

Division Algebras and Supersymmetry

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Abstract

Supersymmetry is deeply related to division algebras. For example, nonabelian Yang–Mills fields minimally coupled to massless spinors are supersymmetric if and only if the dimension of spacetime is 3, 4, 6 or 10. The same is true for the Green–Schwarz superstring. In both cases, supersymmetry relies on the vanishing of a certain trilinear expression involving a spinor field. The reason for this, in turn, is the existence of normed division algebras in dimensions 1, 2, 4 and 8: the real numbers, complex numbers, quaternions and octonions. Here we provide a self-contained account of how this works.

1 Introduction

The minimal supersymmetric extension of pure Yang–Mills theory has the Lagrangian:

$$L = -\frac{1}{4}\langle F, F \rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi \rangle.$$

Here A is a connection on a bundle with semisimple gauge group G , F is the curvature of A , ψ is a \mathfrak{g} -valued spinor field, and \not{D}_A is the covariant Dirac operator associated with A . It is well-known that this theory is supersymmetric if and only if the dimension of spacetime is 3, 4, 6, or 10. Our goal here is to present a self-contained proof of the ‘if’ part of this result, based on the theory of normed division algebras.

This result goes back to the work of Brink, Schwarz, and Sherk [3] and others. The book by Green, Schwarz and Witten [9] contains a standard proof based on the properties of Clifford algebras in various dimensions. But Evans [7] has shown that the supersymmetry of L in dimension $n + 2$ implies the existence of a normed division algebra of dimension n . Conversely, Kugo and Townsend [11] showed how spinors in dimension 3, 4, 6, and 10 derive special properties from the normed division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} . They formulated a supersymmetric model in 6 dimensions using the quaternions, \mathbb{H} . They also speculated about a similar formalism in 10 dimensions using the octonions, \mathbb{O} .

Shortly after Kugo and Townsend’s work, Sudbery [16] used division algebras to construct vectors, spinors and Lorentz groups in Minkowski spacetimes of dimensions 3, 4, 6, and 10. He then refined his construction with Chung [4], and with Manogue [12] he used these ideas to give an octonionic proof of the supersymmetry of the above Lagrangian in dimension 10. This proof was later simplified by Manogue, Dray and Janesky [5]. In the meantime, Schray [13] applied the same tools to the superparticle.

All this work has made it quite clear that normed division algebras explain why the above theory is supersymmetric in dimensions 3, 4, 6, and 10. Technically, what we need to check for supersymmetry is that δL is a total divergence with respect to the supersymmetry transformation

$$\begin{aligned}\delta A &= \epsilon \cdot \psi \\ \delta \psi &= \frac{1}{2} F \epsilon\end{aligned}$$

for any constant spinor field ϵ . (We explain the notation here later; we assume no prior understanding of supersymmetry or normed division algebras.) A calculation that works in any dimension shows that

$$\delta L = \text{tri } \psi + \text{divergence}$$

where $\text{tri } \psi$ is a certain expression depending in a trilinear way on ψ and linearly on ϵ .

So, the marvelous fact that needs to be understood is that $\text{tri } \psi = 0$ in dimensions 3, 4, 6, and 10, thanks to special properties of the normed division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} . Indeed, this fact is responsible for supersymmetry, not only for Yang–Mills fields in these dimensions, but also for superstrings! The same term $\text{tri } \psi$ shows up as the obstruction to supersymmetry in the Green–Schwarz Lagrangian for classical superstrings [8, 9]. So, the vanishing of this term deserves to be understood: clearly, simply, and in as many ways as possible.

Unfortunately, many important pieces of the story are scattered throughout the literature. The treatment of Deligne and Freed [6] is self-contained, and it uses normed division algebras, but it does not use ‘purely equational reasoning’: it proves $\text{tri } \psi = 0$ by first showing that the double cover of the Lorentz group acts transitively on the set of nonzero spinors in dimensions 3, 4, 6, and 10. While this geometrical argument is beautiful and insightful, a purely equational approach has its own charm. The line of work carried out by Sudbery, Manogue, Dray, and collaborators [5, 12, 13] has shown that the equation $\text{tri } \psi = 0$ can be derived from the complete antisymmetry of another trilinear expression, the ‘associator’

$$[a, b, c] = (ab)c - a(bc)$$

in the normed division algebra. Our desire here is to merely present this argument as clearly as we can.

So, here we present an equational proof that $\text{tri } \psi = 0$ in dimensions 3, 4, 6, and 10, based on the complete antisymmetry of the associator for the normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} . In Section 2 we review the properties of normed division algebras that we will need. In Section 3 we start by recalling how to interpret vectors as 2×2 hermitian matrices with entries in \mathbb{K} , and spinors as elements of \mathbb{K}^2 . We then use this language to describe the basic operations involving vectors, spinors and scalars. These include an operation that takes two spinors ψ and ϕ and forms a vector $\psi \cdot \phi$, and an operation that takes a vector A and a spinor ψ and forms a spinor $A\psi$. In Section 4 we prove the fundamental identity that holds only in Minkowski spaces of dimensions 3, 4, 6 and 10:

$$(\psi \cdot \psi)\psi = 0.$$

In Section 5 we introduce a little superalgebra, and explain why we should treat \mathbb{K} as an ‘odd’, or ‘fermionic’, super vector space. In Section 6 we formulate pure super-Yang–Mills theory in terms of normed division algebras, completely avoiding the use of gamma matrices. We explain how the term $\text{tri } \psi$ arises as the obstruction to supersymmetry in this theory. Finally, we use the ‘fundamental identity’ to prove that $\text{tri } \psi = 0$ in dimensions 3, 4, 6 and 10.

2 Normed Division Algebras

By a classic theorem of Hurwitz [10], there are only four normed division algebras: the real numbers, \mathbb{R} , the complex numbers, \mathbb{C} , the quaternions, \mathbb{H} , and the octonions, \mathbb{O} . These algebras have dimen-

sion 1, 2, 4, and 8. For an overview of this subject, including a Clifford algebra proof of Hurwitz's theorem, see [1]. Here we introduce the bare minimum of material needed to reach our goal.

A **normed division algebra** \mathbb{K} is a (finite-dimensional, possibly nonassociative) real algebra equipped with a multiplicative unit 1 and a norm $|\cdot|$ satisfying:

$$|ab| = |a||b|$$

for all $a, b \in \mathbb{K}$. Note this implies that \mathbb{K} has no zero divisors. We will freely identify $\mathbb{R}1 \subseteq \mathbb{K}$ with \mathbb{R} .

In all cases, this norm can be defined using conjugation. Every normed division algebra has a **conjugation** operator—a linear operator $*$: $\mathbb{K} \rightarrow \mathbb{K}$ satisfying

$$a^{**} = a, \quad (ab)^* = b^*a^*$$

for all $a, b \in \mathbb{K}$. Conjugation lets us decompose each element of \mathbb{K} into real and imaginary parts, as follows:

$$\operatorname{Re}(a) = \frac{a + a^*}{2}, \quad \operatorname{Im}(a) = \frac{a - a^*}{2}.$$

Conjugating changes the sign of the imaginary part and leaves the real part fixed. We can write the norm as

$$|a| = \sqrt{aa^*} = \sqrt{a^*a}.$$

This norm can be polarized to give an inner product on \mathbb{K} :

$$(a, b) = \operatorname{Re}(ab^*) = \operatorname{Re}(a^*b).$$

The algebras \mathbb{R} , \mathbb{C} and \mathbb{H} are associative. The octonions \mathbb{O} are not. Yet they come close: the subalgebra generated by any two octonions is associative. Another way to express this fact uses the **associator**:

$$[a, b, c] = (ab)c - a(bc),$$

a trilinear map $\mathbb{K} \otimes \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$. A theorem due to Artin [14] states that for any algebra, the subalgebra generated by any two elements is associative if and only if the associator is alternating (that is, completely antisymmetric in its three arguments). An algebra with this property is thus called **alternative**. The octonions \mathbb{O} are alternative, and so of course are \mathbb{R} , \mathbb{C} and \mathbb{H} : for these three the associator simply vanishes!

In what follows, our calculations make heavy use of the fact that all four normed division algebras are alternative. Besides this, the properties we require are:

Proposition 1. *The associator changes sign when one of its entries is conjugated.*

Proof. Since the subalgebra generated by any two elements is associative, and real elements of \mathbb{K} lie in every subalgebra, $[a, b, c] = 0$ if any one of a, b, c is real. It follows that $[a, b, c] = [\operatorname{Im}(a), \operatorname{Im}(b), \operatorname{Im}(c)]$, which yields the desired result. \square

Proposition 2. *The associator is purely imaginary.*

Proof. Since $(ab)^* = b^*a^*$, a calculation shows $[a, b, c]^* = -[c^*, b^*, a^*]$. By alternativity this equals $[a^*, b^*, c^*]$, which in turn equals $-[a, b, c]$ by the above proposition. So, $[a, b, c]$ is purely imaginary. \square

For any square matrix A with entries in \mathbb{K} , we define its **trace** $\operatorname{tr}(A)$ to be the sum of its diagonal entries. This trace lacks the usual cyclic property, because \mathbb{K} is noncommutative, so in general $\operatorname{tr}(AB) \neq \operatorname{tr}(BA)$. Luckily, taking the real part restores this property:

Proposition 3. *Let $a, b,$ and c be elements of \mathbb{K} . Then*

$$\operatorname{Re}((ab)c) = \operatorname{Re}(a(bc))$$

and this quantity is invariant under cyclic permutations of $a, b,$ and c .

Proof. Proposition 2 implies that $\operatorname{Re}((ab)c) = \operatorname{Re}(a(bc))$. For the cyclic property, it then suffices to prove $\operatorname{Re}(ab) = \operatorname{Re}(ba)$. Since $(a, b) = (b, a)$ and the inner product is defined by $(a, b) = \operatorname{Re}(ab^*) = \operatorname{Re}(a^*b)$, we see:

$$\operatorname{Re}(ab^*) = \operatorname{Re}(b^*a).$$

The desired result follows upon substituting b^* for b . □

Proposition 4. *Let $A, B,$ and C be $k \times \ell, \ell \times m$ and $m \times k$ matrices with entries in \mathbb{K} . Then*

$$\operatorname{Re} \operatorname{tr}((AB)C) = \operatorname{Re} \operatorname{tr}(A(BC))$$

*and this quantity is invariant under cyclic permutations of $A, B,$ and C . We call this quantity the **real trace** $\operatorname{Re} \operatorname{tr}(ABC)$.*

Proof. This follows from the previous proposition and the definition of the trace. □

The reader will have noticed three trilinears in this section: the associator $[a, b, c]$, the real part $\operatorname{Re}((ab)c)$, and the real trace $\operatorname{Re} \operatorname{tr}(ABC)$. This is no coincidence, as they all relate to the star of the show, $\operatorname{tri} \psi$. In fact:

$$\operatorname{tri} \psi = \operatorname{Re} \operatorname{tr}(\psi^\dagger(\epsilon \cdot \psi)\psi).$$

for some suitable matrices $\psi^\dagger, \epsilon \cdot \psi$ and ψ . Of course, we have not yet said how to construct these. We turn to this now.

3 Vectors, Spinors and Intertwiners

It is well-known [1, 11, 16] that given a normed division algebra \mathbb{K} of dimension n , one can construct $(n + 2)$ -dimensional Minkowski spacetime as the space of 2×2 hermitian matrices with entries in \mathbb{K} , with the determinant giving the Minkowski metric. Spinors can then be described as elements of \mathbb{K}^2 . Our goal here is to provide self-contained proofs of these facts, and then develop all the basic operations involving vectors, spinors and scalars using this language.

To begin, let $\mathbb{K}[m]$ denote the space of $m \times m$ matrices with entries in \mathbb{K} . Given $A \in \mathbb{K}[m]$, define its **hermitian adjoint** A^\dagger to be its conjugate transpose:

$$A^\dagger = (A^*)^T.$$

We say such a matrix is **hermitian** if $A = A^\dagger$. Now take the 2×2 hermitian matrices:

$$\mathfrak{h}_2(\mathbb{K}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} : t, x \in \mathbb{R}, y \in \mathbb{K} \right\}.$$

This is an $(n + 2)$ -dimensional real vector space. Moreover, the usual formula for the determinant of a matrix gives the Minkowski norm on this vector space:

$$-\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = -t^2 + x^2 + |y|^2.$$

We insert a minus sign to obtain the signature $(n + 1, 1)$. Note this formula is unambiguous even if \mathbb{K} is noncommutative or nonassociative.

It follows that the double cover of the Lorentz group, $\text{Spin}(n+1, 1)$, acts on $\mathfrak{h}_2(\mathbb{K})$ via determinant-preserving linear transformations. Since this is the ‘vector’ representation, we will often call $\mathfrak{h}_2(\mathbb{K})$ simply V . The Minkowski metric

$$g: V \otimes V \rightarrow \mathbb{R}$$

is given by

$$g(A, A) = -\det(A).$$

There is also a nice formula for the inner product of two different vectors. This involves the **trace reversal** of $A \in \mathfrak{h}_2(\mathbb{K})$, defined by

$$\tilde{A} = A - (\text{tr}A)1.$$

Note we indeed have $\text{tr}(\tilde{A}) = -\text{tr}(A)$. Also note that

$$A = \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \implies \tilde{A} = \begin{pmatrix} -t+x & y \\ y^* & -t-x \end{pmatrix}$$

so trace reversal is really *time reversal*. Moreover:

Proposition 5. *For any vectors $A, B \in V = \mathfrak{h}_2(K)$, we have*

$$A\tilde{A} = \tilde{A}A = -\det(A)1$$

and

$$\frac{1}{2}\text{Re tr}(A\tilde{B}) = \frac{1}{2}\text{Re tr}(\tilde{A}B) = g(A, B)$$

Proof. We check the first equation by a quick calculation. Taking the real trace and dividing by 2 gives

$$\frac{1}{2}\text{Re tr}(A\tilde{A}) = \frac{1}{2}\text{Re tr}(\tilde{A}A) = -\det(A) = g(A, A).$$

Then we use the polarization identity, which says that two symmetric bilinear forms that give the same quadratic form must be equal. \square

Next we consider spinors. As real vector spaces, the spinor representations S_+ and S_- are both just \mathbb{K}^2 . However, they differ as representations of $\text{Spin}(n+1, 1)$. To construct these representations, we begin by defining ways for vectors to act on spinors:

$$\begin{aligned} \gamma: V \otimes S_+ &\rightarrow S_- \\ A \otimes \psi &\mapsto A\psi. \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}: V \otimes S_- &\rightarrow S_+ \\ A \otimes \psi &\mapsto \tilde{A}\psi. \end{aligned}$$

We can also think of these as maps that send elements of V to linear operators:

$$\begin{aligned} \gamma: V &\rightarrow \text{Hom}(S_+, S_-), \\ \tilde{\gamma}: V &\rightarrow \text{Hom}(S_-, S_+). \end{aligned}$$

Here a word of caution is needed: since \mathbb{K} may be nonassociative, 2×2 matrices with entries in \mathbb{K} cannot be identified with linear operators on \mathbb{K}^2 in the usual way. They certainly induce linear operators via left multiplication:

$$L_A(\psi) = A\psi.$$

Indeed, this is how γ and $\tilde{\gamma}$ turn elements of V into linear operators:

$$\begin{aligned} \gamma(A) &= L_A, \\ \tilde{\gamma}(A) &= L_{\tilde{A}}. \end{aligned}$$

However, because of nonassociativity, composing such linear operators is different from multiplying the matrices:

$$L_A L_B(\psi) = A(B\psi) \neq (AB)\psi = L_{AB}(\psi).$$

Since vectors act on elements of S_+ to give elements of S_- and vice versa, they map the space $S_+ \oplus S_-$ to itself. This gives rise to an action of the Clifford algebra $\text{Cliff}(V)$ on $S_+ \oplus S_-$:

Proposition 6. *The vectors $V = \mathfrak{h}_2(\mathbb{K})$ act on the spinors $S_+ \oplus S_- = \mathbb{K}^2 \oplus \mathbb{K}^2$ via the map*

$$\Gamma: V \rightarrow \text{End}(S_+ \oplus S_-)$$

given by

$$\Gamma(A)(\psi, \phi) = (\tilde{A}\phi, A\psi).$$

Furthermore, $\Gamma(A)$ satisfies the Clifford algebra relation:

$$\Gamma(A)^2 = g(A, A)1$$

and so extends to a homomorphism $\Gamma: \text{Cliff}(V) \rightarrow \text{End}(S_+ \oplus S_-)$, i.e. a representation of the Clifford algebra $\text{Cliff}(V)$ on $S_+ \oplus S_-$.

Proof. Suppose $A \in V$ and $\Psi = (\psi, \phi) \in S_+ \oplus S_-$. We need to check that

$$\Gamma(A)^2(\Psi) = -\det(A)\Psi.$$

Here we must be mindful of nonassociativity: we have

$$\Gamma(A)^2(\Psi) = (\tilde{A}(A\psi), A(\tilde{A}\phi)).$$

Yet it is easy to check that the expressions $\tilde{A}(A\psi)$ and $A(\tilde{A}\phi)$ involve multiplying at most two different nonreal elements of \mathbb{K} . These associate, since \mathbb{K} is alternative, so in fact

$$\Gamma(A)^2(\Psi) = ((\tilde{A}A)\psi, (A\tilde{A})\phi).$$

To conclude, we use Proposition 5. □

The action of a vector swaps S_+ and S_- , so acting by vectors twice sends S_+ to itself and S_- to itself. This means that while S_+ and S_- are *not* modules for the Clifford algebra $\text{Cliff}(V)$, they are both modules for the even part of the Clifford algebra, generated by products of pairs of vectors. The group $\text{Spin}(n+1, 1)$ lives in this even part: as is well-known, it is generated by products of pairs of **unit vectors** in V : that is, vectors A with $g(A, A) = \pm 1$. As a result, S_+ and S_- are both representations of $\text{Spin}(n+1, 1)$.

While we will not need this in what follows, one can check that:

- When $\mathbb{K} = \mathbb{R}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\text{Spin}(2, 1)$.
- When $\mathbb{K} = \mathbb{C}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\text{Spin}(3, 1)$.
- When $\mathbb{K} = \mathbb{H}$, S_+ and S_- are the Weyl spinor representations of $\text{Spin}(5, 1)$.
- When $\mathbb{K} = \mathbb{O}$, S_+ and S_- are the Majorana–Weyl spinor representations of $\text{Spin}(9, 1)$.

This counts as a consistency check, because these are precisely the kinds of spinor representations that go into pure super-Yang–Mills theory. But it is important to note that the *differences* between these spinor representations are irrelevant to our argument. What matters is how they are the *same*—they can all be defined on \mathbb{K}^2 .

Now that we have representations of $\text{Spin}(n+1, 1)$ on V , S_+ and S_- , we need to develop the $\text{Spin}(n+1, 1)$ -equivariant maps that relate them. Ultimately, to define the Lagrangian for pure super-Yang–Mills theory, we need:

- An invariant pairing:

$$\langle -, - \rangle: S_+ \otimes S_- \rightarrow \mathbb{R}.$$

- An equivariant map that turns pairs of spinors into vectors:

$$\cdot : S_{\pm} \otimes S_{\pm} \rightarrow V.$$

Another name for an equivariant map between group representations is an ‘intertwining operator’. As a first step, we show that the action of vectors on spinors is itself an intertwining operator:

Proposition 7. *The maps*

$$\begin{aligned} \gamma: V \otimes S_+ &\rightarrow S_- \\ A \otimes \psi &\mapsto A\psi \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}: V \otimes S_- &\rightarrow S_+ \\ A \otimes \psi &\mapsto \tilde{A}\psi \end{aligned}$$

are equivariant with respect to the action of $\text{Spin}(n+1, 1)$.

Proof. Both γ and $\tilde{\gamma}$ are restrictions of the map

$$\Gamma: V \otimes (S_+ \oplus S_-) \rightarrow S_+ \oplus S_-,$$

so it suffices to check that Γ is equivariant. In fact, we will do this not just for the action of $\text{Spin}(n+1, 1)$, but for the larger group $\text{Pin}(n+1, 1)$, the subgroup of $\text{Cliff}(V)$ generated by unit vectors in V . This group acts on V with unit vectors acting by conjugation on $V \subseteq \text{Cliff}(V)$. It acts on $S_+ \oplus S_-$ with a unit vector B acting by $\Gamma(B)$. Both of these representations of $\text{Pin}(n+1, 1)$ restrict to the usual representations of $\text{Spin}(n+1, 1)$.

Thus, we compute:

$$\Gamma(BAB^{-1})\Gamma(B)\Psi = \Gamma(B)(\Gamma(A)\Psi).$$

Here it is important to note that the conjugation BAB^{-1} is taking place in the associative algebra $\text{Cliff}(V)$, not in the algebra of matrices. This equation says that Γ is indeed $\text{Pin}(n+1, 1)$ -equivariant, as claimed. \square

Now we exhibit the key tool: the pairing between S_+ and S_- :

Proposition 8. *The pairing*

$$\begin{aligned} \langle -, - \rangle: S_+ \otimes S_- &\rightarrow \mathbb{R} \\ \psi \otimes \phi &\mapsto \text{Re}(\psi^\dagger \phi) \end{aligned}$$

is invariant under the action of $\text{Spin}(n+1, 1)$.

Proof. Given $A \in V$, we use the fact that the associator is purely imaginary to show that

$$\text{Re} \left((\tilde{A}\phi)^\dagger (A\psi) \right) = \text{Re} \left((\phi^\dagger \tilde{A})(A\psi) \right) = \text{Re} \left(\phi^\dagger (\tilde{A}(A\psi)) \right).$$

As in the proof of the Clifford relation, it is easy to check that the column vector $\tilde{A}(A\psi)$ involves at most two nonreal elements of \mathbb{K} and equals $g(A, A)\psi$. So:

$$\langle \tilde{\gamma}(A)\phi, \gamma(A)\psi \rangle = g(A, A)\langle \psi, \phi \rangle.$$

In particular when A is a unit vector, acting by A swaps the order of ψ and ϕ and changes the sign at most. $\langle -, - \rangle$ is thus invariant under the group in $\text{Cliff}(V)$ generated by products of *pairs* of unit vectors, which is $\text{Spin}(n+1, 1)$. \square

With this pairing in hand, there is a manifestly equivariant way to turn a pair of spinors into a vector. Given $\psi, \phi \in S_+$, there is a unique vector $\psi \cdot \phi$ whose inner product with any vector A is given by

$$g(\psi \cdot \phi, A) = \langle \psi, \gamma(A)\phi \rangle.$$

Similarly, given $\psi, \phi \in S_-$, we define $\psi \cdot \phi \in V$ by demanding

$$g(\psi \cdot \phi, A) = \langle \tilde{\gamma}(A)\psi, \phi \rangle$$

for all $A \in V$. This gives us maps

$$S_{\pm} \otimes S_{\pm} \rightarrow V$$

which are manifestly equivariant.

On the other hand, because $S_{\pm} = \mathbb{K}^2$ and $V = \mathfrak{h}_2(\mathbb{K})$, there is also a naive way to turn a pair of spinors into a vector using matrix operations: just multiply the column vector ψ by the row vector ϕ^\dagger and then take the hermitian part:

$$\psi\phi^\dagger + \phi\psi^\dagger \in \mathfrak{h}_2(\mathbb{K}),$$

or perhaps its trace reversal:

$$\widetilde{\psi\phi^\dagger + \phi\psi^\dagger} \in \mathfrak{h}_2(\mathbb{K}).$$

In fact, these naive guesses match the manifestly equivariant approach described above:

Proposition 9. *The maps $\cdot : S_{\pm} \otimes S_{\pm} \rightarrow V$ are given by:*

$$\begin{aligned} \cdot : S_+ \otimes S_+ &\rightarrow V \\ \psi \otimes \phi &\mapsto \widetilde{\psi\phi^\dagger + \phi\psi^\dagger} \\ \cdot : S_- \otimes S_- &\rightarrow V \\ \psi \otimes \phi &\mapsto \psi\phi^\dagger + \phi\psi^\dagger. \end{aligned}$$

These maps are equivariant with respect to the action of $\text{Spin}(n+1, 1)$.

Proof. First suppose $\psi, \phi \in S_+$. We have already seen that the map $\cdot : S_+ \otimes S_+ \rightarrow V$ is equivariant. We only need to show that this map has the desired form. We start by using some definitions:

$$g(\psi \cdot \phi, A) = \langle \psi, \gamma(A)\phi \rangle = \text{Re}(\psi^\dagger(A\phi)) = \text{Re} \text{tr}(\psi^\dagger A\phi).$$

We thus have

$$g(\psi \cdot \phi, A) = \text{Re} \text{tr}(\psi^\dagger A\phi) = \text{Re} \text{tr}(\phi^\dagger A\psi),$$

where in the last step we took the adjoint of the inside. Applying the cyclic property of the real trace, we obtain

$$g(\psi \cdot \phi, A) = \text{Re} \text{tr}(\phi\psi^\dagger A) = \text{Re} \text{tr}(\psi\phi^\dagger A).$$

Averaging gives

$$g(\psi \cdot \phi, A) = \frac{1}{2} \text{Re} \text{tr}((\psi\phi^\dagger + \phi\psi^\dagger)A).$$

On the other hand, Proposition 5 implies that

$$g(\psi \cdot \phi, A) = \frac{1}{2} \text{Re} \text{tr}(\widetilde{(\psi \cdot \phi)A}).$$

Since both these equations hold for all A , we must have

$$\widetilde{\psi \cdot \phi} = \psi\phi^\dagger + \phi\psi^\dagger.$$

Doing trace reversal twice gets us back where we started, so

$$\psi \cdot \phi = \psi\phi^\dagger + \phi\psi^\dagger$$

as desired. A similar calculation shows that if $\psi, \phi \in S_-$, then $\psi \cdot \phi = \psi\phi^\dagger + \phi\psi^\dagger$. \square

Map	Division algebra notation	Index notation
$g: V \otimes V \rightarrow \mathbb{R}$	$\frac{1}{2} \text{Re tr}(A\tilde{B})$	$A^\mu B_\mu$
$\gamma: V \otimes S_+ \rightarrow S_-$	$A\psi$	$\gamma_\mu A^\mu \psi$
$\tilde{\gamma}: V \otimes S_- \rightarrow S_+$	$\tilde{A}\psi$	$\tilde{\gamma}_\mu A^\mu \psi$
$\cdot: S_+ \otimes S_+ \rightarrow V$	$\widetilde{\psi\phi^\dagger + \phi\psi^\dagger}$	$\overline{\psi}\gamma^\mu\phi$
$\cdot: S_- \otimes S_- \rightarrow V$	$\psi\phi^\dagger + \phi\psi^\dagger$	$\overline{\psi}\tilde{\gamma}^\mu\phi$
$\langle -, - \rangle: S_+ \otimes S_- \rightarrow \mathbb{R}$	$\text{Re}(\psi^\dagger\phi)$	$\overline{\psi}\phi$

Table 1: Division algebra notation vs. index notation

We can summarize our work so far with a table of the basic bilinear maps involving vectors, spinors and scalars. Table 1 shows how to translate between division algebra notation and something more closely resembling standard physics notation. In this table the adjoint spinor $\overline{\psi}$ denotes the spinor dual to ψ under the pairing $\langle -, - \rangle$. The gamma matrix γ^μ denotes a Clifford algebra generator acting on S_+ , while $\tilde{\gamma}^\mu$ denotes the same element acting on S_- . Of course $\tilde{\gamma}$ is not standard physics notation; the standard notation for this depends on which of the four cases we are considering: \mathbb{R} , \mathbb{C} , \mathbb{H} or \mathbb{O} .

4 The Fundamental Identity

Now we prove the fundamental identity that makes supersymmetry tick in dimensions 3, 4, 6, and 10. This identity was dubbed the ‘3- ψ ’s rule’ by Schray [13]. The following proof is based on an argument in the appendix of the paper by Dray, Janesky and Manogue [5]. Note that it is really the alternative law rather than the normed division algebra axioms that does the job:

Theorem 10. *Suppose $\psi \in S_+$. Then $(\psi \cdot \psi)\psi = 0$.*

Proof. By definition,

$$(\psi \cdot \psi)\psi = 2(\widetilde{\psi\psi^\dagger})\psi = 2(\psi\psi^\dagger - \text{tr}(\psi\psi^\dagger)1)\psi.$$

It is easy to check that $\text{tr}(\psi\psi^\dagger) = \psi^\dagger\psi$, so

$$(\psi \cdot \psi)\psi = 2((\psi\psi^\dagger)\psi - (\psi^\dagger\psi)\psi).$$

Since $\psi^\dagger\psi$ is a real number, it commutes with ψ :

$$(\psi \cdot \psi)\psi = 2((\psi\psi^\dagger)\psi - \psi(\psi^\dagger\psi)).$$

Since \mathbb{K} is alternative, every subalgebra of \mathbb{K} generated by two elements is associative. Since $\psi \in \mathbb{K}^2$ is built from just two elements of \mathbb{K} , the right-hand side vanishes. \square

It will be useful to state this result in a somewhat more elaborate form:

Theorem 11. *Define a map*

$$\begin{aligned} T: S_+ \otimes S_+ \otimes S_+ &\rightarrow S_- \\ \psi \otimes \phi \otimes \chi &\mapsto (\psi \cdot \phi)\chi + (\phi \cdot \chi)\psi + (\chi \cdot \psi)\phi. \end{aligned}$$

Then $T = 0$.

Proof. It is easy to check that $\psi \cdot \phi = \phi \cdot \psi$ for all $\psi, \phi \in S_+$, so the map T is completely symmetric in its three arguments. Just as any symmetric bilinear form $B(x, y)$ can be recovered from the corresponding quadratic form $B(x, x)$ by polarization, so too can any symmetric trilinear form be recovered from the corresponding cubic form. Since $T(\psi, \psi, \psi) = 0$ by Theorem 10, it follows that $T = 0$. \square

To see *how* this theorem is the key to supersymmetry for super-Yang–Mills theory, we need a little superalgebra.

5 Superalgebra

So far we have used normed division algebras to construct a number of algebraic structures: vectors as elements of $\mathfrak{h}_2(\mathbb{K})$, spinors as elements of \mathbb{K}^2 , and the various bilinear maps involving vectors, spinors, and scalars. However, to describe supersymmetry, we also need superalgebra. Specifically, we need anticommuting spinors. Physically, this is because spinors are fermions, so we need them to satisfy anticommutation relations. Mathematically, this means that we will do our algebra in the category of ‘super vector spaces’, SuperVect, rather than the category of vector spaces, Vect.

A **super vector space** is a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$ where V_0 is called the **even** or **bosonic** part, and V_1 is called the **odd** or **fermionic** part. Like Vect, SuperVect is a symmetric monoidal category [2]. It has:

- \mathbb{Z}_2 -graded vector spaces as objects;
- Grade-preserving linear maps as morphisms;
- A tensor product \otimes that has the following grading: if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, then $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$;
- A braiding

$$B_{V,W}: V \otimes W \rightarrow W \otimes V$$

defined as follows: $v \in V$ and $w \in W$ are of grade p and q , then

$$B_{V,W}(v \otimes w) = (-1)^{pq} w \otimes v.$$

The braiding encodes the ‘the rule of signs’: in any calculation, when two odd elements are interchanged, we introduce a minus sign.

In what follows *we treat the normed division algebra \mathbb{K} as an odd super vector space*. This turns out to force the spinor representations S_{\pm} to be odd and the vector representation V to be even, as follows.

There is an obvious notion of direct sums for super vector spaces, with

$$(V \oplus W)_0 = V_0 \oplus W_0, \quad (V \oplus W)_1 = V_1 \oplus W_1$$

and also an obvious notion of duals, with

$$(V^*)_0 = (V_0)^*, \quad (V^*)_1 = (V_1)^*.$$

We say a super vector space V is **even** if it equals its even part ($V = V_0$), and **odd** if it equals its odd part ($V = V_1$). Any subspace $U \subseteq V$ of an even (resp. odd) super vector space becomes a super vector space which is again even (resp. odd).

We treat the spinor representations S_{\pm} as super vector spaces using the fact that they are the direct sum of two copies of \mathbb{K} . Since \mathbb{K} is odd, so are S_+ and S_- . Since \mathbb{K}^2 is odd, so is its dual. This

in turn forces the space of linear maps from \mathbb{K}^2 to itself, $\text{End}(\mathbb{K}^2) = \mathbb{K}^2 \otimes (\mathbb{K}^2)^*$, to be even. This even space contains the 2×2 matrices $\mathbb{K}[2]$ as the subspace of maps realized by left multiplication:

$$\begin{aligned} \mathbb{K}[2] &\hookrightarrow \text{End}(\mathbb{K}^2) \\ A &\mapsto L_A. \end{aligned}$$

$\mathbb{K}[2]$ is thus even. Finally, this forces the subspace of hermitian 2×2 matrices, $\mathfrak{h}_2(\mathbb{K})$, to be even. So, the vector representation V is even. All this matches the usual rules in physics, where spinors are fermionic and vectors are bosonic.

6 Super-Yang–Mills Theory

We are now ready to give a division algebra interpretation of the pure super-Yang–Mills Lagrangian

$$L = -\frac{1}{4}\langle F, F \rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi \rangle$$

and use this to prove its supersymmetry. For simplicity, we shall work over Minkowski spacetime, M . This allows us to treat all bundles as trivial, sections as functions, and connections as \mathfrak{g} -valued 1-forms.

At the outset, we fix an invariant inner product on \mathfrak{g} , the Lie algebra of a semisimple Lie group G . We shall use the following standard tools from differential geometry to construct L , none of which need involve spinors or division algebra technology:

- A connection A on a principal G -bundle over M . Since the bundle is trivial we think of this connection as a \mathfrak{g} -valued 1-form.
- The exterior covariant derivative $d_A = d + [A, -]$ on \mathfrak{g} -valued p -forms.
- The curvature $F = dA + \frac{1}{2}[A, A]$, which is a \mathfrak{g} -valued 2-form.
- The usual pointwise inner product $\langle F, F \rangle$ on \mathfrak{g} -valued 2-forms, defined using the Minkowski metric on M and the invariant inner product on \mathfrak{g} .

We also need the following spinorial tools. Because spinors describe fermions, we assume S_+ and S_- are odd objects in SuperVect. So, whenever we switch two spinors, we introduce a minus sign.

- A \mathfrak{g} -valued section ψ of a spin bundle over M . Note that this is, in fact, just a function:

$$\psi: M \rightarrow S_{\pm} \otimes \mathfrak{g}.$$

We call the collection of all such functions $\Gamma(S_{\pm} \otimes \mathfrak{g})$.

- The covariant Dirac operator \not{D}_A derived from D_A . Of course,

$$\not{D}_A: \Gamma(S_{\pm} \otimes \mathfrak{g}) \rightarrow \Gamma(S_{\mp} \otimes \mathfrak{g})$$

and in fact,

$$\not{D}_A = \not{D} + A.$$

- A bilinear pairing

$$\langle -, - \rangle: \Gamma(S_+ \otimes \mathfrak{g}) \otimes \Gamma(S_- \otimes \mathfrak{g}) \rightarrow C^{\infty}(M)$$

built pointwise using our pairing

$$\langle -, - \rangle: S_+ \otimes S_- \rightarrow \mathbb{R}$$

and the invariant inner product on \mathfrak{g} .

The basic fields in our theory are a connection on a principal G -bundle, which we think of as a \mathfrak{g} -valued 1-form:

$$A: M \rightarrow V^* \otimes \mathfrak{g}.$$

and a \mathfrak{g} -valued spinor field, which we think of as a $S_+ \otimes \mathfrak{g}$ -valued function on M :

$$\psi: M \rightarrow S_+ \otimes \mathfrak{g}.$$

All our arguments would work just as well with S_- replacing S_+ .

To show that L is supersymmetric, we need to show δL is a total divergence when δ is the following supersymmetry transformation:

$$\begin{aligned} \delta A &= \epsilon \cdot \psi \\ \delta \psi &= \frac{1}{2} F \epsilon \end{aligned}$$

where ϵ is an arbitrary constant spinor field, treated as odd, but not \mathfrak{g} -valued. By a **supersymmetry transformation** we mean that computationally we treat δ as a derivation. So, it is linear:

$$\delta(\alpha f + \beta g) = \alpha \delta f + \beta \delta g$$

where $\alpha, \beta \in \mathbb{R}$, and it satisfies the product rule:

$$\delta(fg) = \delta(f)g + f\delta g.$$

For a more formal definition of ‘supersymmetry transformation’ see [6].

The above equations require further explanation. The dot in $\epsilon \cdot \psi$ denotes an operation that combines the spinor ϵ with the \mathfrak{g} -valued spinor ψ to produce a \mathfrak{g} -valued 1-form. We build this from our basic intertwiner

$$\cdot : S_+ \otimes S_+ \rightarrow V.$$

We identify V with V^* using the Minkowski inner product g , obtaining

$$\cdot : S_+ \otimes S_+ \rightarrow V^*.$$

Then we tensor both sides with \mathfrak{g} . This gives us a way to act by a spinor field on a \mathfrak{g} -valued spinor field to obtain a \mathfrak{g} -valued 1-form. We take the liberty of also denoting this with a dot:

$$\cdot : \Gamma(S_+) \otimes \Gamma(S_+ \otimes \mathfrak{g}) \rightarrow \Omega^1(M, \mathfrak{g}).$$

We also need to explain how the 2-form F acts on the constant spinor field ϵ . Using the Minkowski metric, we can identify differential forms on M with sections of the Clifford algebra bundle over M :

$$\Omega^*(M) \cong \text{Cliff}(M).$$

Using this, differential forms act on spinor fields. Tensoring with \mathfrak{g} , we obtain a way for \mathfrak{g} -valued differential forms like F to act on spinor fields like ϵ to give \mathfrak{g} -valued spinor fields like $F\epsilon$.

Let us now apply the supersymmetry transformation to each term in the Lagrangian. First, the bosonic term:

Proposition 12. *The bosonic term has:*

$$\delta \langle F, F \rangle = 2(-1)^{n+1} \langle \psi, (\star d_A \star F) \epsilon \rangle + \text{divergence}.$$

Proof. By the symmetry of the inner product, we get:

$$\delta\langle F, F \rangle = 2\langle F, \delta F \rangle.$$

Using the handy formula $\delta F = d_A \delta A$, we have:

$$\langle F, \delta F \rangle = \langle F, d_A \delta A \rangle.$$

Now the adjoint of the operator d_A is $\star d_A \star$, up to a pesky sign: if ν is a \mathfrak{g} -valued $(p-1)$ -form and μ is a \mathfrak{g} -valued p -form, we have

$$\langle \mu, d_A \nu \rangle = (-1)^{dp+d+1+s} \langle \star d_A \star \mu, \nu \rangle + \text{divergence}$$

where d is the dimension of spacetime and s is the signature, i.e., the number of minus signs in the diagonalized metric. It follows that

$$\langle F, \delta F \rangle = \langle F, d_A \delta A \rangle = (-1)^n \langle \star d_A \star F, \delta A \rangle + \text{divergence}$$

where n is the dimension of \mathbb{K} . By the definition of δA , we get

$$\langle \star d_A \star F, \delta A \rangle = \langle \star d_A \star F, \epsilon \cdot \psi \rangle.$$

Now we can use division algebra technology to show:

$$\langle \star d_A \star F, \epsilon \cdot \psi \rangle = \frac{1}{2} \text{Re tr} ((\star d_A \star F)(\epsilon \psi^\dagger + \psi \epsilon^\dagger)) = -\langle \psi, (\star d_A \star F) \epsilon \rangle,$$

using the cyclic property of the real trace in the last step, and introducing a minus sign in accordance with the sign rule. Putting everything together, we obtain the desired result. \square

Even though this proposition involved the bosonic term only, division algebra technology was still a useful tool in its proof. This is even more true in the next proposition, which deals with the fermionic term:

Proposition 13. *The fermionic term has:*

$$\delta\langle \psi, \not{D}_A \psi \rangle = \langle \psi, \not{D}_A (F \epsilon) \rangle + \text{tri } \psi + \text{divergence}$$

where

$$\text{tri } \psi = \langle \psi, (\epsilon \cdot \psi) \psi \rangle.$$

Proof. It is easy to compute:

$$\delta\langle \psi, \not{D}_A \psi \rangle = \langle \delta \psi, \not{D}_A \psi \rangle + \langle \psi, \delta \not{D}_A \psi \rangle + \langle \psi, \not{D}_A \delta \psi \rangle.$$

Now we insert $\delta \not{D}_A = \delta A = \epsilon \cdot \psi$, and thus see that the penultimate term is the trilinear one:

$$\text{tri } \psi = \langle \psi, (\epsilon \cdot \psi) \psi \rangle.$$

So, let us concern ourselves with the remaining terms:

$$\langle \delta \psi, \not{D}_A \psi \rangle + \langle \psi, \not{D}_A \delta \psi \rangle.$$

A computation using the product rule shows that the divergence of the 1-form $\psi \cdot \phi$ is given by $-\langle \phi, \not{D}_A \psi \rangle + \langle \psi, \not{D}_A \phi \rangle$, where the minus sign on the first term arises from using the sign rule with these odd spinors. In the terms under consideration, we can use this identity to move \not{D}_A onto $\delta \psi$:

$$\langle \delta \psi, \not{D}_A \psi \rangle + \langle \psi, \not{D}_A \delta \psi \rangle = 2\langle \psi, \not{D}_A \delta \psi \rangle + \text{divergence}.$$

Substituting $\delta \psi = \frac{1}{2} F \epsilon$, we obtain the desired result. \square

Using these two propositions, it is immediate that

$$\begin{aligned}\delta L &= -\frac{1}{4}\delta\langle F, F\rangle + \frac{1}{2}\delta\langle\psi, \mathbb{D}_A\psi\rangle \\ &= \frac{1}{2}(-1)^n\langle\psi, (\star d_A\star F)\epsilon\rangle + \frac{1}{2}\langle\psi, \mathbb{D}_A(F\epsilon)\rangle + \frac{1}{2}\text{tri}\psi + \text{divergence}\end{aligned}$$

All that remains to show is that $\mathbb{D}_A(F\epsilon) = (-1)^{n+1}(\star d_A\star F)\epsilon$. Indeed, Snygg shows (Eq. 7.6 in [15]) that for an ordinary, non- \mathfrak{g} -valued p -form F

$$\mathcal{D}(F\epsilon) = (dF)\epsilon + (-1)^{d+dp+s}(\star d\star F)\epsilon$$

where d is the dimension of spacetime and s is the signature. This is easily generalized to covariant derivatives and \mathfrak{g} -valued p -forms:

$$\mathbb{D}_A(F\epsilon) = (d_A F)\epsilon + (-1)^{d+dp+s}(\star d_A\star F)\epsilon.$$

In particular, when F is the curvature 2-form, the first term vanishes by the Bianchi identity $d_A F = 0$, and we are left with:

$$\mathbb{D}_A(F\epsilon) = (-1)^{n+1}(\star d_A\star F)\epsilon$$

where n is the dimension of \mathbb{K} . We have thus shown:

Proposition 14. *Under supersymmetry transformations, the Lagrangian L has:*

$$\delta L = \frac{1}{2}\text{tri}\psi + \text{divergence}.$$

The above result actually holds in every dimension, though our proof used division algebras and was thus adapted to the dimensions of interest: 3, 4, 6, and 10. The next result is where division algebra technology becomes really crucial:

Proposition 15. *For Minkowski spacetimes of dimensions 3, 4, 6, and 10, $\text{tri}\psi = 0$.*

Proof. At each point, we can write

$$\psi = \sum \psi^a \otimes g_a,$$

where $\psi^a \in S_+$ and $g_a \in \mathfrak{g}$. When we insert this into $\text{tri}\psi$, we see that

$$\text{tri}\psi = \sum \langle\psi^a, (\epsilon \cdot \psi^b)\psi^c\rangle \langle g_a, [g_b, g_c]\rangle.$$

Since $\langle g_a, [g_b, g_c]\rangle$ is totally antisymmetric, this implies $\text{tri}\psi = 0$ for all ϵ if and only if the part of $\langle\psi^a, (\epsilon \cdot \psi^b)\psi^c\rangle$ that is antisymmetric in a, b and c vanishes for all ϵ . Yet these spinors are odd; for even spinors, we require the part of $\langle\psi^a, (\epsilon \cdot \psi^b)\psi^c\rangle$ that is *symmetric* in a, b and c to vanish for all ϵ .

Now let us bring in some division algebra technology to remove our dependence on ϵ . While we do this, let us replace ψ^a with ψ , ψ^b with ϕ , and ψ^c with χ to lessen the clutter of indices. Substituting in the formulas from Table 1, we have

$$\begin{aligned}\langle\psi, (\epsilon \cdot \phi)\chi\rangle &= \text{Re}(\psi^\dagger(\widetilde{\epsilon\phi^\dagger + \phi\epsilon^\dagger})\chi) \\ &= \text{Re}\text{tr}(\psi^\dagger(\epsilon\phi^\dagger + \phi\epsilon^\dagger - \epsilon^\dagger\phi - \phi^\dagger\epsilon)\chi) \\ &= \langle\epsilon, (\psi \cdot \chi)\phi\rangle,\end{aligned}$$

where again we have employed the cyclic symmetry of the real trace, along with the identity:

$$\text{tr}(\epsilon\phi^\dagger + \phi\epsilon^\dagger) = \text{Re}\text{tr}(\epsilon\phi^\dagger + \phi\epsilon^\dagger) = \phi^\dagger\epsilon + \epsilon^\dagger\phi.$$

This real quantity commutes and associates in any expression. So, if we seek to show that the part of $\langle\psi, (\epsilon \cdot \phi)\chi\rangle$ that is totally symmetric in ψ, ϕ and χ vanishes for all ϵ , it is equivalent to show the totally symmetric part of $(\phi \cdot \chi)\psi$ vanishes. And since the dot operation in $\phi \cdot \chi$ is symmetric, this follows immediately from our main result, Theorem 11. \square

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