 0$
- a graded-antisymmetric map  $[-, -]: L^{\otimes 2} \rightarrow L$  of grade 0, obeying the Jacobi identity up to  $d$  of...

An  $L_\infty$ -algebra is a chain complex

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \cdots \xleftarrow{d} L_n \xleftarrow{d} \cdots$$

equipped with the structure of a Lie algebra 'up to coherent chain homotopy'.

So,  $L$  has:

- a map  $d: L \rightarrow L$  of grade -1 with  $d^2 = 0$
- a graded-antisymmetric map  $[-, -]: L^{\otimes 2} \rightarrow L$  of grade 0, obeying the Jacobi identity up to  $d$  of...
- a graded-antisymmetric map  $[-, -, -]: L^{\otimes 3} \rightarrow L$  of grade 1, obeying its own identity up to  $d$  of...



An  $L_\infty$ -algebra is a chain complex

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \dots \xleftarrow{d} L_n \xleftarrow{d} \dots$$

equipped with the structure of a Lie algebra 'up to coherent chain homotopy'.

So,  $L$  has:

- a map  $d: L \rightarrow L$  of grade -1 with  $d^2 = 0$
- a graded-antisymmetric map  $[-, -]: L^{\otimes 2} \rightarrow L$  of grade 0, obeying the Jacobi identity up to  $d$  of...
- a graded-antisymmetric map  $[-, -, -]: L^{\otimes 3} \rightarrow L$  of grade 1, obeying its own identity up to  $d$  of...
- etc...

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

It can be seen as a *category* with:

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

It can be seen as a *category* with:

- an object for each 0-chain  $x \in L_0$

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

It can be seen as a *category* with:

- an object for each 0-chain  $x \in L_0$
- a morphism  $f: x \rightarrow y$  for each 1-chain  $f \in L_1$  with

$$y - x = df$$

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

It can be seen as a *category* with:

- an object for each 0-chain  $x \in L_0$
- a morphism  $f: x \rightarrow y$  for each 1-chain  $f \in L_1$  with

$$y - x = df$$

So,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = d[x, y, z]$$

says that the Jacobi identity holds *up to isomorphism*.

A **Lie 2-algebra** is an  $L_\infty$ -algebra with only 2 nonzero terms:

$$L_0 \xleftarrow{d} L_1$$

It can be seen as a *category* with:

- an object for each 0-chain  $x \in L_0$
- a morphism  $f: x \rightarrow y$  for each 1-chain  $f \in L_1$  with

$$y - x = df$$

So,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = d[x, y, z]$$

says that the Jacobi identity holds *up to isomorphism*.

Thus a Lie 2-algebra is a ‘categorified’ Lie algebra.

Sati, Schreiber and Stasheff have a theory of connections where  $L_\infty$ -algebras replace Lie algebras.



Sati, Schreiber and Stasheff have a theory of connections where  $L_\infty$ -algebras replace Lie algebras.

Given an  $L_\infty$ -algebra

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \dots \xleftarrow{d} L_n \xleftarrow{d} \dots$$

such a connection can be described locally using:

Sati, Schreiber and Stasheff have a theory of connections where  $L_\infty$ -algebras replace Lie algebras.

Given an  $L_\infty$ -algebra

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \dots \xleftarrow{d} L_n \xleftarrow{d} \dots$$

such a connection can be described locally using:

- an  $L_0$ -valued 1-form  $A$

Sati, Schreiber and Stasheff have a theory of connections where  $L_\infty$ -algebras replace Lie algebras.

Given an  $L_\infty$ -algebra

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \dots \xleftarrow{d} L_n \xleftarrow{d} \dots$$

such a connection can be described locally using:

- an  $L_0$ -valued 1-form  $A$
- an  $L_1$ -valued 2-form  $B$

Sati, Schreiber and Stasheff have a theory of connections where  $L_\infty$ -algebras replace Lie algebras.

Given an  $L_\infty$ -algebra

$$L_0 \xleftarrow{d} L_1 \xleftarrow{d} \dots \xleftarrow{d} L_n \xleftarrow{d} \dots$$

such a connection can be described locally using:

- an  $L_0$ -valued 1-form  $A$
- an  $L_1$ -valued 2-form  $B$
- an  $L_2$ -valued 3-form  $C$
- etc...

We can just as easily consider  $L_\infty$ -superalgebras: now each term is  $\mathbb{Z}_2$ -graded, and we introduce extra signs.

We can just as easily consider  $L_\infty$ -superalgebras: now each term is  $\mathbb{Z}_2$ -graded, and we introduce extra signs.

To describe parallel transport of superstrings, we need a Lie 2-superalgebra

$$L_0 \xleftarrow{d} L_1$$

We can just as easily consider  $L_\infty$ -superalgebras: now each term is  $\mathbb{Z}_2$ -graded, and we introduce extra signs.

To describe parallel transport of superstrings, we need a Lie 2-superalgebra

$$L_0 \xleftarrow{d} L_1$$

**What is this Lie 2-superalgebra?**

One clue comes from the ‘background fields’ in superstring theory. Locally they can be described by:



One clue comes from the ‘background fields’ in superstring theory. Locally they can be described by:

- a 1-form  $A$  valued in the ‘Poincaré Lie superalgebra’  $\mathfrak{siso}(T)$

One clue comes from the ‘background fields’ in superstring theory. Locally they can be described by:

- a 1-form  $A$  valued in the ‘Poincaré Lie superalgebra’  $\mathfrak{siso}(T)$
- a 2-form  $B$  valued in  $\mathbb{R}$

One clue comes from the ‘background fields’ in superstring theory. Locally they can be described by:

- a 1-form  $A$  valued in the ‘Poincaré Lie superalgebra’  $\mathfrak{siso}(T)$
- a 2-form  $B$  valued in  $\mathbb{R}$

So, we want a Lie 2-superalgebra with

$$L_0 = \mathfrak{siso}(T) \quad \text{and} \quad L_1 = \mathbb{R}$$

One clue comes from the ‘background fields’ in superstring theory. Locally they can be described by:

- a 1-form  $A$  valued in the ‘Poincaré Lie superalgebra’  $\mathfrak{siso}(T)$
- a 2-form  $B$  valued in  $\mathbb{R}$

So, we want a Lie 2-superalgebra with

$$L_0 = \mathfrak{siso}(T) \quad \text{and} \quad L_1 = \mathbb{R}$$

(Don’t worry, soon I’ll tell you what  $\mathfrak{siso}(T)$  actually *is!*)

Another clue: classically, we can only write down a Lagrangian for superstrings when spacetime has dimension 3, 4, 6, or 10.

Another clue: classically, we can only write down a Lagrangian for superstrings when spacetime has dimension 3, 4, 6, or 10.

The reason: a certain identity involving spinors holds only in these dimensions.

Another clue: classically, we can only write down a Lagrangian for superstrings when spacetime has dimension 3, 4, 6, or 10.

The reason: a certain identity involving spinors holds only in these dimensions.

In fact, this identity is the equation that a certain bracket

$$[-, -, -]: L_0^{\otimes 3} \rightarrow L_1$$

must obey to give a Lie 2-superalgebra!

Another clue: classically, we can only write down a Lagrangian for superstrings when spacetime has dimension 3, 4, 6, or 10.

The reason: a certain identity involving spinors holds only in these dimensions.

In fact, this identity is the equation that a certain bracket

$$[-, -, -]: L_0^{\otimes 3} \rightarrow L_1$$

must obey to give a Lie 2-superalgebra!

Let's see how it works in detail.



If  $V$  is a finite-dimensional real vector space with a quadratic form  $Q$ , the **Clifford algebra**  $\text{Cliff}(V)$  is the real associative algebra generated by  $V$  with relations

$$v^2 = -Q(v)$$

This algebra is  $\mathbb{Z}_2$ -graded.

If  $V$  is a finite-dimensional real vector space with a quadratic form  $Q$ , the **Clifford algebra**  $\text{Cliff}(V)$  is the real associative algebra generated by  $V$  with relations

$$v^2 = -Q(v)$$

This algebra is  $\mathbb{Z}_2$ -graded.

The double cover of  $\text{SO}(V)$ , the **spin group**  $\text{Spin}(V)$ , sits inside the even part  $\text{Cliff}_0(V)$ .

If  $V$  is a finite-dimensional real vector space with a quadratic form  $Q$ , the **Clifford algebra**  $\text{Cliff}(V)$  is the real associative algebra generated by  $V$  with relations

$$v^2 = -Q(v)$$

This algebra is  $\mathbb{Z}_2$ -graded.

The double cover of  $\text{SO}(V)$ , the **spin group**  $\text{Spin}(V)$ , sits inside the even part  $\text{Cliff}_0(V)$ .

$\text{Cliff}_0(V)$  is either a sum of two matrix algebras, or just one.

If  $V$  is a finite-dimensional real vector space with a quadratic form  $Q$ , the **Clifford algebra**  $\text{Cliff}(V)$  is the real associative algebra generated by  $V$  with relations

$$v^2 = -Q(v)$$

This algebra is  $\mathbb{Z}_2$ -graded.

The double cover of  $\text{SO}(V)$ , the **spin group**  $\text{Spin}(V)$ , sits inside the even part  $\text{Cliff}_0(V)$ .

$\text{Cliff}_0(V)$  is either a sum of two matrix algebras, or just one.

This fact lets us define either two real representations of  $\text{Spin}(V)$ , say  $S_+$  and  $S_-$ , or one, say  $S$ .

If  $V$  is a finite-dimensional real vector space with a quadratic form  $Q$ , the **Clifford algebra**  $\text{Cliff}(V)$  is the real associative algebra generated by  $V$  with relations

$$v^2 = -Q(v)$$

This algebra is  $\mathbb{Z}_2$ -graded.

The double cover of  $\text{SO}(V)$ , the **spin group**  $\text{Spin}(V)$ , sits inside the even part  $\text{Cliff}_0(V)$ .

$\text{Cliff}_0(V)$  is either a sum of two matrix algebras, or just one.

This fact lets us define either two real representations of  $\text{Spin}(V)$ , say  $S_+$  and  $S_-$ , or one, say  $S$ .

In the first case  $S_+ \not\cong S_-$ . In the second, set  $S_+ = S_- = S$ . In either case, let's call  $S_+$  and  $S_-$  **left- and right-handed spinors**.

$\text{Cliff}_0(V)$  acts on  $S_+$  and  $S_-$ . But the whole Clifford algebra acts on  $S_+ \oplus S_-$ , with odd elements interchanging the two parts. So, we can ‘multiply’ a spinor by a vector and get a spinor of the other handedness:

$$\therefore V \otimes S_+ \rightarrow S_-$$

$$\therefore V \otimes S_- \rightarrow S_+$$

$\text{Cliff}_0(V)$  acts on  $S_+$  and  $S_-$ . But the whole Clifford algebra acts on  $S_+ \oplus S_-$ , with odd elements interchanging the two parts. So, we can ‘multiply’ a spinor by a vector and get a spinor of the other handedness:

$$\therefore V \otimes S_+ \rightarrow S_- \qquad \therefore V \otimes S_- \rightarrow S_+$$

When  $V$ ,  $S_+$  and  $S_-$  have the same dimension, we can identify them all and use either of these multiplications to obtain an algebra.

$\text{Cliff}_0(V)$  acts on  $S_+$  and  $S_-$ . But the whole Clifford algebra acts on  $S_+ \oplus S_-$ , with odd elements interchanging the two parts. So, we can ‘multiply’ a spinor by a vector and get a spinor of the other handedness:

$$\therefore V \otimes S_+ \rightarrow S_- \qquad \therefore V \otimes S_- \rightarrow S_+$$

When  $V$ ,  $S_+$  and  $S_-$  have the same dimension, we can identify them all and use either of these multiplications to obtain an algebra.

When  $Q$  is positive definite, this turns out to be a *normed division algebra*.



$\text{Cliff}_0(V)$  acts on  $S_+$  and  $S_-$ . But the whole Clifford algebra acts on  $S_+ \oplus S_-$ , with odd elements interchanging the two parts. So, we can ‘multiply’ a spinor by a vector and get a spinor of the other handedness:

$$\therefore V \otimes S_+ \rightarrow S_- \qquad \therefore V \otimes S_- \rightarrow S_+$$

When  $V$ ,  $S_+$  and  $S_-$  have the same dimension, we can identify them all and use either of these multiplications to obtain an algebra.

When  $Q$  is positive definite, this turns out to be a *normed division algebra*.

Even better, any normed division algebra must arise this way!

$\text{Cliff}_0(V)$  acts on  $S_+$  and  $S_-$ . But the whole Clifford algebra acts on  $S_+ \oplus S_-$ , with odd elements interchanging the two parts. So, we can ‘multiply’ a spinor by a vector and get a spinor of the other handedness:

$$\therefore V \otimes S_+ \rightarrow S_- \qquad \therefore V \otimes S_- \rightarrow S_+$$

When  $V$ ,  $S_+$  and  $S_-$  have the same dimension, we can identify them all and use either of these multiplications to obtain an algebra.

When  $Q$  is positive definite, this turns out to be a *normed division algebra*.

Even better, any normed division algebra must arise this way!

So, let's see when  $\dim(V) = \dim(S_+) = \dim(S_-)$ .

Consider Euclidean space:  $V = \mathbb{R}^k$  with

$$Q(v) = v_1^2 + \cdots + v_k^2$$

Consider Euclidean space:  $V = \mathbb{R}^k$  with

$$Q(v) = v_1^2 + \cdots + v_k^2$$

$V$	$\text{Cliff}(V)$	$\text{Cliff}_0(V)$	$S_{\pm}$
$\mathbb{R}^1$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{R}^2$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{C}$
$\mathbb{R}^3$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$
$\mathbb{R}^4$	$\mathbb{H}[2]$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$ $(S_+ \not\cong S_-)$
$\mathbb{R}^5$	$\mathbb{C}[4]$	$\mathbb{H}[2]$	$\mathbb{H}^2$
$\mathbb{R}^6$	$\mathbb{R}[8]$	$\mathbb{C}[4]$	$\mathbb{C}^4$
$\mathbb{R}^7$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}[8]$	$\mathbb{R}^8$
$\mathbb{R}^8$	$\mathbb{R}[16]$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}^8$ $(S_+ \not\cong S_-)$

Here  $\mathbb{K}[n]$  is the algebra of  $n \times n$  matrices with entries in  $\mathbb{K}$ .

When  $\dim(V) = \dim(S_+) = \dim(S_-)$  we get a normed division algebra:

When  $\dim(V) = \dim(S_+) = \dim(S_-)$  we get a normed division algebra:

$V$	$S_{\pm}$	normed division algebra?
$\mathbb{R}^1$	$\mathbb{R}$	YES: $\mathbb{R}$
$\mathbb{R}^2$	$\mathbb{C}$	YES: $\mathbb{C}$
$\mathbb{R}^3$	$\mathbb{H}$	NO
$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

When  $\dim(V) = \dim(S_+) = \dim(S_-)$  we get a normed division algebra:

$V$	$S_{\pm}$	normed division algebra?
$\mathbb{R}^1$	$\mathbb{R}$	YES: $\mathbb{R}$
$\mathbb{R}^2$	$\mathbb{C}$	YES: $\mathbb{C}$
$\mathbb{R}^3$	$\mathbb{H}$	NO
$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

Increasing  $k$  by 8 multiplies  $\dim(S_{\pm})$  by 16, so these are the *only* normed division algebras!

So: Euclidean space becomes a normed division algebra  $\mathbb{K}$  only in dimensions 1, 2, 4 and 8.



So: Euclidean space becomes a normed division algebra  $\mathbb{K}$  only in dimensions 1, 2, 4 and 8.

Now consider Minkowski spacetime of dimensions 3, 4, 6 and 10. Again, vectors and spinors have a nice description in terms of  $\mathbb{K}$ .

So: Euclidean space becomes a normed division algebra  $\mathbb{K}$  only in dimensions 1, 2, 4 and 8.

Now consider Minkowski spacetime of dimensions 3, 4, 6 and 10. Again, vectors and spinors have a nice description in terms of  $\mathbb{K}$ .

Now vectors  $V$  are the  $2 \times 2$  Hermitian matrices with entries in  $\mathbb{K}$ :

$$V = \left\{ \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} : t, x \in \mathbb{R}, y \in \mathbb{K} \right\}.$$

Now our quadratic form  $Q$  comes from the determinant:

$$\det \begin{pmatrix} t+x & \bar{y} \\ y & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

Now the right-handed spinors  $S_+$  are  $\mathbb{K}^2$ , and the 'multiplication' of vectors and these spinors is just matrix multiplication.

Now the right-handed spinors  $S_+$  are  $\mathbb{K}^2$ , and the ‘multiplication’ of vectors and these spinors is just matrix multiplication.

As reps of  $\text{Spin}(V)$  we have

$$V^* \cong V \qquad S_+^* = S_-$$

so we can take the multiplication

$$\cdot: V \otimes S_+ \rightarrow S_-$$

and use duality to get a ‘bracket’

$$[-, -]: S_+ \otimes S_+ \rightarrow V$$

Now the right-handed spinors  $S_+$  are  $\mathbb{K}^2$ , and the ‘multiplication’ of vectors and these spinors is just matrix multiplication.

As reps of  $\text{Spin}(V)$  we have

$$V^* \cong V \qquad S_+^* = S_-$$

so we can take the multiplication

$$\cdot: V \otimes S_+ \rightarrow S_-$$

and use duality to get a ‘bracket’

$$[-, -]: S_+ \otimes S_+ \rightarrow V$$

Concretely:

$$[\psi, \phi] = \psi\phi^\dagger + \phi\psi^\dagger$$

It’s symmetric!

So, we can define the **translation Lie superalgebra**

$$T = V \oplus S_+$$

with  $V$  as its even part and  $S_+$  as its odd part. We define the bracket to be zero except for  $[-, -]: S_+ \otimes S_+ \rightarrow V$ . The Jacobi identity holds trivially.

So, we can define the **translation Lie superalgebra**

$$T = V \oplus S_+$$

with  $V$  as its even part and  $S_+$  as its odd part. We define the bracket to be zero except for  $[-, -]: S_+ \otimes S_+ \rightarrow V$ . The Jacobi identity holds trivially.

$\text{Spin}(V)$  acts on everything, and its Lie algebra is  $\mathfrak{so}(V)$ , so we can form the semidirect product

$$\mathfrak{siso}(T) = \mathfrak{so}(V) \ltimes T$$

which is called the **Poincaré Lie superalgebra**.

So, we can define the **translation Lie superalgebra**

$$T = V \oplus S_+$$

with  $V$  as its even part and  $S_+$  as its odd part. We define the bracket to be zero except for  $[-, -]: S_+ \otimes S_+ \rightarrow V$ . The Jacobi identity holds trivially.

$\text{Spin}(V)$  acts on everything, and its Lie algebra is  $\mathfrak{so}(V)$ , so we can form the semidirect product

$$\mathfrak{siso}(T) = \mathfrak{so}(V) \ltimes T$$

which is called the **Poincaré Lie superalgebra**.

The corresponding Lie supergroup acts as symmetries of 'Minkowski superspace'.



We get a Poincaré Lie superalgebra whenever we have an invariant symmetric bracket that takes two spinors and gives a vector. What's so special about the dimensions 3, 4, 6 and 10?

We get a Poincaré Lie superalgebra whenever we have an invariant symmetric bracket that takes two spinors and gives a vector. What's so special about the dimensions 3, 4, 6 and 10?

In these dimensions the multiplication

$$\cdot: V \otimes S_+ \rightarrow S_-$$

and bracket

$$[-, -]: S_+ \otimes S_+ \rightarrow V$$

obey the identity

$$[\psi, \psi] \cdot \psi = 0$$

Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using  $\mathbb{K}$ -valued matrices.

We get a Poincaré Lie superalgebra whenever we have an invariant symmetric bracket that takes two spinors and gives a vector. What's so special about the dimensions 3, 4, 6 and 10?

In these dimensions the multiplication

$$\cdot: V \otimes S_+ \rightarrow S_-$$

and bracket

$$[-, -]: S_+ \otimes S_+ \rightarrow V$$

obey the identity

$$[\psi, \psi] \cdot \psi = 0$$

Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using  $\mathbb{K}$ -valued matrices.

In fact, *only* for Minkowski spacetimes of dimension 3, 4, 6, and 10 does this identity hold!

The identity  $[\psi, \psi] \cdot \psi = 0$  lets us extend the Poincaré Lie superalgebra  $\mathfrak{siso}(T)$  to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \xleftarrow{d} \mathbb{R}$$

The identity  $[\psi, \psi] \cdot \psi = 0$  lets us extend the Poincaré Lie superalgebra  $\mathfrak{siso}(T)$  to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \xleftarrow{d} \mathbb{R}$$

The idea:

The identity  $[\psi, \psi] \cdot \psi = 0$  lets us extend the Poincaré Lie superalgebra  $\mathfrak{siso}(T)$  to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \xleftarrow{d} \mathbb{R}$$

The idea:

- $d$  is zero

The identity  $[\psi, \psi] \cdot \psi = 0$  lets us extend the Poincaré Lie superalgebra  $\mathfrak{siso}(T)$  to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \xleftarrow{d} \mathbb{R}$$

The idea:

- $d$  is zero
- $[-, -]$  is zero except for the bracket in  $\mathfrak{siso}(T)$

The identity  $[\psi, \psi] \cdot \psi = 0$  lets us extend the Poincaré Lie superalgebra  $\mathfrak{siso}(T)$  to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \stackrel{d}{\leftarrow} \mathbb{R}$$

The idea:

- $d$  is zero
- $[-, -]$  is zero except for the bracket in  $\mathfrak{siso}(T)$
- $[-, -, -]$  is zero unless two arguments are spinors and one is a vector in  $\mathfrak{siso}(T)$ , and

$$[\psi, \phi, \nu] = g([\psi, \phi], \nu) \in \mathbb{R}$$

where  $\psi, \phi \in S_+$ ,  $\nu \in V$ , and  $g: V \otimes V \rightarrow R$  is the **Minkowski metric**: the bilinear form corresponding to  $Q$ .



To get a Lie 2-superalgebra this way, the ternary bracket must obey an equation. This says that

$$[-, -, -]: \mathfrak{siso}(T)^{\otimes 3} \rightarrow \mathbb{R}$$

is a *3-cocycle* in Lie superalgebra cohomology.

To get a Lie 2-superalgebra this way, the ternary bracket must obey an equation. This says that

$$[-, -, -]: \mathfrak{siso}(T)^{\otimes 3} \rightarrow \mathbb{R}$$

is a *3-cocycle* in Lie superalgebra cohomology.

The equation

$$[\psi, \psi] \cdot \psi = 0$$

is this cocycle condition in disguise.

Let us call the resulting Lie 2-superalgebra **superstring**( $T$ ).

Let us call the resulting Lie 2-superalgebra **superstring**( $T$ ).

### Theorem (John Huerta)

*There is a 2-group in the category of supermanifolds,  $\text{Superstring}(T)$ , whose Lie 2-superalgebra is  $\text{superstring}(T)$ .*

Let us call the resulting Lie 2-superalgebra  $\mathbf{superstring}(T)$ .

### Theorem (John Huerta)

*There is a 2-group in the category of supermanifolds,  $\mathbf{Superstring}(T)$ , whose Lie 2-superalgebra is  $\mathbf{superstring}(T)$ .*

This theorem takes real work to prove. Not every Lie 2-superalgebra has a corresponding ‘Lie 2-supergroup’ in such a simple-minded sense! There are important finite-dimensional Lie 2-algebras that don’t come from 2-groups in the category of manifolds—instead, they come from ‘stacky’ Lie 2-groups.

He also went further:

### Theorem (Huerta)

*In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.*

He also went further:

### Theorem (Huerta)

*In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.*

*This gives a Lie 3-superalgebra*

$$\mathfrak{so}(T) \leftarrow 0 \leftarrow \mathbb{R}$$

*called **2-brane**( $T$ ).*

He also went further:

### Theorem (Huerta)

*In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.*

*This gives a Lie 3-superalgebra*

$$\mathfrak{iso}(T) \leftarrow 0 \leftarrow \mathbb{R}$$

*called **2-brane**( $T$ ).*

*Moreover, there is a 3-group in the category of supermanifolds, **2-Brane**( $T$ ), whose Lie 3-superalgebra is **2-brane**( $T$ ).*



2-Brane( $T$ ) is relevant to the theory of supersymmetric 2-branes in dimension 4, 5, 7 and 11.

2-Brane( $T$ ) is relevant to the theory of supersymmetric 2-branes in dimension 4, 5, 7 and 11.

The 11-dimensional — octonionic! — case also shows up in 11-dimensional supergravity, and thus presumably ' $M$ -theory' (whatever that is).

2-Brane( $T$ ) is relevant to the theory of supersymmetric 2-branes in dimension 4, 5, 7 and 11.

The 11-dimensional — octonionic! — case also shows up in 11-dimensional supergravity, and thus presumably ‘ $M$ -theory’ (whatever that is).

**The buck stops here:** for 12-dimensional Minkowski spacetime, it seems all 5-cocycles on  $s_{15}o(T)$  are trivial. Apparently the nonassociativity of the octonions spoils the calculation!