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Foundations of Categorified Representation Theory

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The text of this dissertation, in part, is a reprint of the material as it appears in *Groupoidification Made Easy* coauthored with John Baez, Christopher Walker and the author, *Higher Dimensional Algebra VII: Groupoidification* coauthored with John Baez, Christopher Walker and the author, and to be published in *Theory and Applications of Categories*, 2010, and *The Hecke Bicategory* authored solely by the author. The co-authors John Baez and Christopher Walker listed above are collaborators in part of the work that forms the basis for this dissertation. The collaborators James Dolan and Todd Trimble have been integral in forming and sharing integral ideas that, in part, form the basis for this dissertation. The collaborator John Baez has supervised this work in its entirety.

To Vera for years of patient love and support. To my parents for getting me through all of the tougher times. To each of my grandparents, especially my Nana.

ABSTRACT OF THE DISSERTATION

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This thesis develops the foundations of the program of groupoidification and presents an application of this program — the Fundamental Theorem of Hecke Operators. In stating this theorem, we develop a theory of enriched bicategories and construct the Hecke bicategory — a categorification of the intertwining operators between permutation representations of a finite group. As an immediate corollary, we obtain a categorification of the Iwahori-Hecke algebra, which leads to solutions of the Zamolodchikov tetrahedron equation. Such solutions are a positive step towards invariants of 2-tangles in 4-dimensional space and constructions of higher-categories with braided structures.

Contents

Chapter 1

Introduction

1.1 Big Picture

There is an inherent iterative nature of category theory that makes the study of higher category theory inevitable. Thus, the beginnings of higher category theory were present at the birth of category theory. Higher structures in algebra and geometry have come from a myriad of sources, and eventually the desire to formalize the landscape of such structures led to Crane and Frenkel's conjectures about quantum groups and topological quantum field theories [15]. Early ideas on *categorification* took hold, and *categorified representation theory* has grown legs and become an important subject at the intersection of geometry, low-dimensional topology, higher-category theory, representation theory and theoretical physics.

This dissertation provides the foundations of a program of categorification initiated in private discussion, in the classroom, and on the internet [4] by Baez, Dolan and Trimble, and described formally by Baez, Walker, and the author in [5, 6]. This program draws on ideas from geometric representation theory, physics, and higher-category theory. As usual, some of the ideas have been introduced in other contexts and other ideas are unique to this work. We hope that the interaction with well-studied concepts will contribute to the continued development of an interesting and long-term program. Replacing linear operators between vector spaces with spans of groupoids this theory of categorification has been developed under the name of 'groupoidification'.

1.2 A Brief Introduction

The program of *groupoidification* is aimed at categorifying various notions from representation theory and mathematical physics. The very simple idea is that one should replace vector spaces by groupoids, i.e., categories with only isomorphisms, and replace linear operators by spans of groupoids. In fact, we define a systematic process called *degroupoidification*:

$$\text{groupoids} \to \text{vector spaces}$$

$$\text{spans of groupoids} \to \text{matrices}$$

In *Higher Dimensional Algebra VII* [6], we suggested some applications of groupoidification to Hall algebras, Hecke algebras, and Feynman diagrams, so that other researchers could begin to categorify *their* favorite notions from representation theory. The present work develops applications related to Hecke algebras.

In this dissertation, we describe *categorified intertwining operators* for representations of a very basic type: the *permutation representations* of a finite group. There is an easy way to categorify the theory of intertwining operators between such representations. Unfortunately, the construction is, in some sense, unnatural. Groupoidification offers a more natural construction. Much of this paper is devoted to explaining how to recover the permutation representations from this categorified structure, and further, to point out how categorified Hecke algebras follow directly from such a construction.

The first tool of representation theory is linear algebra. Vector spaces and linear operators have nice properties, which allow representation theorists to extract a great deal of information about algebraic gadgets ranging from finite groups to Lie groups to Lie algebras and their various relatives and generalizations. We start at the beginning, considering the representation theory of finite groups. Noting the utility of linear algebra in representation theory, this dissertation is fundamentally based on the idea that the heavy dependence of linear algebra on fields, very often the complex numbers, obscures the combinatorial skeleton of the subject. Then, we hope that by peeling back the soft tissue of the continuum, we may expose and examine the bones, revealing new truths by working directly with the underlying combinatorics.

1.3 A Guide to this Work

The present work is composed of two main parts plus an appendix on monoidal bicategories for reference and a bibliography.

Part one, *Groupoidification and Hecke Algebras*, is an expository account of the central problem of this thesis and is given in three chapters. Chapter 2, *Matrices, Spans, and Decategorification*, discusses the relationship between matrices and spans of sets. In particular, we introduce a bicategory of spans of sets as a possible categorification of the intertwining operators between permutation representations of a finite group G. Chapter 3, *Categorified Hecke Operators*, describes the program of groupoidification, enriched bicategories, and a bicategory enriched over the monoidal bicategory of spans of groupoids, which is a natural categorification of intertwining operators and Hecke algebras. Chapter 4, *Applications to Representation Theory and Knot Theory*, describes some applications of the program of groupoidification. In particular, we suggest applications to knot theory in the form of braided monoidal bicategories following from the existence of solutions to the Zamolodchikov tetrahedron equation. There are no proofs in this part of the work.

Part two, *Definitions and Theorems*, provides the background work, precise statements of theorems, and proofs as a rigorous foundation for many of the ideas sketched in the first part. As many of the theorems within involve structures from higher category theory, we attempt to detail the structural morphisms in each proof, but leave the checking of axioms to the reader as these reduce in almost all cases to simple exercises in linear algebra, group theory, or elementary symbol manipulation. Chapter 5, *General Definitions and Theorems*, provides basic background facts about the theory of groupoids and describes the monoidal bicategory of spans of groupoids using the weak pullback construction. Chapter 6, *Degroupoidification*, describes the theory of groupoidification following the work of Baez, Hoffnung, and Walker in [6]. In particular, the monoidal functor, degroupoidification, is described in detail. For computational use in proving some theorems we describe an alternative approach to degroupoidification via a pull-push method. Chapter 7, *Enriched Bicategories*, provides the complete definition of a $\mathcal{V}$-enriched bicategory, for a monoidal bicategory $\mathcal{V}$. This is followed by a change of base theorem, which is used in the following chapter, and we be employed heavily in future work. Chapter 8, *Proof of Fundamental Theorem*, is a detailed statement and proof of the Fundamental Theorem of Hecke Operators along with other major theorems in this thesis. In particular, we discuss and prove a corollary regarding categorified Hecke algebras.

Part three, *Appendix and References*, details the structure of monoidal bicategories following [20, 21, 31]. Finally, a list of relevant papers and books is given in the references.

Part I

First Part: Groupoidification and Hecke Algebras

Chapter 2

Matrices, Spans, and Decategorification

We recall the objects of study and describe the central problem.

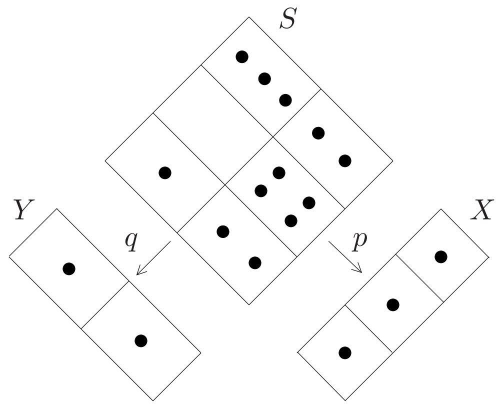
2.1 Spans as Matrices

In this section, we consider the notion of spans of sets, a very simple idea, which is at the heart of categorified representation theory. A **span of sets** from X to Y is a pair of functions with a common domain, like so:

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

We will often denote a span by its apex, when no confusion is likely to arise, or as (p, S, q) when necessary.

A span of sets can be viewed as a a matrix of sets:



For each pair (x, y), we have a set $S_{x,y} = p^{-1}(x) \cap q^{-1}(y)$. In particular, if all the sets $S_{x,y}$ are finite, this can be 'decategorified' to a matrix of natural numbers $|S_{x,y}|$ — a very familiar object in linear algebra. In this sense, a span is a 'categorification' of a matrix. We will consider only *finite* sets throughout this dissertation.

Even better than spans giving rise to matrices, composition of spans gives rise to matrix multiplication. Given a pair of composable spans:

$$\begin{tikzcd}
& T \arrow[dl] \arrow[dr, "q"] & & S \arrow[dl, "p"'] \arrow[dr] & \\
Z & & Y & & X
\end{tikzcd}$$

the composite is the **pullback** of the pair of functions $p\colon S \to Y$ and $q\colon T \to Y$, which is a new span:

$$\begin{tikzcd}
& & TS \arrow[dl] \arrow[dr] & & \\
& T \arrow[dl] \arrow[dr, "q"] & & S \arrow[dl, "p"'] \arrow[dr] & \\
Z & & Y & & X
\end{tikzcd}$$

where TS is the subset of $T \times S$:

$$\{(t, s) \subseteq T \times S \mid p(s) = q(t)\},$$

with the obvious projections to S and T. It is straightforward to check that this process agrees with matrix multiplication after decategorifying.

While we have not defined the notions of categorification and decategorification explicitly, we have been hinting at their role in the relationship between spans of finite sets and matrices of natural numbers. The reason for skirting the definitions is that the notion of 'categorification' is simply a heuristic tool allowing us to 'undo' the process of decategorification. Thus, in the above example, we turn spans of finite sets into matrices of natural numbers simply by counting the number of elements in each set $S_{x,y}$. We note that there is a standard basis of the vector space of linear maps, and each basis element can be realized as a span of finite sets. Thus, we can recover the entire vector space of linear maps by constructing the free vector space on this basis. Further, we can turn a set X into a vector space by constructing the free vector space with basis X. Checking that composition of spans and matrix multiplication agree after taking the cardinality of the set-valued entries is the main step in showing that our decategorification process — spans of sets to linear operators — is *functorial*.

Since we are interested in the relationship between spans and matrices, we expect that a good decategorification process should be 'additive' in some suitable manner. In particular, we should say how to add 'categorified linear operators', or spans. To define *addition of spans*, we consider a pair of spans from a set X to a set Y:

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

We can define the **sum** of S and T as the span:

$$\begin{tikzcd} & S \sqcup T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

where the legs of $S \sqcup T$ are induced by the universal property of the coproduct.

Using this notion of addition, we can write down a 'categorified basis' of spans of finite sets from X to Y — that is, a set of spans whose corresponding matrices span the vector space of linear operators from the free vector space with basis X to the free vector space with basis Y. These *categorified basis vectors* are the spans:

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

where there is one span corresponding to each element $(x, y) \in X \times Y$. These are the **irreducible spans** — those that cannot be written as the sum of two 'non-trivial spans'. A **non-trivial span** is a span whose apex is the empty set. Colloquially we say that *spans of finite sets categorify linear operators between finite dimensional vector spaces.*

2.2 Permutation Representations

Again we start with a very simple idea. We want to study the actions of a finite group G on finite sets — *finite G-sets*. These extend to *permutation representations of G*. We fix the field of real numbers and consider only real representations throughout this paper.

Definition 1. *A* **permutation representation** *of a finite group G is a finite-dimensional representation of G together with a chosen basis such that the action of G maps basis vectors to basis vectors.*

Thus, finite G-sets can be linearized to obtain permutation representations of G. In fact, we have described a relationship between the objects of the *category* of finite G-sets and the objects of the *category* of permutation representations of G. Given a finite group G, the category $G\mathrm{Set}$ of finite G-sets has:

- finite G-sets as objects,
- G-equivariant functions as morphisms,

and the category $\mathrm{PermRep}(G)$ of permutation representations has:

- permutation representations of G as objects,
- intertwining operators as morphisms.

Definition 2. *An* **intertwining operator** $f\colon V \to W$ *between permutation representations of a finite group G is a linear operator from V to W that is G-equivariant, i.e., commutes with the actions of G.*

The main goal of this paper is to categorify the very special algebras of Hecke operators called the *Iwahori-Hecke algebras* [13, 34]. Of course, an algebra is a Vect-enriched category with exactly one object, and the Hecke algebras are isomorphic to certain one-object subcategories of the Vect-enriched category of permutation representations. Thus, we consider the morphisms of the category $\mathrm{PermRep}(G)$ to be *Hecke operators* and refer to the category as the *Hecke algebroid* — a many-object generalization of the Hecke algebra. We will construct a bicategory — or more precisely, an *enriched bicategory* — called the *Hecke bicategory* that categorifies the Hecke algebroid for any finite group G.

There is a functor from finite G-sets to permutation representations of G. As stated above, the maps between G-sets are G-equivariant functions — that is, functions between G-sets X and Y that respect the actions of G. Such a function $f\colon X \to Y$ gives rise to a G-equivariant linear map (or intertwining operator) $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$. However, there are many more intertwining operators from $\widetilde{X}$ to $\widetilde{Y}$ than there are G-equivariant maps from X to Y. In particular, the former is a vector space over the real numbers, while the latter is a finite set. For example, an intertwining operator $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ may take a basis vector $x \in \widetilde{X}$ to any $\mathbb{R}$-linear combination of basis vectors in $\widetilde{Y}$, whereas a map of G-sets does not have the freedom of scaling or adding basis elements.

So, in the language of category theory the process of linearizing finite G-sets to obtain permutation representations is a faithful, essentially surjective functor, which is not at all full.

2.3 Spans of G-Sets

In the previous section, we discussed the relationship between finite G-sets and permutation representations. In Section 2.1, we saw a basis for the vector space of linear operators between the free vector spaces on a pair of finite sets X and Y coming from spans between X and Y. Thinking of finite sets as G-sets with a trivial action of G suggests that we can generalize this story to obtain a basis for the vector space of intertwining operators between permutation representations from *spans of finite G-sets*.

A **span of finite G-sets** from a finite G-set X to a finite G-set Y is a pair of maps with a common domain like so:

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

where S is a finite G-set, and p and q are G-equivariant maps.

Since the category of finite G-sets and G-equivariant maps has a coproduct, we can define addition of spans of finite G-sets as we did for finite sets in Section 2.1. Further, we obtain a basis of intertwining operators from $\widetilde{X}$ to $\widetilde{Y}$ as spans of G-sets from X to Y. These *categorified basis vectors* are spans:

$$\begin{tikzcd}
 & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Y & & X
\end{tikzcd}$$

where $S \subseteq X \times Y$ is the G-orbit of some point in $X \times Y$, and p and q are the obvious projections.

At this point, we see the first hints of the existence of a categorified Hecke algebroid. Having found a basis coming from spans of finite G-sets is promising because spans of sets — and, similarly G-sets — naturally form a bicategory.

2.4 Bicategories of Spans

The development of bicategories by Benabou [10] is an early example of categorification. A (small) category consists of a *set of objects* and a *set of morphisms*. A bicategory is a categorification of this concept, so there is a new layer of structure [31]. In particular, a (small) bicategory $\mathcal{B}$ consists of:

- a set of objects $x, y, z \ldots$,
- for each pair of objects a set of morphisms,
- for each pair of morphisms a set of 2-morphisms,

and given any pair of objects x, y, there is a hom-category $\hom(x, y)$ which has:

- 1-morphisms $x \to y$ of $\mathcal{B}$ as objects,
- 2-morphisms:

$$\begin{tikzcd}
x \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & \bullet\, y
\end{tikzcd}$$

of $\mathcal{B}$ as morphisms. There is a *vertical composition* of 2-morphisms,

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] & \bullet
\end{tikzcd}$$

as well as a *horizontal composition*,

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

and these are required to satisfy certain coherence axioms, which make these operations simultaneously well-defined. See Appendix 9 for complete definitions of bicategories and maps of bicategories.

Benabou's definition followed from several important examples of bicategories, which he presented in [10], and which are very familiar in categorified and geometric representation theory. The first example is the bicategory of spans of sets, which has:

- sets as objects,
- spans of sets as morphisms,
- maps of spans of sets as 2-morphisms.

We defined spans of sets in Section 2.1. A **map of spans of sets** from a span S to a span T is a function $f\colon S \to T$ such that the following diagram commutes:

$$\begin{tikzcd}
& S \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\
Y & & X \\
& T \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}$$

For each finite group G, there is a closely related bicategory Span(GSet), which has:

- finite G-sets as objects,
- spans of finite G-sets as morphisms,
- maps of spans of finite G-sets as 2-morphisms.

The definitions are the same as in the bicategory of spans of sets, except for the obvious finiteness condition and that every arrow should be made G-equivariant.

This bicategory seems to be a good candidate for a categorification of PermRep(G). In the next section, we define a process of decategorification, and see that we do *not* recover the Hecke algebroid. Of course, that is not the end of the story. This requires the development of groupoidification.

2.5 Decategorification

In this section, we describe a functor from the bicategory of spans of G-sets to the category of permutation representations of G.

Consider the bicategory of spans of finite G-sets from the previous section. We have seen that such spans can be interpreted as categorified intertwining operators between permutation representations, i.e., matrices whose entries are sets. However, while counting the number of elements of these sets produces a matrix with natural number entries, we have not specified a decategorification process, which takes the bicategory of spans of finite G-sets to the category $\mathrm{PermRep}(G)$. Our goal is to obtain the entire category of permutation representations.

Let us propose such a process and see what goes wrong when we apply it to $\mathrm{Span}(G\mathrm{Set})$. Since the bicategory of spans has a coproduct it is natural to apply a functor, which one might call an 'additive Grothendieck construction' in analogy with the usual split Grothendieck group construction on abelian categories. This is a functor:

$$\mathcal{L}\colon \mathrm{Span}(G\mathrm{Set}) \to \mathrm{PermRep}(G)$$

We note that the objects of the bicategory are finite G-sets. Thus, the functor only needs to linearize these to obtain permutation representations of G as described in Section 2.3:

$$X \mapsto \widetilde{X}$$

The interesting part of this process is turning the hom-categories consisting of spans of finite G-sets and maps between these spans into vector spaces. To do this, we take the free vector space on the set of matrices corresponding to isomorphism classes of *irreducible spans* — those spans which cannot be written as a coproduct of two non-trivial spans.

However, there are many more isomorphism classes of irreducible spans of G-sets from X to Y than needed to span the space of intertwining operators between the permutation representations $\widetilde{X}$ and $\widetilde{Y}$. This is most obvious if we take $X = Y = 1$ and take G to be non-trivial. Then the space of intertwining operators is 1-dimensional, but there are many more isomorphism classes of irreducible spans of G-sets from X to Y. Such a span has an apex that is a G-set with just a single orbit.

Fortunately, the problem is clear. Given spans of finite G-sets S and T from X to Y, we say S and T are the same as matrices if they are the same in each matrix entry, i.e., in each fiber over a pair $(x, y) \in X \times Y$. This is true precisely when there is a bijection $f: S \to T$ making the following diagram commute:

$$\begin{tikzcd}
& S \arrow[dl] \arrow[dr] \arrow[dd, "f"] & \\
Y & & X \\
& T \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The problem arises when we consider only the G-equivariant maps from S to T. In the next section, we see that relaxing this requirement solves this problem.

2.6 The Hecke Bicategory — Take One

This section introduces the Hecke bicategory as a bicategory of spans of finite G-sets. We alter the bicategory Span(GSet) by considering a larger class of 2-morphisms. We then show that extending the decategorification functor $\mathcal{L}$ introduced in the last section to this new bicategory, we obtain precisely the Hecke algebroid as its image. Unfortunately, this leaves us with a less than desirable solution, which we now describe.

We consider the bicategory $\mathrm{Span}^*(G\mathrm{Set})$ consisting of:

- finite G-sets as objects,
- spans of finite G-sets as morphisms,
- *not necessarily G-equivariant* maps of spans as 2-morphisms.

The raised asterisk is there to remind us of the new description of the 2-morphisms. Of course, our decategorification functor $\mathcal{L}$ can be applied equally well to any bicategory of spans of finite sets. Thus, we have, for each finite group G, the functor:

$$\mathcal{L}: \mathrm{Span}^*(G\mathrm{Set}) \to \mathrm{PermRep}(G)$$

The bicategory $\mathrm{Span}^*(G\mathrm{Set})$ categorifies the Hecke algebroid $\mathrm{PermRep}(G)$, or more precisely:

Claim 3. *Given a finite group G,*

$$\mathcal{L}\left(\mathrm{Span}^*(G\mathrm{Set})\right) \simeq \mathrm{PermRep}(G)$$

as Vect*-enriched categories.*

Unfortunately, the use of *not-necessarily equivariant* maps of G-sets makes this construction appear artificial. One goal of this paper is to solve this problem by giving a more natural description of the categorified Hecke algebroid. We do this in Section 8.1.

Chapter 3

Categorified Hecke Operators

We summarize the new concepts presented in this thesis while sketching the solution to the central problem. The following sections introduce the necessary machinery to present a more natural description of the Hecke bicategory. Enriched bicategories are described for use in Section 8.2 to construct the Hecke bicategory and state the Fundamental Theorem of Hecke Operators.

3.1 Action Groupoids and Groupoid Cardinality

In this section, we draw a connection between G-sets and groupoids via the 'action groupoid' construction. We then recall *groupoid cardinality* [3], which makes this connection explicit. Groupoid cardinality is discussed in greater detail in Section 6.1.

For any G-set, there exists a corresponding groupoid, called the *action groupoid*, *transformation groupoid*, or *weak quotient*:

Definition 4. *Given a group G and a G-set X, the* **action groupoid** $X//G$ *is the category which has:*

- *elements of X as objects,*
- *pairs $(g, x) \in G \times X$ as morphisms $(g, x)\colon x \to x'$, where $g \cdot x = x'$.*

Composition of morphisms is defined by the product on G.

Of course, associativity follows from associativity in G and the construction defines a groupoid since any morphism $(g, x)\colon x \to x'$ has an inverse $(g^{-1}, x')\colon x' \to x$.

So every finite G-set defines a groupoid, and we will see in Section 3.4 that the weak quotient of G-sets plays an important role in understanding categorified permutation representations.

Next, we recall the definition of groupoid cardinality [3]:

Definition 5. *Given a (small) groupoid $\mathcal{G}$, its* **groupoid cardinality** *is defined as:*

$$|\mathcal{G}| = \sum_{\text{isomorphism classes of objects [x]}} \frac{1}{|\mathrm{Aut}(x)|}$$

If this sum diverges, we say $|\mathcal{G}| = \infty$.

In this paper, we will only consider *finite groupoids* — groupoids with a finite set of objects and finite set of morphisms. In general, we could allow groupoids with infinitely many isomorphism classes of objects, and the cardinality of a groupoid would take values in the non-negative real numbers when the sum converges. Generalized cardinalities have been studied by a number of authors [18, 30, 32, 41].

Groupoid cardinality makes explicit the relationship between a G-set and the corresponding action groupoid. In particular, we have the following equation:

$$|X//G| = |X|/|G|$$

whenever G is a finite group acting on a finite set X. Using ordinary set cardinality, the equality:

$$|X/G| = |X|/|G|$$

fails to hold unless the group action is free.

Weakening the quotient $(X \times Y)/G$, we obtain the action groupoid $(X \times Y)//G$, which will be central in categorifying the permutation representation category of a finite group G. In the next section, we define degroupoidification using the notion of groupoid cardinality.

3.2 Degroupoidification

In this section, we recall some of the main ideas of groupoidification. Of course, in practice this means we will discuss the corresponding process of decategorification — the degroupoidification functor.

To define degroupoidification in [6], we considered a functor from the category of spans of groupoids to the category of linear operators between vector spaces. In the present setting, we will need to extend degroupoidification to a functor between bicategories.

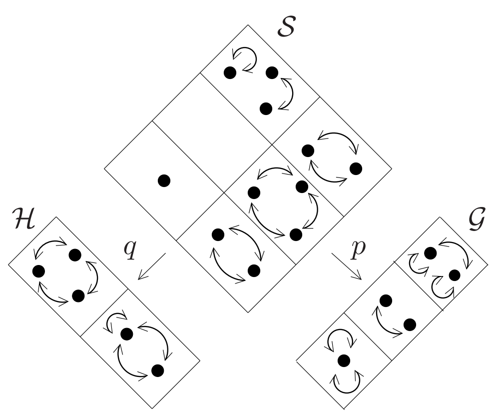
We extend the functor to a bicategory Span, which has:

- (finite) groupoids as objects;
- spans of (finite) groupoids as 1-morphisms;
- 'isomorphism classes of equivalences' of spans of (finite) groupoids as 2-morphisms.

Since all groupoids that show up in this paper arise from the action groupoid construction on finite G-sets, there is no problem restricting our attention to finite groupoids.

Arbitrary spans of groupoids form a tricategory, which has not only *maps of spans* as 2-morphisms, but also *maps of maps of spans* as 3-morphisms. Thus, it takes some work to restrict this structure to a bicategory. While there are more sophisticated ways of obtaining a bicategory such as requiring that the legs of our spans be fibrations, we do so by taking *isomorphism classes of equivalences of spans* as 2-morphisms. Equivalences of spans and isomorphisms between these will be defined in Section 5.3.

Spans of finite groupoids are categorified matrices of non-negative rational numbers in the same way that spans of finite sets are categorified matrices of natural numbers. A *span of groupoids* is a pair of functors with common domain, and we can picture one of these roughly as follows:



Whereas one uses set cardinality to realize spans of sets as matrices, we can use groupoid cardinality to obtain a matrix from a span of groupoids.

We have seen evidence that a span of groupoids is a categorified matrix, so a groupoid must be a categorified vector space. To make these notions precise, we define a monoidal functor — the degroupoidification functor:

$$\mathcal{D}\colon \mathrm{Span} \to \mathrm{Vect},$$

as follows. Given a groupoid $\mathcal{G}$, we obtain a vector space $\mathcal{D}(\mathcal{G})$, called the **degroupoidification** of $\mathcal{G}$, by taking the free vector space on the set of isomorphism classes of objects of $\mathcal{G}$.

We say a groupoid $\mathcal{V}$ *over* a groupoid $\mathcal{G}$:

$$\begin{tikzcd} \mathcal{V} \arrow[d, "p"] \\ \mathcal{G} \end{tikzcd}$$

is a **groupoidified vector**. In particular, from the functor p we can produce a vector in $\mathcal{D}(\mathcal{G})$ in the following way.

The **full inverse image** of an object x in $\mathcal{G}$ is the groupoid $p^{-1}(x)$, which has:

- objects v of $\mathcal{V}$, such that $p(v) \cong x$, as objects,
- morphisms $v \to v'$ in $\mathcal{V}$ as morphisms.

We note that this construction depends only on the isomorphism class of x. Since the set of isomorphism classes of $\mathcal{G}$ determine a basis of the corresponding vector space, the vector determined by p can be defined as:

$$\sum_{\text{isomorphism classes of objects [x]}} |p^{-1}(x)|[x],$$

where $|p^{-1}(x)|$ is the groupoid cardinality of $p^{-1}(x)$. We note that a 'groupoidified basis' can be obtained in this way as a set of functors from the terminal groupoid $\mathbf{1}$ to a representative object of each isomorphism class of $\mathcal{G}$. A **groupoidified basis** of $\mathcal{G}$ is a set of groupoids $\mathcal{V} \to \mathcal{G}$ over $\mathcal{G}$ such that the corresponding vectors give a basis of the vector space $\mathcal{D}(\mathcal{G})$.

Given a span of groupoids,

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

we want to produce a linear map $\mathcal{D}(\mathcal{S})\colon \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{H})$. The details are checked in Section 6.1. Here we show only that given a basis vector of $\mathcal{D}(\mathcal{G})$, the span $\mathcal{S}$ determines a vector in $\mathcal{D}(\mathcal{H})$. To do this, we need the notion of the weak pullback of groupoids — a categorified version of the pullback of sets.

Given a diagram of groupoids:

$$\begin{tikzcd}
\mathcal{H} \arrow[dr, "q"] & & \mathcal{G} \arrow[dl, "p"'] \\
& \mathcal{K} &
\end{tikzcd}$$

the **weak pullback** of $p\colon \mathcal{G} \to \mathcal{K}$ and $q\colon \mathcal{H} \to \mathcal{K}$ is the diagram:

$$\begin{tikzcd}
& \mathcal{H}\mathcal{G} \arrow[dl] \arrow[dr] & \\
\mathcal{H} \arrow[dr, "q"] & & \mathcal{G} \arrow[dl, "p"'] \\
& \mathcal{K} &
\end{tikzcd}$$

where $\mathcal{H}\mathcal{G}$ is a groupoid whose objects are triples (h, g, α) consisting of an object $h \in \mathcal{H}$, an object $g \in \mathcal{G}$, and an isomorphism $\alpha\colon p(g) \to q(h)$ in $\mathcal{K}$. A morphism in $\mathcal{H}\mathcal{G}$ from (h, g, α) to (h', g', α') consists of a morphism $f\colon g \to g'$ in $\mathcal{G}$ and a morphism $f'\colon h \to h'$ in $\mathcal{H}$ such that the following square commutes:

$$\begin{tikzcd}
p(g) \arrow[r, "\alpha"] \arrow[d, "p(f)"'] & q(h) \arrow[d, "q(f')"] \\
p(g') \arrow[r, "\alpha'"'] & q(h')
\end{tikzcd}$$

As in the case of the pullback of sets, the maps out of $\mathcal{H}\mathcal{G}$ are the obvious projections. Further, this construction should satisfy a certain universal property, which we describe in Section 5.3.

Now, given our span and a chosen *groupoidified basis vector*:

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl] \arrow[dr] & & \mathbf{1} \arrow[dl] \\
\mathcal{H} & & \mathcal{G} &
\end{tikzcd}$$

we obtain a groupoid over $\mathcal{H}$ by constructing the weak pullback:

$$\begin{tikzcd}
& & \mathcal{S}\mathbf{1} \arrow[dl] \arrow[dr] & \\
& \mathcal{S} \arrow[dl] \arrow[dr] & & \mathbf{1} \arrow[dl] \\
\mathcal{H} & & \mathcal{G} &
\end{tikzcd}$$

Now, $\mathcal{S}\mathbf{1}$ is a groupoid over $\mathcal{H}$, and we can compute the resulting vector. In general, to guarantee that this process defines a linear operator, we need to restrict to the so-called 'tame' spans defined in [6]. However, spans of finite groupoids are automatically tame, so we can safely ignore this issue.

One can check that this process defines a linear operator from a span of groupoids, and, further, that this process is functorial. This is done in Section refprocess and Section 6.3.2, respectively. This is the degroupoidification functor. In Section 6.2, we check that equivalent spans are sent to the same linear operators. It is then straightforward to extend this to our bicategory of spans of groupoids by adding identity 2-morphisms to the category of vector spaces and sending all 2-morphisms between spans of groupoids to the corresponding identity 2-morphism.

In the next section, we define a notion of *enriched bicategories*. We will see that constructing an enriched bicategory depends heavily on having a monoidal bicategory in hand. The bicategory Span defined above is, in fact, a monoidal bicategory — that is, Span has a tensor product, which is a functor

$$\otimes \colon \mathrm{Span} \times \mathrm{Span} \to \mathrm{Span},$$

along with further structure and satisfying some coherence relations.

We describe the main components of the tensor product on Span. Given a pair of groupoids $\mathcal{G}, \mathcal{H}$, the tensor product $\mathcal{G} \times \mathcal{H}$ is the product in Cat. Further, for each pair of pairs of groupoids $(\mathcal{G}, \mathcal{H}), (\mathcal{J}, \mathcal{K})$ there is a functor:

$$\otimes \colon \mathrm{Span}(\mathcal{G}, \mathcal{H}) \times \mathrm{Span}(\mathcal{J}, \mathcal{K}) \to \mathrm{Span}(\mathcal{G} \times \mathcal{J}, \mathcal{H} \times \mathcal{K}),$$

defined as follows:

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & & & \mathcal{T} \arrow[dl, "v"'] \arrow[dr, "u"] & & & \mathcal{S}\times\mathcal{T} \arrow[dl, "q\times v"'] \arrow[dr, "p\times u"] & \\
\mathcal{H} & & \mathcal{G}\ , & \mathcal{K} & & \mathcal{J} \arrow[r, mapsto] & \mathcal{H}\times\mathcal{K} & & \mathcal{G}\times\mathcal{J}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd] & & & \mathcal{T} \arrow[dl, "v"'] \arrow[dr, "u"] \arrow[dd] & & & \mathcal{S}\times\mathcal{T} \arrow[dl, "q\times v"'] \arrow[dr, "p\times u"] \arrow[dd] & \\
\mathcal{H} \arrow[r, phantom, "\Downarrow\nu"] & & \mathcal{G}\ , \arrow[l, phantom, "\Downarrow\mu"'] & \mathcal{K} \arrow[r, phantom, "\Downarrow\nu'"] & & \mathcal{J} \arrow[l, phantom, "\Downarrow\mu'"'] \arrow[r, mapsto] & \mathcal{H}\times\mathcal{K} \arrow[r, phantom, "\Downarrow"] & & \mathcal{G}\times\mathcal{J} \arrow[l, phantom, "\Downarrow"'] \\
& \mathcal{S}' \arrow[ul, "q'"] \arrow[ur, "p'"'] & & & \mathcal{T}' \arrow[ul, "v'"] \arrow[ur, "u'"'] & & & \mathcal{S}'\times\mathcal{T}' \arrow[ul, "q'\times v'"] \arrow[ur, "p'\times u'"'] &
\end{tikzcd}$$

3.3 Enriched Bicategories

A monoidal structure such as the tensor product on Span discussed in the previous section is the crucial ingredient for defining *enriched bicategories*. In particular, given a monoidal bicategory $\mathcal{V}$ with the tensor product $\otimes$, a $\mathcal{V}$-enriched bicategory has for each pair of objects x, y, an object $\hom(x, y)$ of $\mathcal{V}$. Further, composition involves the tensor product in $\mathcal{V}$:

$$\circ\colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

Monoidal bicategories were defined by Gordon, Powers, and Street [20] in 1994 as a special case of tricategories [16, 21].

In this section, we give a definition of enriched bicategories followed by a *change of base* theorem, which says which sort of map $f\colon \mathcal{V} \to \mathcal{V}'$ lets us turn a $\mathcal{V}$-enriched bicategory into a $\mathcal{V}'$-enriched bicategory.

Remember that for each finite group G there is a category of permutation representations $\mathrm{PermRep}(G)$. Enriched bicategories allow us to define a Span-enriched bicategory called the Hecke bicategory, and denoted $\mathrm{Hecke}(G)$, which categorifies $\mathrm{PermRep}(G)$. The importance of introducing a theory of enriched bicategories is twofold. First, the composition structure of the Hecke bicategory is given by spans of groupoids, not functors between groupoids. It follows that the Hecke bicategory is not a bicategory in the traditional sense. Second, a trivial application of the change of base theorem is the main tool employed in proving the Fundamental Theorem of Hecke Operators via degroupoidification. However, change of base is important, and this is illustrated in [22]. In particular, using some basic topos theory, the change of base theorem also provides a connection between groupoidification and the alternative view of categorified intertwining operators as spans of G-sets as discussed in Section 2.6.

Before giving the definition of an *enriched bicategory*, we recall the definition of an enriched category — that is, a category enriched over a monoidal category $\mathcal{V}$ [29]. An *enriched category* consists of:

- a set of objects $x, y, z \ldots$,
- for each pair of objects x, y, an object $\hom(x, y) \in \mathcal{V}$,
- composition and identity-assigning maps that are morphisms in $\mathcal{V}$.

For example, PermRep(G) is a category enriched over the monoidal category of vector spaces.

We now define enriched bicategories, which are simply a categorified version of enriched categories.

Definition 6 ((Enriched bicategory))**.** *Let $\mathcal{V}$ be a monoidal bicategory. A $\mathcal{V}$-**bicategory** $\mathcal{B}$ consists of the following data subject to the following axioms:*
Data:

- *A set* $\mathrm{Ob}(\mathcal{B})$ *of objects* $x, y, z, \ldots$*;*
- *for every pair of objects x, y, a* **hom-object** $\hom(x, y) \in \mathcal{V}$*, which we will often denote (x, y), while suppressing the tensor product when necessary;;*
- *a morphism called* **composition**

$$c = c_{xyz} \colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

for each triple f objects $x, y, z \in \mathcal{B}$;

- *an* **identity-assigning** *morphism*

$$i_x \colon I \to \hom(x, x)$$

for each object $a \in \mathcal{B}$;

- *an invertible 2-morphism called the* **associator**

$$\begin{tikzcd}
((w,x)\otimes(x,y))\otimes(y,z) \arrow[rr, "a"'] \arrow[d, "c\otimes 1"'] & \arrow[d, phantom, "\alpha_{abcd}\,\Swarrow"] & (w,x)\otimes((x,y)\otimes(y,z)) \arrow[d, "1\otimes c"] \\
(w,y)\otimes(y,z) \arrow[dr, "c"'] & & (w,x)\otimes(x,z) \arrow[dl, "c"] \\
& (w,z) &
\end{tikzcd}$$

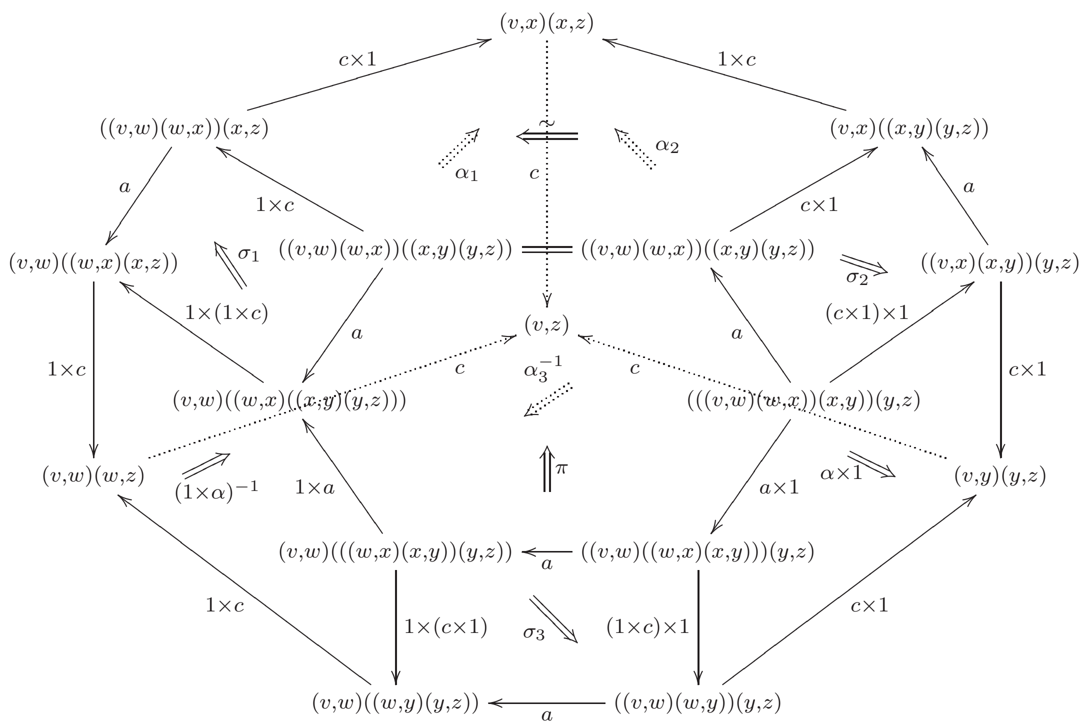
for each quadruple of objects $w, x, y, z \in \mathcal{B}$;

- *and invertible* 2*-morphisms called the* **right unitor** *and* **left unitor**

$$
\begin{tikzcd}
 & (x,x)\otimes(x,y) \arrow[dl, "c_{xxy}"'] \\
(x,y) & I\otimes(x,y) \arrow[l, "r_{xy}"] \arrow[u, "i_x\otimes 1"'] \arrow[ul, phantom, "\Leftarrow \rho_{xy}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(x,y)\otimes(y,y) \arrow[dr, "c_{xyy}"] & \\
(x,y)\otimes I \arrow[u, "1\otimes i_y"] \arrow[r, "l_{xy}"'] \arrow[ur, phantom, "\Rightarrow \lambda_{xy}"] & (x,y)
\end{tikzcd}
$$

for every pair of objects $x, y \in \mathcal{B}$*;*

Axioms:



$$
\begin{tikzcd}
((x,y)\otimes(y,y))\otimes(y,z) \arrow[rrr, "a"] \arrow[dr, "c\otimes 1"] & & & (x,y)\otimes((y,y)\otimes(y,z)) \arrow[dr, "1\otimes c"] & \\
& (x,y)\otimes(y,z) \arrow[r, "c"] \arrow[rrr, bend right, "1"'] \arrow[l, phantom, "\rho\otimes 1\ \Searrow"] & (x,z) \arrow[u, phantom, "\Swarrow\alpha"] \arrow[d, phantom, "\overset{\sigma}{\Searrow}\quad \mu_{abc}\Searrow"] & & (x,y)\otimes(y,z) \arrow[ll, "c"'] \\
((x,y)\otimes I)\otimes(y,z) \arrow[uu, "(1\otimes i_y)\otimes 1"] \arrow[ur, "r\otimes 1"'] \arrow[rrr, "a"] & & & (x,y)\otimes(I\otimes(y,z)) \arrow[uu, "1\otimes(i_y\otimes 1)"] \arrow[ur, "1\otimes l"'] \arrow[ur, phantom, "\Nwarrow (1\otimes\lambda)^{-1}"] &
\end{tikzcd}
$$

σ is the component 2-cell expressing the pseudo naturality of the associator for the tensor product in the monoidal bicategory $\mathcal{V}$, and the arrow marked $\sim$ is just the associator natural isomorphism in the underlying bicategory of $\mathcal{V}$.

We note that diagrams similar to our axioms of enriched bicategories appeared in the work of Aguilar and Mahajan [1].

Given a monoidal bicategory $\mathcal{V}$, which has only identity 2-morphisms, then every $\mathcal{V}$-bicategory is a $\mathcal{V}$-category in the obvious way, and every $\mathcal{V}$-enriched category can be trivially extended to a $\mathcal{V}$-bicategory. This flexibility will allow us to think of PermRep(G) as either a Vect-enriched category or as a Vect-enriched bicategory.

Now we state a *change of base* construction which allows us to change a $\mathcal{V}$-enriched bicategory to a $\mathcal{V}'$-enriched bicategory.

Theorem 7. *Given a lax-monoidal homomorphism of monoidal bicategories $f\colon \mathcal{V} \to \mathcal{V}'$ and a $\mathcal{V}$-bicategory $\mathcal{B}_{\mathcal{V}}$, then there is a $\mathcal{V}'$-bicategory*

$$\bar{f}(\mathcal{B}_{\mathcal{V}}).$$

Proof. This is Theorem 72 in Section 7.1.3. $\square$

A monoidal homomorphism is just the obvious sort of map between monoidal bicategories [20, 21]. A *lax*-monoidal homomorphism f is a bit more general: it need not preserve the tensor product up to isomorphism. Instead, it preserves the tensor product only up to a *morphism*:

$$f(x) \otimes' f(y) \to f(x \otimes y).$$

The data of the enriched bicategory $\bar{f}(\mathcal{B}_{\mathcal{V}})$ is straightforward to write down and the proof of the claim is a trivial, yet tedious surface diagram chase. Here we just point out the most important idea. The new enriched bicategory $\bar{f}(\mathcal{B}_{\mathcal{V}})$ has the same objects as $\mathcal{B}_{\mathcal{V}}$, and for each pair of objects x, y, the hom-category of $\bar{f}(\mathcal{B}_{\mathcal{V}})$ is:

$$\hom_{\bar{f}(\mathcal{B}_{\mathcal{V}})}(x,y) := f(\hom_{\mathcal{B}_{\mathcal{V}}}(x,y))$$

This theorem will allow us to pass from the more natural definition of the Hecke bicategory, which we define in the next section, to our original definition of the Hecke bicategory as the bicategory of spans of finite G-sets $\mathrm{Span}^*(G\mathrm{Set})$.

3.4 The Hecke Bicategory — Take Two

We are now in a position to present a more satisfactory categorification of the intertwining operators between permutation representations of a finite group G. This is the Span-enriched category $\mathrm{Hecke}(G)$ — the *Hecke bicategory.*

Theorem 8. *Given a finite group G, there is a* Span*-enriched bicategory* $\mathrm{Hecke}(G)$ *which has:*

- *finite G-sets $X, Y, Z \ldots$ as objects,*
- *for each pair of finite G-sets X, Y, an object of* Span*, the action groupoid:*

$$\hom(X,Y) = (X \times Y)//G,$$

- *composition*

$$\circ\colon (X \times Y)//G \times (Y \times Z)//G \to (X \times Z)//G$$

  is the span of groupoids,

$$\begin{tikzcd} & (X \times Y \times Z)//G \arrow[dl, "p_X \times p_Z"'] \arrow[dr, "(p_X \times p_Y)\times(p_Y \times p_Z)"] & \\ (X \times Z)//G & & (X \times Y)//G \times (Y \times Z)//G \end{tikzcd}$$

- *for each finite G-set X, an identity assigning span from the terminal groupoid* 1 *to $(X \times X)//G$,*
- *invertible* 2*-morphisms in* Span *assuming the role of the associator and left and right unitors.*

Proof. This is Theorem 73 in Section 8.1. □

Given this structure one needs to check that the axioms of an enriched bicategory are satisfied; however, we will not prove this here. Combining the degroupoidification functor of Section 6.1, the change of base theorem of Section 3.3, and the enriched bicategory Hecke(G) described above, we can now state the Fundamental Theorem of Hecke Operators. This is the content of the next section.

3.5 The Fundamental Theorem of Hecke Operators

In this section, we make the relationship between the *Hecke algebroid* PermRep(G) of permutation representations of a finite group G and the Hecke bicategory Hecke(G) precise. The idea is that for each finite group G, the Hecke bicategory Hecke(G) categorifies PermRep(G).

We recall the functor *degroupoidification*:

$$\mathcal{D}\colon \mathrm{Span} \to \mathrm{Vect}$$

which replaces groupoids with vector spaces and spans of groupoids with linear operators. With this functor in hand, we can apply the change of base theorem to the Span-enriched bicategory Hecke(G). In other words, for each finite group G there is a Vect-enriched bicategory:

$$\bar{\mathcal{D}}\left(\mathrm{Hecke}(G)\right),$$

which has

- permutation representations $X, Y, Z, \ldots$ of G as objects,
- for each pair of permutation representations X, Y, the vector space

$$\hom(X, Y) = \mathcal{D}\left((X \times Y)//G\right)$$

with G-orbits of $X \times Y$ as basis. Of course, a Vect-enriched bicategory is also a Vect-enriched category. The following is the statement of the Fundamental Theorem of Hecke Operators, an equivalence of Vect-enriched categories.

Theorem 9. *Given a finite group G,*

$$\bar{\mathcal{D}}\left(\mathrm{Hecke}(G)\right) \simeq \mathrm{PermRep}(G)$$

as Vect-*enriched categories.*

Proof. This is Theorem 74 in Section 8.2. □

More explicitly, this says that given two permutation representations X and Y, the vector space of intertwining operators between them can be constructed as the degroupoidification of the groupoid $(X \times Y)//G$.

An important corollary of the Fundamental Theorem of Hecke Operators is that for certain G-sets, which are the flag varieties X associated to Dynkin diagrams, the hom-groupoid $\mathrm{Hecke}(X, X)$ categorifies the associated Hecke algebra.

Corollary 10. *Hecke algebras*

Proof. This is Corollary 75 in Section 8.3. □

We will describe these Hecke algebras in Section 4.2 and make the relationship to the Hecke bicategory and some of its applications explicit in Section 4.3.

Chapter 4

Applications to Representation Theory and Knot Theory

Degroupoidification is a systematic process; groupoidification is the attempt to undo this process. The previous section explains degroupoidification—but not why groupoidification is interesting. The interest lies in its applications to concrete examples. So, let us sketch an application to Hecke algebras. See [6] for a sketch of applications to Feynman diagrams and Hall algebras.

4.1 Hecke Algebras

The main theorem of this paper was the Fundamental Theorem of Hecke Operators. This is, in fact, a statement about categorified Hecke algebras. There is a nice collection of literature on categorified Hecke algebras as they have played a central role in the development of categorified and geometric representation theory.

In Section 2.3, we motivated the notion of a categorification of permutation representations and intertwining operators via connections with spans of sets — especially finite G-sets. This leads us to an awkward first construction of the Hecke bicategory as $\mathrm{Span}^*(G\mathrm{Set})$. We spent the rest of the paper building a more natural construction of the Hecke bicategory, and a way to relate said construction $\mathrm{Hecke}(G)$ to $\mathrm{Span}^*(G\mathrm{Set})$.

In applying the categorified Hecke algebra to knot theory, the more concrete description of the Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$ as spans of finite G-sets, allows a hands-on approach to the 2-morphisms, i.e., the not-necessarily G-equivariant maps of spans. We show that in certain cases these are Yang-Baxter operators that satisfy the

Zamolodchikov tetrahedron equation.

4.2 Hecke Algebras as Intertwining Algebras

In this section, we recall some descriptions of the Hecke algebra and note a categorification of these algebras in the context of the Fundamental Theorem of Hecke Operators. Categorified Hecke algebras have been studied by a number of authors in various contexts including Soergel bimodules citeSoe, a recent diagrammatic interpretation of the work of Soergel by Elias and Khovanov [17], and a geometric interpretation by Webster and Williamson [40]. Further, Hecke categories have been studied in the context of the Kazhdan-Lusztig conjectures [28]. Hecke algebras are close relatives of quantum groups, which have provided the major thrust in research in categorified representation theory. See [19, 26, 37], for example.

Hecke algebras are constructed from a Dynkin diagram and a prime power. Moreover, they are algebras of Hecke operators. The term 'Hecke operator' is largely confined to the realm of number theory and modular forms, but it makes sense to say that the Hecke algebras with which we are concerned at present consist of Hecke operators. That is, the notion of Hecke operator can be interpreted quite broadly as the Hecke algebra is just a special example of a vector space of intertwining operators between permutation representations.

There are several well-known equivalent descriptions of the Hecke algebra $\mathcal{H}(\Gamma, q)$ obtained from a Dynkin diagram Γ and a prime power q. One kind of Hecke algebra, commonly referred to as the *Iwahori-Hecke algebra*, is a q-deformation of the group algebra of the Coxeter group of Γ. A standard example of a Coxeter group associated to a Dynkin diagram is the symmetric group on n letters S_n, which is the Coxeter group of the A_{n-1} Dynkin diagram. We will return to this definition in Section 4.3 and see that it lends itself to combinatorial applications of the Hecke algebra. This combinatorial aspect comes from the close link between the Coxeter group and its associated Coxeter complex, a finite simplicial complex which plays an essential role in the theory of buildings [12].

Hecke algebras have an alternative definition as algebras of intertwining operators between certain coinduced representations [13]. Given a Dynkin diagram Γ and prime power q, there is an associated simple algebraic group $G = G(\Gamma, q)$. Choosing a Borel subgroup $B \subset G$, i.e., a maximal solvable subgroup, we can construct the corresponding flag variety $X = G/B$, a transitive G-set.

Now, for a finite group G and a representation V of a subgroup $H \subset G$, the *coinduced representation* of G from H is defined as the V-valued functions on G, which commute with the action of H:

$$\mathrm{CoInd}_H^G = \{f\colon G \to V \mid h \cdot f(g) = f(hg)\}$$

The action of $g \in G$ is defined on a function $f\colon G \to V$ as $g \cdot f(g') = f(g'g^{-1})$. A standard fact about finite groups says that the representation coinduced from the trivial representation of any subgroup is the permutation representation on the cosets of that subgroup.

Thus, from the trivial representation of a Borel subgroup B, we obtain the permutation representation on the cosets of B, i.e., the flag variety X. Then the Hecke algebra is defined as the algebra of intertwining operators from $\widetilde{X}$ to itself:

$$\mathrm{PermRep}(G)(\widetilde{X}, \widetilde{X}) := \mathcal{H}(\Gamma, q),$$

where $G = G(\Gamma, q)$ and we use the notation $\mathcal{C}(A, B)$ to denote $\hom(A, B)$ in the category $\mathcal{C}$.

Given this definition of the Hecke algebra, we have an immediate corollary to Theorem 9:

Corollary 11. *Given a finite group $G = G(\Gamma, q)$, the* hom-*category* $\mathrm{Hecke}(G)(\widetilde{X}, \widetilde{X})$ *categorifies $\mathcal{H}(\Gamma, q)$.*

4.3 The Categorified Hecke Algebra and 2-Tangles

Now that we have developed the machinery of the Fundamental Theorem of Hecke Operators, and we have seen a categorification of Hecke algebras abstractly as a corollary, we can look at a concrete example. The categorified Hecke algebra is particularly easy to understand from our original definition of the Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$ as the bicategory of finite G-sets, spans of G-sets, and not-necessarily equivariant maps of spans. Further, in this categorified picture we can see relationships with 2-tangles in 4-dimensional space.

While we found it useful in considering the Fundamental Theorem of Hecke Operators to view Hecke algebras as algebras of intertwining operators, viewing the Hecke algebra as a q-deformation of a Coxeter group [23] is helpful in examples.

Any Dynkin diagram gives rise to a simple Lie group, and the Weyl group of this simple Lie group is a Coxeter group. Let Γ be a Dynkin diagram. We write $d \in \Gamma$ to mean that d is a dot in this diagram. Associated to each unordered pair of dots $d, d' \in \Gamma$ is a number $m_{dd'} \in \{2, 3, 4, 6\}$. In the usual Dynkin diagram conventions:

- $m_{dd'} = 2$ is drawn as no edge at all,
- $m_{dd'} = 3$ is drawn as a single edge,
- $m_{dd'} = 4$ is drawn as a double edge,
- $m_{dd'} = 6$ is drawn as a triple edge.

For any prime power q, our Dynkin diagram Γ gives a Hecke algebra. The *Hecke algebra* $\mathcal{H}(\Gamma, q)$ corresponding to this data is the associative $\mathbb{R}$-algebra with one generator σ_d for each $d \in \Gamma$, and relations:

$$\sigma_d^2 = (q-1)\sigma_d + q$$

for all $d \in \Gamma$, and

$$\sigma_d \sigma_{d'} \sigma_d \cdots = \sigma_{d'} \sigma_d \sigma_{d'} \cdots$$

for all $d, d' \in \Gamma$, where each side has $m_{dd'}$ factors.

When $q = 1$, this Hecke algebra is simply the group algebra of the Coxeter group associated to Γ: that is, the group with one generator s_d for each dot $d \in \Gamma$, and relations

$$s_d^2 = 1, \qquad (s_d s_{d'})^{m_{dd'}} = 1.$$

So, the Hecke algebra can be thought of as a q-deformation of this Coxeter group.

We recall the flag variety $X = G/B$ from Section 4.2. This set is a smooth algebraic variety, but we only need the fact that it is a finite set equipped with a transitive action of the finite group G. Starting from just this G-set X, we can see an explicit picture of the categorified Hecke algebra of spans of G-sets from X to X.

The key is that for each dot $d \in \Gamma$ there is a special span of G-sets that corresponds to the generator $\sigma_d \in \mathcal{H}(\Gamma, q)$. To illustrate these ideas, let us consider the simplest nontrivial example, the Dynkin diagram A_2:

$$\bullet \!-\!\!\!-\!\!\!-\! \bullet$$

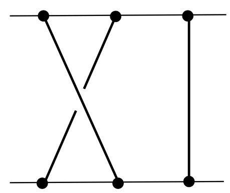
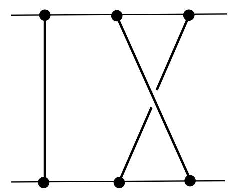
The Hecke algebra associated to A_2 has two generators, which we call P and L, for reasons soon to be revealed:

$$P = \sigma_1, \qquad L = \sigma_2.$$

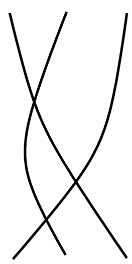
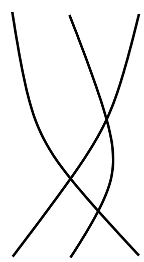
The relations are

$$P^2 = (q-1)P + q, \qquad L^2 = (q-1)P + q, \qquad PLP = LPL.$$

It follows that this Hecke algebra is a quotient of the group algebra of the 3-strand braid group, which has two generators P and L, which we can draw as tangles in 3-dimensional space:

$P =$  $L =$ 

and one relation $PLP = LPL$:

 $=$ 

called the *Yang–Baxter equation* or *third Reidemeister move*. This is why Jones could use traces on the A_n Hecke algebras to construct invariants of knots [25]. This connection to knot theory makes it especially interesting to categorify Hecke algebras.

So, let us see what the categorified Hecke algebra looks like, and where the Yang–Baxter equation comes from. The algebraic group corresponding to the A_2 Dynkin diagram and the prime power q is $G = \mathrm{SL}(3, \mathbb{F}_q)$, and we can choose the Borel subgroup B to consist of upper triangular matrices in $\mathrm{SL}(3, \mathbb{F}_q)$. Recall that a complete flag in the vector space $\mathbb{F}_q^3$ is a pair of subspaces

$$0 \subset V_1 \subset V_2 \subset \mathbb{F}_q^3.$$

The subspace V_1 must have dimension 1, while V_2 must have dimension 2. Since G acts transitively on the set of complete flags, while B is the subgroup stabilizing a chosen flag, the flag variety $X = G/B$ in this example is just the set of complete flags in $\mathbb{F}_q^3$—hence its name.

We can think of $V_1 \subset \mathbb{F}_q^3$ as a point in the projective plane $\mathbb{F}_q\mathrm{P}^2$, and $V_2 \subset \mathbb{F}_q^3$ as a line in this projective plane. From this viewpoint, a complete flag is a chosen point lying on a chosen line in $\mathbb{F}_q\mathrm{P}^2$. This viewpoint is natural in the theory of 'buildings', where each Dynkin diagram corresponds to a type of geometry [12]. Each dot in the Dynkin diagram then stands for a 'type of geometrical figure', while each edge stands for an 'incidence relation'. The A_2 Dynkin diagram corresponds to projective plane geometry. The dots in this diagram stand for the figures 'point' and 'line':

$$\text{point} \bullet \!\!-\!\!-\!\!-\!\!-\!\! \bullet \text{ line}$$

The edge in this diagram stands for the incidence relation 'the point p lies on the line ℓ'.

We can think of P and L as special elements of the A_2 Hecke algebra, as already described. But when we categorify the Hecke algebra, P and L correspond to irreducible spans of G-sets – that is, not a coproduct of two non-trivial spans of G-sets. Let us describe these spans and explain how the Hecke algebra relations arise in this categorified setting.

The objects P and L can be defined by giving irreducible spans of G-sets:

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

In general, any span of G-sets

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & X \end{tikzcd}$$

such that $q \times p \colon S \to X \times X$ is injective can be thought of as G-invariant binary relation between elements of X. Irreducible G-invariant spans are always injective in this sense. So, such spans can also be thought of as G-invariant relations between flags. In these terms, we define P to be the relation that says two flags have the same line, but different points:

$$P = \{((p,\ell),(p',\ell)) \in X \times X \mid p \neq p'\}$$

Similarly, we think of L as a relation saying two flags have different lines, but the same point:

$$L = \{((p,\ell),(p,\ell')) \in X \times X \mid \ell \neq \ell'\}.$$

Given this, we can check that

$$P^2 \cong (q-1) \times P + q \times 1, \qquad L^2 \cong (q-1) \times L + q \times 1, \qquad PLP \cong LPL.$$

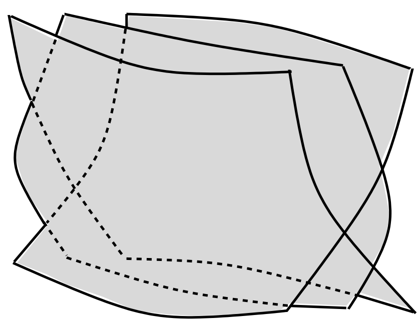
Here both sides refer to spans of G-sets. Addition of spans is defined using coproduct, while 1 denotes the identity span from X to X. We use 'q' to stand for a fixed q-element set, and similarly for '$q-1$'. We compose spans of G-sets using the ordinary pullback.

To check the existence of the first two isomorphisms above, we just need to count. In $\mathbb{F}_q\mathrm{P}^2$, the are $q+1$ points on any line. So, given a flag we can change the point in q different ways. To change it again, we have a choice: we can either send it back to the original point, or change it to one of the $q-1$ other points. So, $P^2 \cong (q-1) \times P + q \times 1$. Since there are also $q+1$ lines through any point, similar reasoning shows that $L^2 \cong (q-1) \times L + q \times 1$.

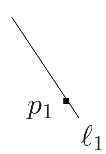
The Yang–Baxter isomorphism

$$PLP \cong LPL$$

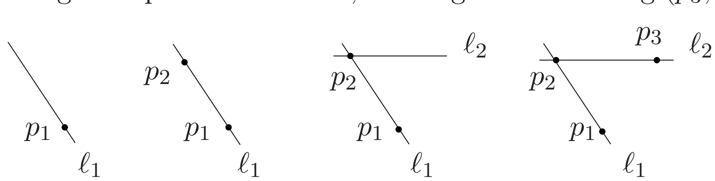
is more interesting. For this isomorphism we will draw the corresponding 2-tangle in 4-dimensional space [8]:



We construct it as follows. First consider the left-hand side, PLP. So, start with a complete flag called (p_1, ℓ_1):

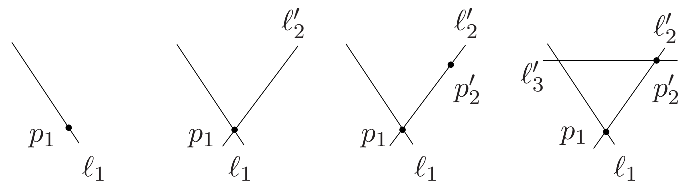


Then, change the point to obtain a flag (p_2, ℓ_1). Next, change the line to obtain a flag (p_2, ℓ_2). Finally, change the point once more, which gives us the flag (p_3, ℓ_2):



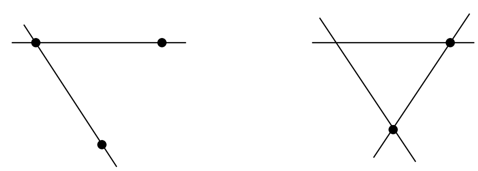
The figure on the far right is a typical element of PLP.

On the other hand, consider LPL. So, start with the same flag as before, but now change the line, obtaining (p_1, ℓ_2'). Next change the point, obtaining the flag (p_2', ℓ_2'). Finally, change the line once more, obtaining the flag (p_2', ℓ_3'):



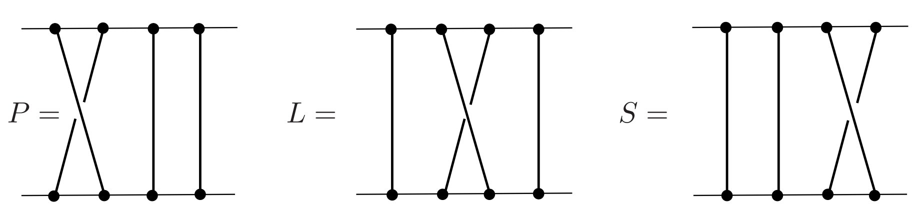
The figure on the far right is a typical element of LPL.

Now, the axioms of projective plane geometry say that any two distinct points lie on a unique line, and any two distinct lines intersect in a unique point. So, any figure of the sort shown on the left below determines a unique figure of the sort shown on the right, and vice versa:

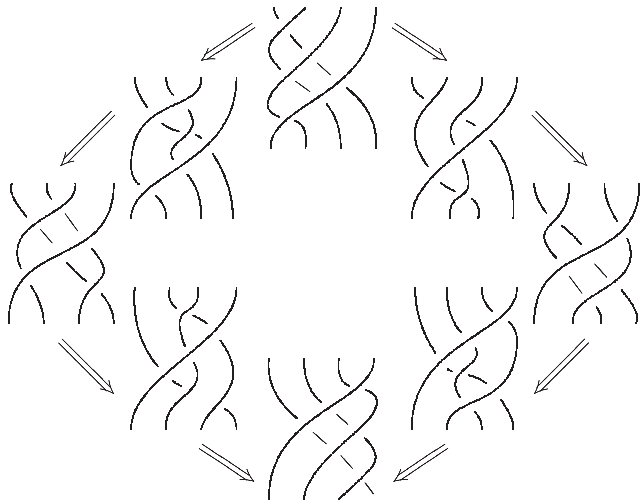


Comparing this with the pictures above, we see this bijection induces an isomorphism of spans $PLP \cong LPL$. So, we have derived the Yang–Baxter isomorphism from the axioms of projective plane geometry!

While the Yang-Baxter *equation* is present in the generators and relations description of the Hecke algebra, we have seen that the categorified setting allows us to view these equations as *isomorphisms* of spans of G-sets. As such, these *Yang-Baxter operators* satisfy an equation of their own – the *Zamolodchikov tetrahedron equation* [27]. However, this equation only appears in the categorified A_n Hecke algebra, for $n \geq 3$. We can assign braids on four strands to the generators of the A_3 Hecke algebra:



where composition of spans, or multiplication in the Hecke algebra, corresponds to stacking of braid diagrams. Then we can express the Zamolodchikov equation – as an equation in the categorified Hecke algebra – in the form of a commutative diagram of braids [2, 14]:



This is just the beginning of a wonderful story involving Dynkin diagrams of more general types, incidence geometries, logic, braided monoidal 2-categories [9, 35], knot invariants, topological quantum field theories, geometric representation theory, and more!

Part II

Second Part: Definitions and Theorems

Chapter 5

General Definitions and Theorems

5.1 Basics of Groupoids

Definition 12. *A* **groupoid** $\mathcal{G}$ *is a category in which all morphisms are invertible.*

Definition 13. *We denote the set of objects in a groupoid* $\mathcal{G}$ *by* $\mathrm{Ob}(G)$ *and the set of morphisms by* $\mathrm{Mor}(\mathcal{G})$.

In the present work, all of the groupoids we encounter will be **finite** — they have a *finite set* of objects — and are **locally finite** — given any pair of objects, the set of morphisms from one object to the other is finite.

Definition 14. *A* **functor** $p\colon \mathcal{G} \to \mathcal{H}$ *between categories is a pair of functions* $p\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{H})$ *and* $p\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{H})$ *such that* $p(1_x) = 1_{p(x)}$ *for* $x \in \mathrm{Ob}(\mathcal{G})$ *and* $p(fg) = p(f)p(g)$ *for* $f, g \in \mathrm{Mor}(\mathcal{G})$.

Definition 15. *A* **natural transformation** $\alpha\colon p \Rightarrow q$ *between functors* $p, q\colon \mathcal{G} \to \mathcal{H}$ *consists of a morphism* $\alpha_x\colon p(x) \to q(x)$ *in* $\mathrm{Mor}(\mathcal{H})$ *for each* $x \in \mathrm{Ob}(\mathcal{G})$ *such that for each morphism* $f\colon x \to x'$ *in* $\mathrm{Mor}(\mathcal{G})$ *the following naturality square commutes:*

$$\begin{tikzcd} p(x) \arrow[r, "\alpha_x"] \arrow[d, "p(f)"'] & q(x) \arrow[d, "q(f)"] \\ p(x') \arrow[r, "\alpha_{x'}"'] & q(x') \end{tikzcd}$$

Definition 16. *A* **natural isomorphism** *is a natural transformation $\alpha\colon p \Rightarrow q$ between functors $p, q\colon \mathcal{G} \to \mathcal{H}$ such that for each $x \in \mathrm{Ob}(\mathcal{G})$, the morphism α_x is invertible.*

Note that a natural transformation between functors between *groupoids* is necessarily a natural isomorphism.

Notation 17. *We will use* Grpd *to denote the* 2*-category of groupoids, functors, and natural isomorphisms.*

In what follows, and throughout the paper, we write $x \in \mathcal{G}$ as shorthand for $x \in \mathrm{Ob}(\mathcal{G})$. Also, several places throughout this paper we have used the notation $\alpha \cdot p$ or $p \cdot \alpha$ to denote operations combining a functor p and a natural transformation α. These operations are called 'whiskering':

Definition 18. *Given groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$, functors $p\colon \mathcal{G} \to \mathcal{H}$, $q\colon \mathcal{H} \to \mathcal{J}$ and $q'\colon \mathcal{H} \to \mathcal{J}$, and a natural transformation $\alpha\colon q \Rightarrow q'$, there is a natural transformation $\alpha \cdot p\colon qp \Rightarrow q'p$ called the* **right whiskering** *of α by q. This assigns to any object $x \in \mathcal{G}$ the morphism $\alpha_{p(x)}\colon q(p(x)) \to q'(p(x))$ in $\mathcal{H}$, which we denote as $(\alpha \cdot p)_x$. Similarly, given a groupoid $\mathcal{K}$ and a functor $r\colon \mathcal{H} \to \mathcal{K}$, there is a natural transformation $r \cdot \alpha\colon rq \Rightarrow rq'$ called the* **left whiskering** *of α by r. This assigns to any object $y \in \mathcal{H}$ the morphism $r(\alpha_y)\colon rq(y) \to rq'(y)$ in $\mathcal{K}$, which we denote as $(r \cdot \alpha)_y$.*

Definition 19. *A functor $p\colon \mathcal{G} \to \mathcal{H}$ between groupoids is called an* **equivalence** *if there exists a functor $q\colon \mathcal{H} \to \mathcal{G}$, called the* **weak inverse** *of p, and natural isomorphisms $\eta\colon qp \Rightarrow 1_{\mathcal{G}}$ and $\rho\colon pq \Rightarrow 1_{\mathcal{H}}$. In this case we say $\mathcal{G}$ and $\mathcal{H}$ are* **equivalent**.

Definition 20. *A functor $p\colon \mathcal{G} \to \mathcal{H}$ between groupoids is called* **faithful** *if for each pair of objects $x, y \in \mathcal{G}$ the function $p\colon \hom(x, y) \to \hom(p(x), p(y))$ is injective.*

Definition 21. *A functor $p\colon \mathcal{G} \to \mathcal{H}$ between groupoids is called* **full** *if for each pair of objects $x, y \in \mathcal{G}$, the function $p\colon \hom(x, y) \to \hom(p(x), p(y))$ is surjective.*

Definition 22. *A functor $p\colon \mathcal{G} \to \mathcal{H}$ between groupoids is called* **essentially surjective** *if for each object $y \in \mathcal{H}$, there exists an object $x \in \mathcal{G}$ and a morphism $f\colon p(x) \to y$ in $\mathcal{H}$.*

A functor has all three of the above properties if and only if the functor is an equivalence. It is often convenient to prove two groupoids are equivalent by exhibiting a functor which is full, faithful and essentially surjective.

5.2 Spans of Groupoids

Definition 23. *A* **span** *of groupoids is a pair of functors in* Grpd *with a common source object.*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

We think of such a span as a morphism from $\mathcal{G}$ to $\mathcal{H}$. We can also talk about *maps between spans* of groupoids:

Definition 24. *Given a pair of parallel spans in a* Grpd*, a* **map between spans** *is a triple (f, μ, ν) consisting of a functor $f\colon \mathcal{S} \to \mathcal{S}'$ together with a pair of natural isomorphisms $\mu\colon p \Rightarrow p'f$ and $\nu\colon q \Rightarrow q'f$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\ \mathcal{H} \arrow[r, phantom, "\nu\Downarrow"] & & \mathcal{G} \arrow[l, phantom, "\Downarrow\mu"] \\ & \mathcal{S}' \arrow[ul, "q'"] \arrow[ur, "p'"'] & \end{tikzcd}$$

Maps between spans can be composed both *vertically* and *horizontally* suggesting the existence of a bicategory of spans of groupoids. We discuss these composition operations in detail below as the *weak pullback* for horizontal composition and in Section 5.4 for vertical composition. However, maps of spans as defined above are not quite the right type of 2-morphism for our purposes. We, in fact, want equivalences of spans as defined here.

Definition 25. *An* **equivalence** *of spans of groupoids*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{S}' \arrow[dl, "q'"'] \arrow[dr, "p'"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

is a map of spans (f, μ, ν) from $\mathcal{S}$ to $\mathcal{S}'$ together with a map of spans (g, μ', ν') from $\mathcal{S}'$ to $\mathcal{S}$ and natural isomorphisms $\gamma\colon gf \Rightarrow 1$ and $\gamma'\colon fg \Rightarrow 1$ such that the following equations hold:

$$1_p = (p \cdot \gamma)(\mu' \cdot f)\mu \qquad 1_q = (q \cdot \gamma)(\nu' \cdot f)\nu$$
$$1_{p'} = (p' \cdot \gamma')(\mu \cdot g)\mu' \qquad 1_{q'} = (q' \cdot \gamma')(\nu \cdot g)\nu'.$$

There is a category consisting of:

- groupoids
- equivalence classes of spans of groupoids.

The composition of morphisms in this category is defined via the 'weak pullback'. The data needed to construct a weak pullback of groupoids is a 'cospan':

Definition 26. *Given groupoids $\mathcal{G}$ and H, a* **cospan** *is a diagram*

$$\begin{tikzcd} \mathcal{H} \arrow[dr, "q"'] & & \mathcal{G} \arrow[dl, "p"] \\ & \mathcal{J} & \end{tikzcd}$$

in Grpd*.*

Definition 27. *Given a cospan in the 2-category* Grpd*:*

$$\begin{tikzcd} \mathcal{H} \arrow[dr, "q"'] & & \mathcal{G} \arrow[dl, "p"] \\ & \mathcal{J} & \end{tikzcd}$$

the **weak pullback** *is the groupoid where:*

- *an object is a triple (y, α, x), where $x \in \mathcal{G}$, $y \in \mathcal{H}$, and $\alpha \colon p(x) \to q(y)$;*
- *a morphism from (y, α, x) to (y', α', x') is a pair of morphisms $\sigma \colon x \to x'$ in X and $\tau \colon y \to y'$ in Y such that the following diagram commutes:*

$$\begin{tikzcd} q(y) \arrow[d, "q(\tau)"'] & p(x) \arrow[l, "\alpha"'] \arrow[d, "p(\sigma)"] \\ q(y') & p(x') \arrow[l, "\alpha'"] \end{tikzcd}$$

  as morphisms.

We will discuss weak pullbacks and their universal property in greater detail in Section 5.3. Given a pair of composable spans:

$$\begin{tikzcd} & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{K} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

we write the composite as the weak pullback:

$$\begin{tikzcd}
& & \mathcal{TS} \arrow[dl] \arrow[dr] & & \\
& \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathcal{K} & & \mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

This composition process only makes sense, however, if we *choose* a weak pullback for each cospan in Grpd. We also have *maps between maps between spans*:

Definition 28. *Given a parallel pair of maps of spans $f = (f, \mu, \nu)$ and $g = (g, \mu', \nu')$ in* Grpd*:*

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, bend left, "f"] \arrow[dd, dashed, bend right, "g"'] & \\
\mathcal{H} & \arrow[l, phantom, "\nu'\Downarrow \quad \nu\Downarrow"] \arrow[r, phantom, "\overset{\alpha}{\Leftarrow} \quad \mu'\Swarrow \quad \mu\Swarrow"] & \mathcal{G} \\
& \mathcal{S}' \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}$$

then a **map of maps of spans** $\alpha \colon f \Rrightarrow g$ *consists of a natural isomorphism $\alpha \colon f \Rightarrow g$ making the above diagram commute, i.e. $(p' \cdot \alpha)\mu = \mu'$ and $(q' \cdot \alpha)\nu = \nu'$.*

Maps of maps of spans compose in the obvious way. We describe this in Section 5.4.

Definition 29. *Given a pair of maps of maps of spans $\alpha \colon f \Rrightarrow g$ and $\beta \colon g \Rrightarrow f$, if $\beta\alpha = 1_f$ and $\alpha\beta = 1_g$, then β is called the* **right inverse** *of α and α is called the* **left inverse** *of β. In this case we say α is an* **isomorphism**. *Then f and g are* **isomorphic** *as maps of spans. We write $[f]$ to indicate the isomorphism class of maps of spans represented by (f, μ, ν).*

It is now straightforward to define a functorial composition process for spans of groupoids and the maps between spans. This will be the horizontal composition for the bicategory consisting of:

- groupoids
- spans of groupoids
- isomorphism classes of equivalences of spans

which we define explicitly in Section 5.3.

Consider the parallel pairs of composable spans with a pair of maps of spans between them:

$$\begin{tikzcd}
 & \mathcal{T} \arrow[dl, "u"'] \arrow[dr, "r"] \arrow[dd, "g"] & & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\
\mathcal{J} \arrow[r, phantom, "\Rightarrow\nu'"] & \arrow[r, phantom, "\Leftarrow\mu'"] & \mathcal{H} \arrow[r, phantom, "\Rightarrow\nu"] & \arrow[r, phantom, "\Leftarrow\mu"] & \mathcal{G} \\
 & \mathcal{T}' \arrow[ul, "u'"] \arrow[ur, "r'"'] & & \mathcal{S}' \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}$$

We want composition to yield a map between the composite spans in a functorial manner. That is, the *weak pullback* should induce functors called *composition*:

$$\mathrm{Span}(\mathcal{G},\mathcal{H}) \times \mathrm{Span}(\mathcal{H},\mathcal{J}) \to \mathrm{hom}(\mathcal{G},\mathcal{J}).$$

This is due to the fact that the weak pullback is equivalent to a certain limit in the 2-category Grpd — composition yields:

$$\begin{tikzcd}
 & \mathcal{T}\mathcal{S} \arrow[dl, "u\pi_{\mathcal{T}}"'] \arrow[dr, "p\pi_{\mathcal{S}}"] \arrow[dd, "\widetilde{fg}"] & \\
\mathcal{J} \arrow[r, phantom, "\Rightarrow\nu'"] & \arrow[r, phantom, "\Leftarrow\mu"] & \mathcal{G} \\
 & \mathcal{T}'\mathcal{S}' \arrow[ul, "u'\pi_{\mathcal{T}'}"] \arrow[ur, "p'\pi_{\mathcal{S}'}"'] &
\end{tikzcd}$$

with $\widetilde{fg}$ defined by:

$$(t, \alpha, s) \mapsto (g(t), \mu'_t \alpha \nu_s^{-1}, f(s)).$$

5.3 Weak Pullbacks and Equivalences of Spans

We described pullbacks and weak pullbacks in [6]. We will describe weak pullbacks and their universal property here in detail and give a more complete description of the bicategory of spans of groupoids. As with weak pullbacks in a category, we want to understand how to pull back cospans, but this time in a bicategory, however the definition of cospan does not change.

Definition 30. *A* **cospan** *in the* 2*-category* Grpd *is a pair of functors with common codomain.*

We define the weak pullback as a limit in the 2-category Grpd. Actually, the definition is equivalent to the limit of the cospan, although slightly modified.

Definition 31. *Given a cospan:*

$$\begin{tikzcd} \mathcal{H} \arrow[dr, "g"'] & & \mathcal{G} \arrow[dl, "f"] \\ & \mathcal{J} & \end{tikzcd}$$

in the 2*-category* Grpd*, the* **weak pullback** *is an object* $\mathcal{P}$ *in* Grpd *with projection* 1*-morphisms and a* 2*-isomorphism, like so:*

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} \arrow[dr, "g"'] & \arrow[l, phantom, "\overset{\phi}{\Longleftarrow}"] & \mathcal{G} \arrow[dl, "f"] \\ & \mathcal{J} & \end{tikzcd}$$

such that:

- *for any object* $\mathcal{Q}$ *in* Grpd *with* 1*-morphisms and a* 2*-isomorphism, like so:*

$$\begin{tikzcd} & \mathcal{Q} \arrow[dl, "n"'] \arrow[dr, "m"] & \\ \mathcal{H} \arrow[dr, "g"'] & \arrow[l, phantom, "\overset{\psi}{\Longleftarrow}"] & \mathcal{G} \arrow[dl, "f"] \\ & \mathcal{J} & \end{tikzcd}$$

then there exists a 1*-morphism* $h\colon \mathcal{Q} \to \mathcal{P}$ *and a pair of* 2*-isomorphisms* $\alpha\colon ph \Rightarrow m$ *and* $\beta\colon qh \Rightarrow n$ *such that*

$$f \cdot \alpha \circ \phi \cdot h \circ g \cdot \beta^{-1} = \psi,$$

- *and given any pair of* 1*-morphisms* $h, j\colon \mathcal{Q} \to \mathcal{P}$ *and* 2*-morphisms* $\alpha'\colon ph \Rightarrow pj$ *and* $\beta'\colon qh \Rightarrow qj$ *such that:*

  $$\phi \cdot j \circ g \cdot \beta' = f \cdot \alpha' \circ \phi \cdot h,$$

  then there exists a unique 2*-morphism* $\gamma\colon h \Rightarrow j$ *such that* $p \cdot \gamma = \alpha'$ *and* $q \cdot \gamma = \beta'$.

The universal property of weak pullbacks should guarantee that given a cospan in a bicategory $\mathcal{B}$, any two pullbacks, i.e, quadruples (q, P, p, ϕ) and (n, Q, m, ψ), should be equivalent in some sense. In fact, the next proposition shows that the equivalence between any two weak pullbacks in Grpd is an equivalence of spans, i.e., the spans (q, P, p) and (n, Q, m) from $\mathcal{G}$ to $\mathcal{H}$ are equivalent as spans in the sense described in Section 5.3.

Proposition 32. *The weak pullback is defined up to equivalence of spans in* Span(Grpd).

Proof. Consider a cospan:

$$\begin{tikzcd} \mathcal{H} \arrow[dr, "g"'] & & \mathcal{G} \arrow[dl, "f"] \\ & \mathcal{J} & \end{tikzcd}$$

in Grpd and a pair of weak pullback datum (q, P, p, ϕ) and (n, Q, m, ψ) in Grpd.

From the universal property we have a pair of maps of spans:

$$(\beta, h, \alpha)\colon (n, Q, m) \to (q, P, p)$$

such that:

$$g \cdot \beta \circ \phi \cdot h \circ f \cdot \alpha^{-1} = \psi$$

and

$$(\widetilde{\beta}, \widetilde{h}, \widetilde{\alpha})\colon (q, P, p) \to (n, Q, m)$$

such that:

$$g \cdot \widetilde{\beta} \circ \psi \cdot \widetilde{h} \circ f \cdot \widetilde{\alpha}^{-1} = \phi.$$

Then to induce a unique 2-cell from the universal property, we consider the parallel 1-morphisms:

$$1_P, h\widetilde{h} \colon P \to P,$$

and the 2-morphisms:

$$\alpha' := \widetilde{\alpha} \circ \alpha \cdot \widetilde{h} \colon ph\widetilde{h} \Rightarrow p \;\text{ and }\; \beta' := \widetilde{\beta} \circ \beta \cdot \widetilde{h} \colon qh\widetilde{h} \Rightarrow q.$$

Substituting the above equation for ψ into the above equation for ϕ gives:

$$\phi = g \cdot \widetilde{\beta} \circ [(g \cdot \beta \circ \phi \cdot h \circ f \cdot \alpha^{-1}) \cdot \widetilde{h}] \circ f \cdot \widetilde{\alpha}^{-1}.$$

It is then clear that the equation:

$$\phi \circ f \cdot \alpha' = g \cdot \beta' \circ \phi \cdot h\widetilde{h}$$

holds. Looking back to the universal property of weak pullbacks, we now obtain the desired unique 2-morphism

$$\gamma \colon h\widetilde{h} \Rightarrow 1_P$$

with the property that:

$$p \cdot \gamma = \alpha' \;\text{ and }\; q \cdot \gamma = \beta'.$$

We can draw this as:

$$\begin{tikzcd}
 & P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, dashed, bend right, "1_P"] \arrow[dd, bend left, "h\widetilde{h}"'] & \\
A \arrow[r, phantom, "Id \Downarrow \quad \alpha' \Downarrow \quad \overset{\gamma}{\Leftarrow}"] & \arrow[r, phantom, "\Swarrow Id \quad \Swarrow \beta'"] & B \\
 & P \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

In particular,

$$\gamma \colon (\beta', h\widetilde{h}, \alpha') \Rrightarrow (1, 1_P, 1)$$

is a map of maps of spans.

Swapping $h\widetilde{h}$ and 1_P in this argument and taking the inverses α'^{-1} and β'^{-1} of the corresponding 2-morphisms, we similarly obtain a unique map:

$$\gamma' \colon 1_P \Rightarrow h\widetilde{h},$$

with the property that:

$$p \cdot \gamma' = \alpha'^{-1} \text{ and } q \cdot \gamma' = \beta'^{-1}.$$

This gives a map of spans:

$$\gamma' \colon (1, 1_P, 1) \Rrightarrow (\beta'^{-1}, h\widetilde{h}, \alpha'^{-1}),$$

which composed with γ gives the unique map of maps of spans

$$\gamma'\gamma \colon (\beta', h\widetilde{h}, \alpha') \Rightarrow (\beta'^{-1}, h\widetilde{h}, \alpha'^{-1}).$$

This implies that $\gamma'\gamma$ is the identity. Reversing this argument we see that $\gamma\gamma'$ is also the identity, and thus, $\gamma^{-1} = \gamma'$. i.e., γ is an isomorphism. We can reverse the full argument to see that $\widetilde{h}h$ is isomorphic to the identity map of spans on (n, Q, m).

This completes the proof that any two pullbacks of a given cospan are equivalent as spans. $\square$

Proposition 33. *The weak pullback in the* 2*-category* Grpd *described in Definition 27 has the same universal property described above.*

Proof. Given the weak pullback diagram:

$$\begin{tikzcd}
 & \mathcal{H}\times_{\mathcal{J}}\mathcal{G} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathcal{H} \arrow[dr, "f"'] & \overset{\cong}{\underset{\phi}{\Longleftarrow}} & \mathcal{G} \arrow[dl, "g"] \\
 & \mathcal{J} &
\end{tikzcd}$$

and any other weak pullback diagram:

$$\begin{tikzcd}
 & \mathcal{Q} \arrow[dl, "n"'] \arrow[dr, "m"] & \\
\mathcal{H} \arrow[dr, "f"'] & \overset{\cong}{\underset{\psi}{\Longleftarrow}} & \mathcal{G} \arrow[dl, "g"] \\
 & \mathcal{J} &
\end{tikzcd}$$

there is a functor:

$$h\colon \mathcal{Q} \to \mathcal{H} \times_{\mathcal{J}} \mathcal{G}$$

defined by:

$$x \mapsto (m(x), n(x), \psi_x)$$

$$(\rho\colon x \to x') \mapsto (m(\rho), n(\rho))\colon (m(x), n(x), \psi_x) \to (m(x'), n(x'), \psi_{x'})$$

and the requisite diagram commutes by naturality of ψ.

$$\begin{tikzcd} m(x) \arrow[r, "m(\rho)"] \arrow[d, "\psi_x"'] & m(x') \arrow[d, "\psi_{x'}"] \\ n(x) \arrow[r, "n(\rho)"'] & n(x') \end{tikzcd}$$

It is not difficult to check that this process preserves identities and composition. The required natural transformations are the identities, since $ph = m$ and $qh = n$, and the first equality of the universal property is easily checked as $\phi \cdot h = \psi$.

Given a pair of functors $j, h\colon \mathcal{Q} \to \mathcal{H} \times_{\mathcal{J}} \mathcal{G}$ and natural transformations $\alpha'\colon ph \Rightarrow pj$ and $\beta'\colon qh \Rightarrow qj$ such that:

$$\phi \cdot j \circ f \cdot \alpha' = g \cdot \beta' \circ \phi \cdot h,$$

then there is a natural transformation:

$$\gamma\colon h \to j$$

whose components $\gamma_x\colon h(x) \to j(x)$ are defined by:

$$(ph(x), qh(x), \psi_{h(x)}) \mapsto (pj(x), qj(x), \phi_{j(x)})$$

and the requisite square commutes by the above equality:

$$\begin{tikzcd} fph(x) \arrow[r, "f(\alpha'_x)"] \arrow[d, "\phi_{h(x)}"'] & fpj(x) \arrow[d, "\phi_{j(x)}"] \\ gqh(x) \arrow[r, "g(\beta'_x)"'] & gqj(x) \end{tikzcd}$$

Then it is clear that $p \cdot \gamma = \alpha'$ and $q \cdot \gamma = \beta'$. $\square$

5.4 The Bicategory of Spans

We prove a special case of a general fact concerning monoidal bicategories of spans arising from bicategories with finite limits. In particular, we consider the 2-category Grpd. We will see that inducing the monoidal structure on the span bicategory requires a certain relationship between binary products and pullbacks. Here, all limits are taken in the weak sense of limits in a bicategory.

Claim 34. *Given a bicategory $\mathcal{C}$ with weak pullbacks, then there is a tricategory* $\mathrm{Span}(\mathcal{C})$ *consisting of:*

- *objects of $\mathcal{C}$ as objects,*
- *spans in $\mathcal{C}$ as morphisms,*
- *maps of spans as* 2*-morphisms and*
- *maps of maps of spans as* 3*-morphisms.*

We do not use the full structure of this tricategory at any point in this work. Instead we prove a similar theorem, where we work with the 2-category Grpd and take isomorphism classes of equivalences of spans to obtain a bicategory:

Theorem 35. *Given the* 2*-category* Grpd*, there is a bicategory of spans* Span(Grpd) *which has:*

- *objects of* Grpd *as objects,*
- *spans in* Grpd *as morphisms,*
- *isomorphism classes of equivalences of spans in* Grpd *as* 2*-morphisms.*

Proof. We give the structure of the bicategory below. □

We give an explicit statement of the structure of the 2-category Span(Grpd):

- the objects are groupoids $\mathcal{G}, \mathcal{H}, \mathcal{J}, \ldots$;
- given a pair of groupoids $\mathcal{G}$,$\mathcal{H}$, there is a category $\mathrm{Span}(\mathcal{G}, \mathcal{H})$ with:

– spans from $\mathcal{G}$ to $\mathcal{H}$ in Grpd as objects — these are the 1-morphisms of Span(Grpd):

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

– isomorphism classes of equivalences of spans $[f]\colon \mathcal{S} \Rightarrow \mathcal{S}'$ as morphisms — these are the 2-morphisms of Span(Grpd), i.e. diagrams

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] \arrow[dd, "f"] & \\ \mathcal{H} \arrow[r, phantom, "\Searrow"] & & \mathcal{G} \arrow[l, phantom, "\Swarrow"] \\ & \mathcal{S}' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commuting up to natural isomorphisms in Grpd and such that f is an equivalence of groupoids satisfying the extra equations in the definition of equivalence of spans.

Composition and the identity in Span$(\mathcal{G}, \mathcal{H})$

- vertical composition of 2-morphisms is done as follows: given composable 2-morphisms:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[d, "f"] \arrow[dl, phantom, shift left=4, "\beta \Searrow"] \arrow[dr, phantom, shift right=4, "\Swarrow \alpha"] & \\ \mathcal{H} & \mathcal{S}' \arrow[l, "q'"] \arrow[r, "p'"'] \arrow[d, "f'"] & \mathcal{G} \\ & \mathcal{S}'' \arrow[ul, "q''"] \arrow[ur, "p''"'] \arrow[ul, phantom, shift right=4, "\beta' \Searrow"] \arrow[ur, phantom, shift left=4, "\Downarrow \alpha'"] & \end{tikzcd}$$

define the composite to be $[f'f]\colon \mathcal{S} \to \mathcal{S}''$ with natural isomorphisms:

$$(f \cdot \alpha') \circ \alpha\colon p \to p''f'f$$

and

$$(f \cdot \beta') \circ \beta\colon q \to q''f'f,$$

and given a span

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

- the identity 2-morphism for this composition consists of the identity functor $1_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$ and the pair of natural isomorphisms canonically isomorphic to the identities:

$$1_{[p]} \colon p \Rightarrow p1_{\mathcal{S}}$$

and

$$1_{[q]} \colon q \Rightarrow q1_{\mathcal{S}} :$$

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "1_{\mathcal{S}}"] & \\ \mathcal{H} \arrow[r, phantom, "\Downarrow"] & & \mathcal{G} \arrow[l, phantom, "\Downarrow"] \\ & \mathcal{S} \arrow[ul, "q"] \arrow[ur, "p"'] & \end{tikzcd}$$

The composition functor

- there is a functor called (horizontal) composition:

$$\circ_{\mathcal{G},\mathcal{H},\mathcal{J}} \colon \mathrm{Span}(\mathcal{G},\mathcal{H}) \times \mathrm{Span}(\mathcal{H},\mathcal{J}) \to \mathrm{Span}(\mathcal{G},\mathcal{J})$$

given by a choice of weak pullback for each cospan in Grpd.

The identity functor

- there is an identity functor $Id_{\mathcal{G}} \colon 1 \to \mathrm{Span}(\mathcal{G},\mathcal{G})$ for each groupoid $\mathcal{G}$, where 1 is the terminal category whose one object is sent to the span:

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "1_{\mathcal{G}}"'] \arrow[dr, "1_{\mathcal{G}}"] & \\ \mathcal{G} & & \mathcal{G} \end{tikzcd}$$

and the one morphism is sent to the isomorphism class of maps of spans: $[1_{\mathcal{G}}] \colon \mathcal{G} \Rightarrow \mathcal{G}$, which consists of the identity functor for $\mathcal{G}$ in Grpd and the identity natural isomorphisms $1_{1_{\mathcal{G}}} \colon 1_{\mathcal{G}} \Rightarrow 1_{\mathcal{G}}$ in Grpd.

The left and right unitor isomorphisms

- for each pair of groupoids, $\mathcal{G}, \mathcal{H}$ there are left and right unitor natural isomorphisms. We show the construction of the left unitor. The right unitor follows similarly. In components, there is an invertible isomorphism class of maps of spans for each span:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

That is, for each pair of groupoids $\mathcal{G}, \mathcal{H}$, a natural isomorphism:

$$\lambda_{\mathcal{G},\mathcal{H}} \colon \circ_{\mathcal{G},\mathcal{G},\mathcal{H}} \bar{\circ}(1 \times Id_{\mathcal{H}}) \Rightarrow 1,$$

whose component:

$$\lambda_{\mathcal{S}} \colon \circ (Id_{\mathcal{H}}, \mathcal{S}) \overset{\cong}{\Rightarrow} \mathcal{S},$$

for the given span is a map between the following span:

$$\begin{tikzcd} & & \mathcal{HS} \arrow[dl, "\pi_{\mathcal{H}}"'] \arrow[dr, "\pi_{\mathcal{S}}"] & & \\ & \mathcal{H} \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

and the given span:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

This map is straightforward to write down.

The associator isomorphism

- for each quadruple of groupoids, $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{K}$ there is an associator natural isomorphism. In components, there is an invertible isomorphism class of maps of spans for each triple of composable spans:

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "q''"'] \arrow[dr, "p''"] & & \mathcal{T} \arrow[dl, "q'"'] \arrow[dr, "p'"] & & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{K} & & \mathcal{J} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

That is, for each quadruple of groupoids $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{K}$, a natural isomorphism:

$$\alpha_{\mathcal{G},\mathcal{H},\mathcal{J},\mathcal{K}} \colon \circ_{\mathcal{G},\mathcal{H},\mathcal{K}} \bar{\circ}(1 \times \circ_{\mathcal{H},\mathcal{J},\mathcal{K}}) \Rightarrow \circ_{\mathcal{G},\mathcal{J},\mathcal{K}} \bar{\circ}(\circ_{\mathcal{G},\mathcal{H},\mathcal{J}} \times 1),$$

whose component:

$$\alpha_{\mathcal{S},\mathcal{T},\mathcal{U}} \colon \circ\,(\mathcal{S}, \circ(\mathcal{T},\mathcal{U})) \overset{\cong}{\Rightarrow} \circ(\circ(\mathcal{S},\mathcal{T}),\mathcal{U}),$$

a map between the following spans:

$$\begin{tikzcd}
 & & & \mathcal{S}(\mathcal{T}\mathcal{U}) \arrow[dl] \arrow[ddrr] & & & \\
 & & \mathcal{T}\mathcal{U} \arrow[dl] \arrow[dr] & & & & \\
 & \mathcal{U} \arrow[dl] \arrow[dr] & & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathcal{K} & & \mathcal{J} & & \mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

and:

$$\begin{tikzcd}
 & & & (\mathcal{S}\mathcal{T})\mathcal{U} \arrow[ddll] \arrow[dr] & & & \\
 & & & & \mathcal{S}\mathcal{T} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{U} \arrow[dl] \arrow[dr] & & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathcal{K} & & \mathcal{J} & & \mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

This map is straightforward to write down.

Now we show that the axioms hold:

- Since the object and morphisms of $\mathrm{Span}(\mathcal{G}, \mathcal{H})$ are the morphisms and 2-morphisms of $\mathrm{Span}(\mathrm{Grpd})$, it is clear that the source and target maps are well-defined for $\mathrm{Span}(\mathcal{G}, \mathcal{H})$.
- Given a span:

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

there is an identity morphism $[1_{\mathcal{S}}]\colon \mathcal{S} \to \mathcal{S}$ consisting of the identity $1_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{S}$ in Grpd and the canonical 2-morphisms $p \Rightarrow p1_{\mathcal{S}}$ and $q \Rightarrow q1_{\mathcal{S}}$,

- Composition of morphisms in $\mathrm{Span}(\mathcal{G}, \mathcal{H})$ is the same as composition of 2-morphisms in Span(Grpd) and was described above. Associativity follows from the associativity of composition of 2-morphisms in Grpd.

- Similarly, the left and right identity laws follow from the identity laws in Grpd. Thus, $\mathrm{Span}(\mathcal{G}, \mathcal{H})$ is a category.

- Given objects $\mathcal{G}, \mathcal{H}, \mathcal{J}$ in Span(Grpd) we define a composition functor

$$\circ_{\mathcal{G},\mathcal{H},\mathcal{J}}\colon \mathrm{Span}(\mathcal{G}, \mathcal{H}) \times \mathrm{Span}(\mathcal{H}, \mathcal{J}) \to \mathrm{Span}(\mathcal{G}, \mathcal{J}),$$

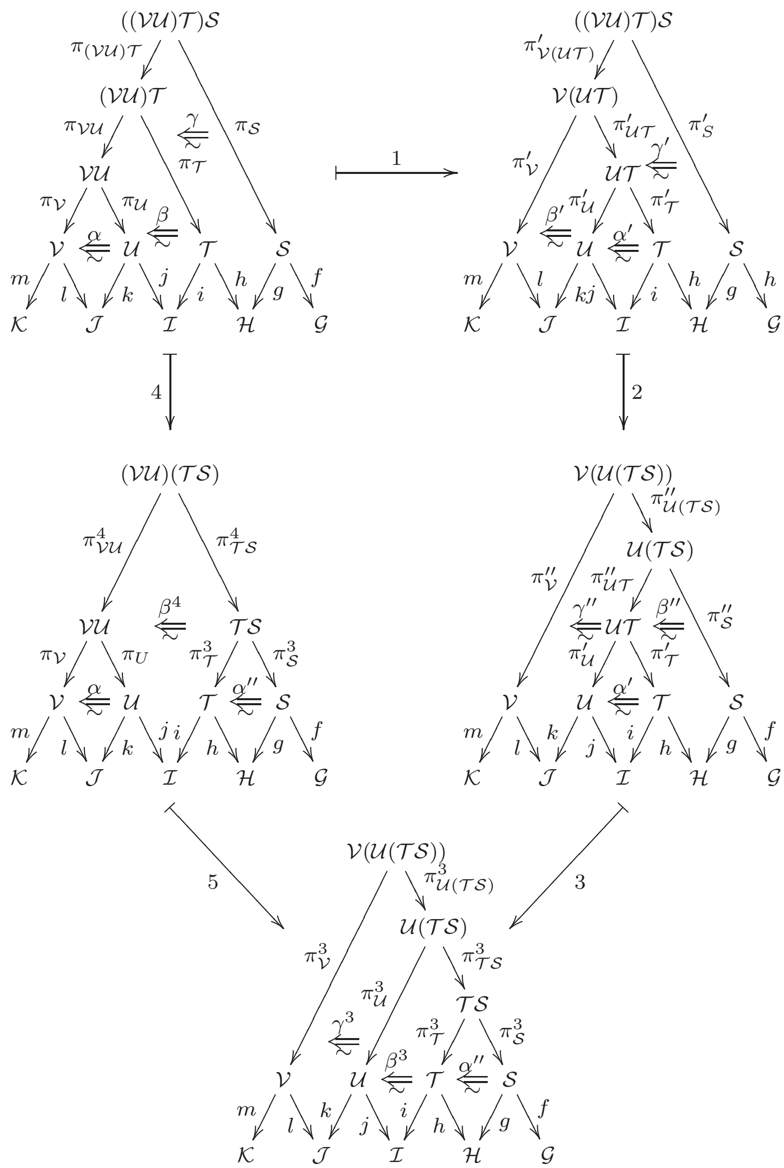
  by weak pullback of spans. Composition of maps between spans is given by the universal property of weak pullbacks as follows. First, given two pairs of composable spans with a map between them:

$$\begin{tikzcd}
 & & \mathcal{T}\mathcal{S} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{T} \arrow[dl] \arrow[dr] \arrow[dd] \arrow[rr, phantom, "\Rightarrow"] & & \mathcal{S} \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathcal{J} \arrow[r, phantom, "\Searrow\ \Swarrow"] & & \mathcal{H} \arrow[r, phantom, "\Searrow\ \Swarrow"] & & \mathcal{G} \\
 & \mathcal{T}' \arrow[ul] \arrow[ur] & & \mathcal{S}' \arrow[ul] \arrow[ur] \arrow[ll, phantom, "\Leftarrow"] & \\
 & & \mathcal{T}'\mathcal{S}' \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

  the composition functor gives the above picture. Applying the universal property to the lower diamond and the hexagon formed in the middle with top $\mathcal{T}\mathcal{S}$ and bottom $\mathcal{T}'\mathcal{S}'$, we obtain a map from $\mathcal{T}\mathcal{S}$ to $\mathcal{T}'\mathcal{S}'$ and a pair of 2-morphisms. This data is the image of the composition functor on a map of spans. From this description it can easily be seen that the identity 2-morphism is preserved. Also, the preservation of composition of maps of spans can be seen with one more application of the universal property of weak pullbacks.

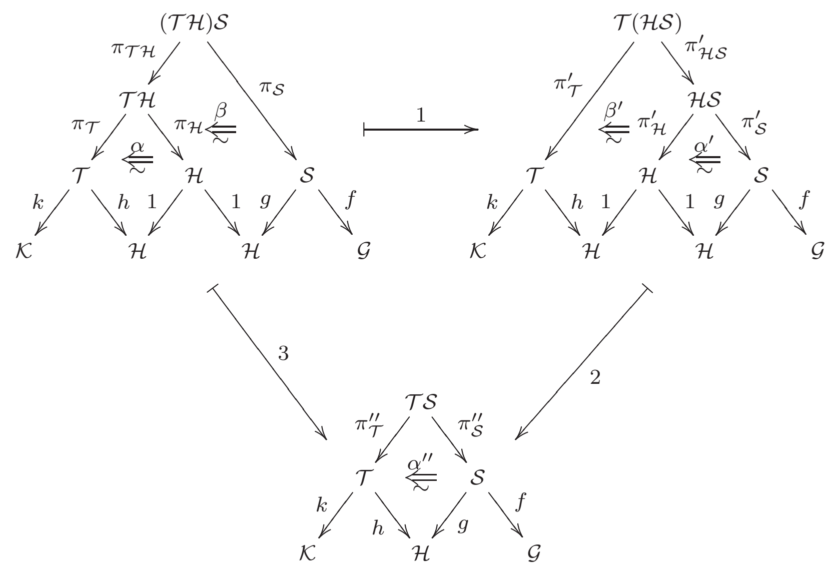
- Given an object $\mathcal{G}$ in Span(Grpd), there is a functor from the terminal category to $\mathrm{Span}(\mathcal{G}, \mathcal{G})$ defined by taking the one object to the identity span from $\mathcal{G}$ to $\mathcal{G}$ and the identity morphism to the isomorphism class of maps of spans $[1_{\mathcal{G}}]$. It is clear that this is a functor.

- We need to show that the following diagram commutes up to isomorphism. This tells us that any two ways of putting parentheses in a composite of spans will be isomorphic.



To check this axiom one only needs to write down explicit expressions for the maps between these diagrams. For each of the five arrows this will involve finding first one or two auxiliary maps to yield the final map. For this one needs the construction outlined in the description of the associator. It will be clear from construction that each of these maps is an equivalence of spans. We can then see that the pentagon commutes up to isomorphism. Thus, taking isomorphism classes of equivalences of spans as morphisms gives us strict commutativity of the pentagon.

We now check that the unitor triangle:



commutes. This is proved very similarly to the proof of the previous axiom.

5.5 The Monoidal Structure

We detail the monoidal structure of Span(Grpd) — the underlying bicategory. We will shorten our notation by denoting this bicategory simply by Span, when necessary. This consists of:

The monoidal product

- a homomorphism of bicategories:

$$\otimes\colon \mathrm{Span(Grpd)} \times \mathrm{Span(Grpd)} \to \mathrm{Span(Grpd)}$$

  called *composition* given by:

  - a function taking pairs of groupoids to their product in Grpd:

  $$(\mathcal{G}, \mathcal{H}) \longmapsto \mathcal{G} \times \mathcal{H},$$

  - for each pair of pairs of groupoids $(\mathcal{G}, \mathcal{H}), (\mathcal{G}', \mathcal{H}')$, a functor:

  $$\otimes\colon \mathrm{Span}(\mathcal{G}, \mathcal{H}) \times \mathrm{Span}(\mathcal{G}', \mathcal{H}') \to \mathrm{Span}(\mathcal{G} \times \mathcal{G}', \mathcal{H} \times \mathcal{H}')$$

  defined by the universal property of products of groupoids. Although many of the following constructions follow from this universal property, we choose to give explicit constructions, as proof by universal property is slick, yet monotonous.

$$\begin{tikzcd}[column sep=small]
 & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & & & \mathcal{S}' \arrow[dl, "q'"'] \arrow[dr, "p'"] & & & & \mathcal{S}\times\mathcal{S}' \arrow[dl, "\widetilde{q\times q'}"'] \arrow[dr, "\widetilde{p\times p'}"] & \\
\mathcal{H} & & \mathcal{G} & \mathcal{H}' & & \mathcal{G}' \arrow[r, mapsto, shift left=3ex] & & \mathcal{H}\times\mathcal{H}' & & \mathcal{G}\times\mathcal{G}'
\end{tikzcd}$$

where $\widetilde{p \times p'}$ and $\widetilde{q \times q'}$ are the functors induced by the universal property as shown in the following diagrams:

$$\begin{tikzcd}
\mathcal{S} \arrow[d, "p"'] & \mathcal{S}\times\mathcal{S}' \arrow[l, "\pi_{\mathcal{S}}"'] \arrow[r, "\pi_{\mathcal{S}'}"] \arrow[d, dashed, "\widetilde{p\times p'}"] \arrow[dl, phantom, "\alpha^{\sim}\Nwarrow"] \arrow[dr, phantom, "\Nearrow\alpha'^{\sim}"] & \mathcal{S}' \arrow[d, "p'"] \\
\mathcal{G} & \mathcal{G}\times\mathcal{G}' \arrow[l, "\pi_{\mathcal{G}}"] \arrow[r, "\pi_{\mathcal{G}'}"'] & \mathcal{G}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S} \arrow[d, "p"'] & \mathcal{S}\times\mathcal{S}' \arrow[l, "\pi_{\mathcal{S}}"'] \arrow[r, "\pi_{\mathcal{S}'}"] \arrow[d, dashed, "\widetilde{q\times q'}"] \arrow[dl, phantom, "\beta^{\sim}\Nwarrow"] \arrow[dr, phantom, "\Nearrow\beta'^{\sim}"] & \mathcal{S}' \arrow[d, "p'"] \\
\mathcal{H} & \mathcal{H}\times\mathcal{H}' \arrow[l, "\pi_{\mathcal{H}}"] \arrow[r, "\pi_{\mathcal{H}'}"'] & \mathcal{H}'
\end{tikzcd}$$

In fact, $\widetilde{p \times p'}$ is isomorphic to $p \times p'$, we only write down the diagrams for their utility in defining the isomorphism class of equivalences of spans, from a pair of such isomorphism classes. In particular, we have:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd] & & & \mathcal{S}' \arrow[dl, "q'"'] \arrow[dr, "p'"] \arrow[dd] & & & & \mathcal{S}\times\mathcal{S}' \arrow[dl, "\widetilde{q\times q'}"'] \arrow[dr, "\widetilde{p'\times p'}"] \arrow[dd] & \\
\mathcal{H} \arrow[r, phantom, "\Downarrow\sigma'"] & & \mathcal{G} \arrow[l, phantom, "\sigma\Downarrow"] & \mathcal{H}' \arrow[r, phantom, "\Downarrow\tau'"] & & \mathcal{G}' \arrow[l, phantom, "\tau\Downarrow"] \arrow[rr, mapsto] & & \mathcal{H}\times\mathcal{H}' \arrow[r, phantom, "\Downarrow"] & & \mathcal{G}\times\mathcal{G}' \arrow[l, phantom, "\Downarrow"] \\
 & \mathcal{T} \arrow[ul, "n"] \arrow[ur, "m"'] & & & \mathcal{T}' \arrow[ul, "n'"] \arrow[ur, "m'"'] & & & & \mathcal{T}\times\mathcal{T}' \arrow[ul, "\widetilde{n\times n'}"] \arrow[ur, "\widetilde{m\times m'}"'] &
\end{tikzcd}$$

where the map of spans is:

$$\begin{tikzcd}[column sep=small]
 & & & \mathcal{S} \arrow[dlll, bend right, "p"'] \arrow[dll, dashed] \arrow[dl, "q"'] \arrow[dd] & \mathcal{S}\times\mathcal{S}' \arrow[l, "\pi_{\mathcal{S}}"'] \arrow[r, "\pi_{\mathcal{S}'}"] \arrow[dd, dashed] & \mathcal{S}' \arrow[dr, "p'"] \arrow[drr, dashed] \arrow[drrr, bend left, "n'"] \arrow[dd] & & & \\
\mathcal{G} & \mathcal{G}\times\mathcal{G}' \arrow[l, "\pi_{\mathcal{G}'}"] \arrow[r, "\pi_{\mathcal{G}}"'] & \mathcal{G}' & & & & \mathcal{H} & \mathcal{H}\times\mathcal{H}' \arrow[l, "\pi_{\mathcal{H}}"] \arrow[r, "\pi_{\mathcal{H}'}"'] & \mathcal{H}' \\
 & & & \mathcal{T} \arrow[ulll, bend left, "m"] \arrow[ull, dashed] \arrow[ul, "n"] & \mathcal{T}\times\mathcal{T}' \arrow[l, "\pi_{\mathcal{T}}"] \arrow[r, "\pi_{\mathcal{T}'}"'] & \mathcal{T}' \arrow[ur, "m'"] \arrow[urr, dashed] \arrow[urrr, bend right, "q'"'] & & &
\end{tikzcd}$$

where the specification of the 2-cells is completely straightforward from the discussion above.

– for each triple of pairs of groupoids $(\mathcal{G}, \mathcal{G}'), (\mathcal{H}, \mathcal{H}'), (\mathcal{K}, \mathcal{K}')$, a natural isomorphism:

$$\phi \colon c \circ (\otimes \times \otimes) \Rightarrow \otimes \circ (c \times c)$$

that is, for each pair of pairs of composable spans:

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & & & \mathcal{T}' \arrow[dl] \arrow[dr] & & \mathcal{S}' \arrow[dl] \arrow[dr] & \\
\mathcal{K} & & \mathcal{H} & & \mathcal{G} & \mathcal{K}' & & \mathcal{H}' & & \mathcal{G}'
\end{tikzcd}
$$

there is an equivalence of spans, which is necessarily invertible, between:

$$
\begin{tikzcd}[column sep=small]
 & & (\mathcal{T}\times\mathcal{T}')(\mathcal{S}\times\mathcal{S}') \arrow[dl] \arrow[dr] & & & & \\
 & \mathcal{T}\times\mathcal{T}' \arrow[dl] \arrow[dr] & & \mathcal{S}\times\mathcal{S}' \arrow[dl] \arrow[dr] & & \mathcal{S}\mathcal{T}\times\mathcal{S}'\mathcal{T}' \arrow[dl] \arrow[dr] & \\
\mathcal{K}\times\mathcal{K}' & & \mathcal{H}\times\mathcal{H}' & & \mathcal{G}\times\mathcal{G}' \quad \mathcal{K}\times\mathcal{K}' & & \mathcal{G}\times\mathcal{G}'
\end{tikzcd}
$$

This map is completely straightforward to write down.

– for each pair of groupoids $\mathcal{G}, \mathcal{H}$, a natural isomorphism:

$$\phi \colon I_{\otimes} \Rightarrow I \times I \circ \otimes$$

which is just the identity map of spans on:

$$
\begin{tikzcd}
 & \mathcal{G}\times\mathcal{H} \arrow[dl, "1"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & \mathcal{G}\times\mathcal{H}
\end{tikzcd}
$$

– check axioms

The monoidal unit

- a homomorphism of bicategories:

$$I \colon \mathbf{1} \to \mathrm{Span}(\mathrm{Grpd})$$

where $\mathbf{1}$ denotes the unit bicategory, given by:

– the terminal groupoid 1,

– the span and map of spans consisting of only the terminal groupoid and identity natural isomorphisms.

– check axioms

The monoidal associator

- a pseudo natural equivalence $\mathbf{a}$

$$\begin{tikzcd}
\mathrm{Span}\times\mathrm{Span}\times\mathrm{Span} \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\mathbf{a}"] & \mathrm{Span}\times\mathrm{Span} \arrow[d, "\otimes"] \\
\mathrm{Span}\times\mathrm{Span} \arrow[r, "\otimes"'] & \mathrm{Span}
\end{tikzcd}$$

in $\mathbf{Bicat}(\mathrm{Span}\times\mathrm{Span}\times\mathrm{Span},\mathrm{Span})$ given by:

  - for each triple of groupoids $\mathcal{G},\mathcal{H},\mathcal{K}$, a span $(\mathcal{G}\times\mathcal{H})\times\mathcal{K}$:

$$\begin{tikzcd}
& (\mathcal{G}\times\mathcal{H})\times\mathcal{K} \arrow[dl, "a"'] \arrow[dr, "1"] & \\
\mathcal{G}\times(\mathcal{H}\times\mathcal{K}) & & (\mathcal{G}\times\mathcal{H})\times\mathcal{K}
\end{tikzcd}$$

    where a is the associator in Grpd,

  - for each pair of triples of groupoids $(\mathcal{G},\mathcal{H},\mathcal{K}),(\mathcal{G}',\mathcal{H}',\mathcal{K}')$, a natural isomorphism

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{G},\mathcal{G}')\times\mathrm{Span}(\mathcal{H},\mathcal{H}')\times\mathrm{Span}(\mathcal{K},\mathcal{K}') \arrow[r, "\otimes\circ(\otimes\times 1)"] \arrow[d, "\otimes\circ(1\times\otimes)"'] \arrow[dr, phantom, "\nearrow"] & \mathrm{Span}((\mathcal{G}\times\mathcal{H})\times\mathcal{K},(\mathcal{G}'\times\mathcal{H}')\times\mathcal{K}') \arrow[d, "((\mathcal{G}'\times\mathcal{H}')\times\mathcal{K}')_*"] \\
\mathrm{Span}(\mathcal{G}\times(\mathcal{H}\times\mathcal{K}),\mathcal{G}'\times(\mathcal{H}'\times\mathcal{K}')) \arrow[r, "((\mathcal{G}\times\mathcal{H})\times\mathcal{K})^*"'] & \mathrm{Span}((\mathcal{G}\times\mathcal{H})\times\mathcal{K},\mathcal{G}\times(\mathcal{H}\times\mathcal{K}))
\end{tikzcd}$$

    that is, for each triple of spans:

$$\begin{tikzcd}
& \mathcal{R} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{T} \arrow[dl] \arrow[dr] & \\
\mathcal{G}' & & \mathbf{X}\ \mathcal{H}' & & \mathcal{H}\ \ \mathcal{K}' & & \mathcal{K}
\end{tikzcd}$$

    there is an invertible isomorphism class of equivalences of spans from the composite of the first of the following diagrams to the composite of the second:

$$\begin{tikzcd}
& \mathcal{R}\times(\mathcal{S}\times\mathcal{T}) \arrow[dl] \arrow[dr] & & (\mathcal{G}\times\mathcal{H})\times\mathcal{K} \arrow[dl] \arrow[dr] & \\
\mathcal{G}'\times(\mathcal{H}'\times\mathcal{K}') & & \mathcal{G}\times(\mathcal{H}\times\mathcal{K}) & & (\mathcal{G}\times\mathcal{H})\times\mathcal{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{G}'\times\mathcal{H}')\times\mathcal{K}' \arrow[dl] \arrow[dr] & & (\mathcal{R}\times\mathcal{S})\times\mathcal{T} \arrow[dl] \arrow[dr] & \\
\mathcal{G}'\times(\mathcal{H}'\times\mathcal{K}') & & (\mathcal{G}'\times\mathcal{H}')\times\mathcal{K}' & & (\mathcal{G}\times\mathcal{H})\times\mathcal{K}
\end{tikzcd}$$

    Again this is straightforward to write down.

– Check axioms.

The monoidal unitors

- pseudo natural equivalences $\mathbf{l}$ and $\mathbf{r}$

$$\begin{tikzcd} & \mathrm{Span}\times\mathrm{Span} \arrow[dr, "\otimes"] & \\ \mathrm{Span} \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & \Downarrow\mathbf{l} & \mathrm{Span} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Span}\times\mathrm{Span} \arrow[dr, "\otimes"] & \\ \mathrm{Span} \arrow[ur, "1\times I"] \arrow[rr, "1"'] & \Downarrow\mathbf{r} & \mathrm{Span} \end{tikzcd}$$

in $\mathbf{Bicat}(\mathrm{Span}, \mathrm{Span})$

– For each groupoid $\mathcal{G}$, spans for $\mathbf{l}$ and $\mathbf{r}$, respectively:

$$\begin{tikzcd} & 1\times\mathcal{G} \arrow[dl, "1"'] \arrow[dr, "l"] & \\ \mathcal{G} & & 1\times\mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{G}\times 1 \arrow[dl, "1"'] \arrow[dr, "r"] & \\ \mathcal{G} & & \mathcal{G}\times 1 \end{tikzcd}$$

where l and r are the left and right unitors, respectively, in Grpd. For each span:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

there is an invertible isomorphism class of maps of spans for $\mathbf{l}$ from the composite of the first of the following diagrams to the composite of the second:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & & 1\times\mathcal{G} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} & & 1\times\mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & 1\times\mathcal{H} \arrow[dl] \arrow[dr] & & 1\times\mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & 1\times\mathcal{H} & & 1\times\mathcal{G} \end{tikzcd}$$

and similarly for $\mathbf{r}$:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{G}\times 1 \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} & & \mathcal{G}\times 1 \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{H}\times 1 \arrow[dl] \arrow[dr] & & \mathcal{S}\times 1 \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{H}\times 1 & & \mathcal{G}\times 1 \end{tikzcd}$$

Again these are straightforward to write down.

– Check axioms.

The pentagonator

- an invertible modification π

$$
\begin{tikzcd}
 & \mathrm{Span}^4 \arrow[rr, "\otimes\times 1\times 1"] \arrow[dl, "1\times 1\times\otimes"'] \arrow[dr, "1\times\otimes\times 1"] & & \mathrm{Span}^3 \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow a\times 1"] & \\
\mathrm{Span}^3 \arrow[dr, "1\times\otimes"'] \arrow[rr, phantom, "\overset{1\times a}{\Leftarrow}"] & & \mathrm{Span}^3 \arrow[rr, "\otimes\times 1"] \arrow[dl, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow a"] & & \mathrm{Span}^2 \arrow[dl, "\otimes"] \\
 & \mathrm{Span}^2 \arrow[rr, "\otimes"'] & & \mathrm{Span} &
\end{tikzcd}
\overset{\pi}{\Rrightarrow}
\begin{tikzcd}
 & \mathrm{Span}^4 \arrow[rr, "\otimes\times 1\times 1"] \arrow[dl, "1\times 1\times\otimes"'] \arrow[dr, phantom, "="] & & \mathrm{Span}^3 \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & \\
\mathrm{Span}^3 \arrow[rr, "\otimes\times 1"] \arrow[dr, "1\times\otimes"'] & & \mathrm{Span}^2 \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{a}{\Leftarrow}"] \arrow[d, phantom, "\Downarrow a"] & & \mathrm{Span}^2 \arrow[dl, "\otimes"] \\
 & \mathrm{Span}^2 \arrow[rr, "\otimes"'] & & \mathrm{Span} &
\end{tikzcd}
$$

in the bicategory **Bicat**$(\mathrm{Span}^4, \mathrm{Span})$, for example;

  - This modification can be written as an arrow between composites:

    $$\pi\colon \otimes\cdot(1\times\alpha)\circ\alpha\cdot(1\times\otimes\times 1)\circ\otimes\cdot(\alpha\times 1) \Rrightarrow \alpha\cdot(1\times 1\times\otimes)\circ\otimes\cdot 1\circ\alpha\cdot(\otimes\times 1\times 1)$$

    that is, for each quadruple of groupoids $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{K}$, there is an invertible isomorphism class of maps of spans (defined up to associativity in Grpd by the choice associativity of composition) between the following triples of composable spans:

$$
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{G}\times(\mathcal{H}\times\mathcal{J}))\times\mathcal{K} \arrow[dl, "a\times 1"'] \arrow[dr, "1"] & & \mathcal{G}\times((\mathcal{H}\times\mathcal{J})\times\mathcal{K}) \arrow[dl, "a"'] \arrow[dr, "1"] & & \mathcal{G}\times(\mathcal{H}\times(\mathcal{J}\times\mathcal{K})) \arrow[dl, "1\times a"'] \arrow[dr, "1"] & \\
((\mathcal{G}\times\mathcal{H})\times\mathcal{J})\times\mathcal{K} & & (\mathcal{G}\times(\mathcal{H}\times\mathcal{J}))\times\mathcal{K} & & \mathcal{G}\times((\mathcal{H}\times\mathcal{J})\times\mathcal{K}) & & \mathcal{G}\times(\mathcal{H}\times(\mathcal{J}\times\mathcal{K}))
\end{tikzcd}
$$

    and

$$
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{G}\times\mathcal{H})\times(\mathcal{J}\times\mathcal{K}) \arrow[dl, "a"'] \arrow[dr, "1"] & & (\mathcal{G}\times\mathcal{H})\times(\mathcal{J}\times\mathcal{K}) \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{G}\times(\mathcal{H}\times(\mathcal{J}\times\mathcal{K})) \arrow[dl, "a"'] \arrow[dr, "1"] & \\
((\mathcal{G}\times\mathcal{H})\times\mathcal{J})\times\mathcal{K} & & (\mathcal{G}\times\mathcal{H})\times(\mathcal{J}\times\mathcal{K}) & & (\mathcal{G}\times\mathcal{H})\times(\mathcal{J}\times\mathcal{K}) & & \mathcal{G}\times(\mathcal{H}\times(\mathcal{J}\times\mathcal{K}))
\end{tikzcd}
$$

    Again this map is straightforward to write down.

  - Check axioms

The monoidal middle

- an invertible modification μ:

$$
\begin{tikzcd}
\mathrm{Span}^2 \arrow[dr, "1\times I\times 1"] \arrow[rr, bend left, "1"] \arrow[ddr, bend right, "1"'] & \arrow[d, phantom, "\Uparrow r\times 1"] & \mathrm{Span}^2 \arrow[dd, "\otimes"] \\
\arrow[r, phantom, "\Leftarrow 1\times l"] & \mathrm{Span}^3 \arrow[ur, "\otimes\times 1"'] \arrow[d, "1\times\otimes"] \arrow[r, phantom, "\Downarrow a"] & \\
 & \mathrm{Span}^2 \arrow[r, "\otimes"'] & \mathrm{Span}
\end{tikzcd}
\overset{\mu}{\Rrightarrow}
\begin{tikzcd}
\mathrm{Span}^2 \arrow[rr, bend left, "1"] \arrow[dr, bend right, "1"'] & & \mathrm{Span}^2 \arrow[d, "\otimes"] \arrow[dl, phantom, "\overset{1}{\Leftarrow}"] \\
 & \mathrm{Span}^2 \arrow[r, "\otimes"'] & \mathrm{Span}
\end{tikzcd}
$$

– This modification can be written as an arrow between composites:

$$\mu\colon \otimes\cdot(1\times l)\circ a\cdot(1\times I\times 1)\circ\otimes\cdot(r\times 1)^{-1}\Rrightarrow 1$$

that is, for each pair of groupoids $\mathcal{G},\mathcal{H}$, there is an invertible isomorphism class of maps of spans (defined up to associativity in Grpd by the choice associativity of composition) between the composite of the following diagram:

$$\begin{tikzcd}[column sep=small]
& \mathcal{G}\times(1\times\mathcal{H}) \arrow[dl, "1\times l"'] \arrow[dr, "1"] & & (\mathcal{G}\times 1)\times\mathcal{H} \arrow[dl, "a"'] \arrow[dr, "1"] & & \mathcal{G}\times\mathcal{H} \arrow[dl, "r^{-1}\times 1"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & \mathcal{G}\times(1\times\mathcal{H}) & & (\mathcal{G}\times 1)\times\mathcal{H} & & \mathcal{G}\times\mathcal{H}
\end{tikzcd}$$

and the identity isomorphism class on the span $\mathcal{G}\times\mathcal{H}$. Again this map is straightforward to write down.

– Check axioms

The left unitorator

- an invertible modification λ:

$$\begin{tikzcd}[column sep=small]
& \mathrm{Span}^3 \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow l\times 1"] & \\
\mathrm{Span}^2 \arrow[ur, "I\times 1\times 1"] \arrow[rr, "1"'] \arrow[d, "\otimes"'] & {} \arrow[d, phantom, "="] & \mathrm{Span}^2 \arrow[d, "\otimes"] \\
\mathrm{Span} \arrow[rr, "1"'] & {} & \mathrm{Span}
\end{tikzcd}
\overset{\lambda}{\Rrightarrow}
\begin{tikzcd}[column sep=small]
& \mathrm{Span}^3 \arrow[dr, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & \\
\mathrm{Span}^2 \arrow[ur, "I\times 1\times 1"] \arrow[d, "\otimes"'] \arrow[r, phantom, "="] & \mathrm{Span}^2 \arrow[dr, "\otimes"] \arrow[r, phantom, "\Downarrow a"] \arrow[d, phantom, "\Downarrow l"] & \mathrm{Span}^2 \arrow[d, "\otimes"] \\
\mathrm{Span} \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & {} & \mathrm{Span}
\end{tikzcd}$$

– This modification can be written as an arrow between composites:

$$\lambda\colon 1\circ\otimes\cdot(l\times 1)\Rrightarrow l\cdot\otimes\circ\otimes\cdot 1\circ a\cdot(I\times 1\times 1)$$

that is, for each pair of groupoids $\mathcal{G},\mathcal{H}$, there is an invertible isomorphism class of maps of spans (defined up to associativity in Grpd by the choice associativity of composition) between the composites of the following diagrams:

$$\begin{tikzcd}[column sep=small]
& \mathcal{G}\times\mathcal{H} \arrow[dl, "1"'] \arrow[dr, "1"] & & (1\times\mathcal{G})\times\mathcal{H} \arrow[dl, "l\times 1"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & \mathcal{G}\times\mathcal{H} & & (1\times\mathcal{G})\times\mathcal{H}
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
& 1\times(\mathcal{G}\times\mathcal{H}) \arrow[dl, "l"'] \arrow[dr, "1"] & & 1\times(\mathcal{G}\times\mathcal{H}) \arrow[dl, "1"'] \arrow[dr, "1"] & & 1\times(\mathcal{G}\times\mathcal{H}) \arrow[dl, "a"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & 1\times(\mathcal{G}\times\mathcal{H}) & & 1\times(\mathcal{G}\times\mathcal{H}) & & (1\times\mathcal{G})\times\mathcal{H}
\end{tikzcd}$$

Again this map is straightforward to write down.

– check axioms.

The right unitorator

- an invertible modification ρ:

$$
\begin{tikzcd}
\mathrm{Span} \arrow[rr, "1"] \arrow[drr, phantom, "="] & & \mathrm{Span} \arrow[r, phantom, "\overset{\rho}{\Rrightarrow}"] & \mathrm{Span} \arrow[rr, "1"] \arrow[dr, "1\times I"'] & \arrow[d, phantom, "\Uparrow r"] & \mathrm{Span} \\
\mathrm{Span}^2 \arrow[u, "\otimes"] \arrow[rr, "1"] \arrow[dr, "1\times 1\times I"'] & \arrow[d, phantom, "\Uparrow 1\times r"] & \mathrm{Span}^2 \arrow[u, "\otimes"'] & \mathrm{Span}^2 \arrow[u, "\otimes"] \arrow[dr, "1\times 1\times I"'] \arrow[r, phantom, "="] & \mathrm{Span}^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\Rightarrow a"] & \mathrm{Span}^2 \arrow[u, "\otimes"'] \\
& \mathrm{Span}^3 \arrow[ur, "1\times\otimes"'] & & & \mathrm{Span}^3 \arrow[u, "\otimes\times 1"] \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
$$

  – This modification can be written as an arrow between composites:

$$\rho\colon 1 \circ \otimes \cdot (1 \times r) \Rrightarrow r \cdot \otimes \circ \otimes \cdot 1 \circ a^{-1} \cdot (1 \times 1 \times I)$$

  that is, for each pair of groupoids $\mathcal{G}, \mathcal{H}$, there is an invertible isomorphism class of maps of spans (defined up to associativity in Grpd by the choice associativity of composition) between the composites of the following diagrams:

$$
\begin{tikzcd}
& \mathcal{G}\times\mathcal{H} \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{G}\times(\mathcal{H}\times 1) \arrow[dl, "1\times r"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & \mathcal{G}\times\mathcal{H} & & \mathcal{G}\times(\mathcal{H}\times 1)
\end{tikzcd}
$$

  and

$$
\begin{tikzcd}
& (\mathcal{G}\times\mathcal{H})\times 1 \arrow[dl, "r"'] \arrow[dr, "1"] & & (\mathcal{G}\times\mathcal{H})\times 1 \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{G}\times(\mathcal{H}\times 1) \arrow[dl, "a^{-1}"'] \arrow[dr, "1"] & \\
\mathcal{G}\times\mathcal{H} & & (\mathcal{G}\times\mathcal{H})\times 1 & & (\mathcal{G}\times\mathcal{H})\times 1 & & \mathcal{G}\times(\mathcal{H}\times 1)
\end{tikzcd}
$$

  Again this map is straightforward to write down.

  – check axioms.

Chapter 6

Degroupoidification

In this section we describe a systematic process for turning groupoids into vector spaces and spans into linear operators. This process, 'degroupoidification', is in fact a kind of functor. 'Groupoidification' is the attempt to *undo* this functor. To 'groupoidify' a piece of linear algebra means to take some structure built from vector spaces and linear operators and try to find interesting groupoids and spans that degroupoidify to give this structure. So, to understand groupoidification, we need to master degroupoidification.

6.1 Defining Degroupoidification

We begin by describing how to turn a groupoid into a vector space.

Definition 36. *Given a groupoid $\mathcal{G}$, let $\underline{\mathcal{G}}$ be the set of isomorphism classes of objects of $\mathcal{G}$.*

Definition 37. *Given a groupoid $\mathcal{G}$, let the* **degroupoidification** *of $\mathcal{G}$ be the vector space*

$$\mathbb{R}[\underline{\mathcal{G}}] = \{\sum_{x \in \underline{\mathcal{G}}} c_x x \mid c_x \in \mathbb{R}\}.$$

A groupoid over a groupoid $v\colon \mathcal{V} \to \mathcal{G}$ will give a vector in $\mathbb{R}[\underline{\mathcal{G}}]$. To construct this, we use the concept of groupoid cardinality:

Definition 38. *The* **cardinality** *of a groupoid $\mathcal{G}$ is*

$$|\mathcal{G}| = \sum_{[x] \in \underline{\mathcal{G}}} \frac{1}{|\mathrm{Aut}(x)|}$$

where $|\mathrm{Aut}(x)|$ is the cardinality of the automorphism group of an object x in $\mathcal{G}$.

Since all of our groupoids are finite in this work, the cardinality of a groupoid is a well-defined nonnegative rational number.

Lemma 39. *Given equivalent groupoids $\mathcal{G}$ and $\mathcal{H}$, $|\mathcal{G}| = |\mathcal{H}|$.*

Proof. From a functor $p\colon \mathcal{G} \to \mathcal{H}$ between groupoids, we can obtain a function $\underline{p}\colon \underline{\mathcal{G}} \to \underline{\mathcal{H}}$. If p is an equivalence, $\underline{p}$ is a bijection. Since these are the indexing sets for the sum in the definition of groupoid cardinality, we just need to check that for a pair of elements $[x] \in \underline{\mathcal{G}}$ and $[y] \in \underline{\mathcal{H}}$ such that $\underline{p}([x]) = [y]$, we have $|\mathrm{Aut}(x)| = |\mathrm{Aut}(y)|$. This follows from p being full and faithful, and that the cardinality of automorphism groups is an invariant of an isomorphism class of objects in a groupoid. Thus,

$$|\mathcal{G}| = \sum_{x \in \underline{\mathcal{G}}} \frac{1}{|\mathrm{Aut}(x)|} = \sum_{y \in \underline{\mathcal{H}}} \frac{1}{|\mathrm{Aut}(y)|} = |\mathcal{H}|.$$

□

With the concept of groupoid cardinality in hand, we now describe how to obtain a vector in $\mathbb{R}[\underline{\mathcal{G}}]$ from a groupoid over $\mathcal{G}$.

Definition 40. *Given a groupoid $\mathcal{G}$, a* **groupoid over** $\mathcal{G}$ *is a groupoid $\mathcal{V}$ equipped with a functor $v\colon \mathcal{V} \to \mathcal{G}$.*

Definition 41. *Given a groupoid over $\mathcal{G}$, say $v\colon \mathcal{V} \to G$, and an object $x \in G$, we define the* **full inverse image** *of x, denoted $v^{-1}(x)$, to be the groupoid where:*

- *an object is an object $a \in \mathcal{V}$ such that $v(a) \cong x$;*
- *a morphism $f\colon a \to a'$ is any morphism in $\mathcal{V}$ from a to a'.*

We sometimes loosely say that $\mathcal{V}$ is a groupoid over $\mathcal{G}$. When we do this, we are referring to a functor $v\colon \mathcal{V} \to \mathcal{G}$.

Definition 42. *Given a groupoid over $\mathcal{G}$, say $v\colon \mathcal{V} \to \mathcal{G}$, there is a vector $\mathcal{D}(V) \in \mathbb{R}[\underline{\mathcal{G}}]$ defined by the coefficients $|v^{-1}(x)|$ for each basis vector $[x]$.*

Both addition and scalar multiplication of vectors have groupoidified analogues. We can add two groupoids $\mathcal{V}, \mathcal{U}$ over $\mathcal{G}$ by taking their coproduct, i.e., the disjoint union of $\mathcal{V}$ and $\mathcal{U}$ with the obvious map to $\mathcal{G}$:

$$\begin{tikzcd} \mathcal{V}+\mathcal{U} \arrow[d] \\ \mathcal{G} \end{tikzcd}$$

We then have:

Proposition. *Given groupoids $\mathcal{V}$ and $\mathcal{U}$ over $\mathcal{G}$,*

$$\mathcal{D}(\mathcal{V}+\mathcal{U}) = \mathcal{D}(\mathcal{V}) + \mathcal{D}(\mathcal{U}).$$

Proof. This will appear later as part of Lemma 46, which also considers infinite sums. $\square$

We can also multiply a groupoid over $\mathcal{G}$ by a 'scalar' — that is, a fixed groupoid. Given a groupoid over $\mathcal{G}$, say $v\colon \mathcal{V} \to \mathcal{G}$, and a groupoid Λ, the cartesian product $\Lambda \times \mathcal{V}$ becomes a groupoid over $\mathcal{G}$ as follows:

$$\begin{tikzcd} \Lambda \times \mathcal{V} \arrow[d, "v\pi_2"] \\ \mathcal{G} \end{tikzcd}$$

where $\pi_2\colon \Lambda \times \mathcal{V} \to \mathcal{V}$ is projection onto the second factor. We then have:

Proposition. *Given a groupoid Λ and a groupoid $\mathcal{V}$ over $\mathcal{G}$, the groupoid $\Lambda \times \mathcal{V}$ over $\mathcal{G}$ satisfies*

$$\mathcal{D}(\Lambda \times \mathcal{V}) = |\Lambda|\mathcal{D}(\mathcal{V}).$$

Proof. This is proved as Proposition 55. $\square$

We have seen how degroupoidification turns a groupoid $\mathcal{G}$ into a vector space $\mathbb{R}[\underline{\mathcal{G}}]$. Degroupoidification also turns any span of groupoids into a linear operator.

Definition 43. *Given groupoids $\mathcal{G}$ and $\mathcal{H}$, a* **span** *from $\mathcal{G}$ to $\mathcal{H}$ is a diagram*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

where $\mathcal{S}$ is groupoid and $p\colon \mathcal{S} \to \mathcal{G}$ and $q\colon \mathcal{S} \to \mathcal{H}$ are functors.

To turn a span of groupoids into a linear operator, we employ the weak pullback. This construction will let us apply a span from $\mathcal{G}$ to $\mathcal{H}$ to a groupoid over $\mathcal{G}$ in order to obtain a groupoid over $\mathcal{H}$. Then, since a groupoid over $\mathcal{G}$ gives a vector in $\mathbb{R}[\underline{\mathcal{G}}]$, while a groupoid over $\mathcal{H}$ gives a vector in $\mathbb{R}[\underline{\mathcal{H}}]$, a span from $\mathcal{G}$ to $\mathcal{H}$ will give a map from $\mathbb{R}[\underline{\mathcal{G}}]$ to $\mathbb{R}[\underline{\mathcal{H}}]$. Moreover, this map will be linear.

Given a span of groupoids:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

and a groupoid over $\mathcal{G}$:

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "v"'] \\ \mathcal{G} & \end{tikzcd}$$

we can take the weak pullback, which we call $\mathcal{SV}$:

$$\begin{tikzcd} & & \mathcal{SV} \arrow[dl, "\pi_{\mathcal{S}}"'] \arrow[dr, "\pi_{\mathcal{V}}"] & \\ & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & & \mathcal{V} \arrow[dl, "v"] \\ \mathcal{H} & & \mathcal{G} & \end{tikzcd}$$

and think of $\mathcal{SV}$ as a groupoid over $\mathcal{H}$:

$$\begin{tikzcd} & \mathcal{SV} \arrow[dl, "q\pi_{\mathcal{S}}"'] \\ \mathcal{H} & \end{tikzcd}$$

This process will determine a linear operator from $\mathbb{R}[\underline{\mathcal{G}}]$ to $\mathbb{R}[\underline{\mathcal{H}}]$.

Theorem. *Given a span:*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

there exists a unique linear operator

$$\mathcal{D}(\mathcal{S})\colon \mathbb{R}[\underline{\mathcal{G}}] \to \mathbb{R}[\underline{\mathcal{H}}]$$

such that

$$\mathcal{D}(\mathcal{S})\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{SV})$$

whenever $\mathcal{V}$ is a groupoid over $\mathcal{G}$.

Proof. This is Theorem 49. $\square$

Theorem 52 gives an explicit formula for the the operator corresponding to a span $\mathcal{S}$ from $\mathcal{G}$ to $\mathcal{H}$. Since $\underline{\mathcal{G}}$ and $\underline{\mathcal{H}}$ are finite, then $\mathbb{R}[\underline{\mathcal{G}}]$ has a basis given by the isomorphism classes $[x]$ in $\mathcal{G}$, and similarly for $\mathbb{R}[\underline{\mathcal{H}}]$. With respect to these bases, the matrix entries of $\mathcal{D}(\mathcal{S})$ are given as follows:

$$\mathcal{D}(\mathcal{S})_{[y][x]} = \sum_{[s]\in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)|} \tag{6.1}$$

where $|\mathrm{Aut}(x)|$ is the set cardinality of the automorphism group of $x \in \mathcal{G}$, and similarly for $|\mathrm{Aut}(s)|$.

As with vectors, there are groupoidified analogues of addition and scalar multiplication for operators. Given two spans from $\mathcal{G}$ to $\mathcal{H}$:

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

we can add them as follows. By the universal property of the coproduct we obtain from the right legs of the above spans a functor from the disjoint union $\mathcal{S}+\mathcal{T}$ to $\mathcal{G}$. Similarly, from the left legs of the above spans, we obtain a functor from $\mathcal{S}+\mathcal{T}$ to $\mathcal{H}$. Thus, we obtain a span

$$\begin{tikzcd} & \mathcal{S}+\mathcal{T} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

This addition of spans is compatible with degroupoidification:

Proposition. *If $\mathcal{S}$ and $\mathcal{T}$ are spans from $\mathcal{G}$ to $\mathcal{H}$, then so is $\mathcal{S}+\mathcal{T}$, and*

$$\mathcal{D}(\mathcal{S}+\mathcal{T}) = \mathcal{D}(\mathcal{S}) + \mathcal{D}(\mathcal{T}).$$

Proof. This is proved as Proposition 53. □

We can also multiply a span by a 'scalar': that is, a fixed groupoid. Given a groupoid Λ and a span

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

we can multiply them to obtain a span

$$\begin{tikzcd} & \Lambda \times \mathcal{S} \arrow[dl, "q\pi_2"'] \arrow[dr, "p\pi_2"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

Again, we have compatibility with degroupoidification:

Proposition. *Given a groupoid* Λ *and a span*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

then

$$\underset{\sim}{\Lambda \times S} = |\Lambda| \underset{\sim}{\mathcal{S}}.$$

Proof. This is proved as Proposition 56. ☐

Next we turn to the all-important process of *composing* spans. This is the groupoidified analogue of matrix multiplication. Suppose we have a span from $\mathcal{G}$ to $\mathcal{H}$ and a span from $\mathcal{H}$ to $\mathcal{J}$:

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & \\ \mathcal{J} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

Then we say these spans are **composable**. In this case we can form a weak pullback in the middle:

$$\begin{tikzcd} & & \mathcal{T}\mathcal{S} \arrow[dl, "\pi_{\mathcal{T}}"'] \arrow[dr, "\pi_{\mathcal{S}}"] & & \\ & \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & \\ \mathcal{J} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

which gives a span from $\mathcal{G}$ to $\mathcal{J}$:

$$\begin{tikzcd} & \mathcal{T}\mathcal{S} \arrow[dl, "q_{\mathcal{T}}\pi_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{S}}\pi_{\mathcal{S}}"] & \\ \mathcal{J} & & \mathcal{G} \end{tikzcd}$$

called the **composite** $\mathcal{T}\mathcal{S}$.

When all the groupoids involved are discrete, the spans $\mathcal{S}$ and $\mathcal{T}$ are just matrices of sets, as explained in Section 2.1. We urge the reader to check that in this case, the process of composing spans is really just matrix multiplication, with cartesian product of sets taking the place of multiplication of numbers, and disjoint union of sets taking the place of addition:

$$(\mathcal{T}\mathcal{S})^k_j = \coprod_{j \in \mathcal{H}} \mathcal{T}^k_j \times \mathcal{S}^j_i .$$

So, composing spans of groupoids is a generalization of matrix multiplication, with weak pullback playing the role of summing over the repeated index j in the formula above.

So, it should not be surprising that degroupoidification sends a composite of spans to the composite of their corresponding operators:

Proposition. *If $\mathcal{S}$ and $\mathcal{T}$ are composable spans:*

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & \\ \mathcal{J} & & \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

then the composite span

$$\begin{tikzcd} & \mathcal{T}\mathcal{S} \arrow[dl, "q_{\mathcal{T}}\pi_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{S}}\pi_{\mathcal{S}}"] & \\ \mathcal{J} & & \mathcal{G} \end{tikzcd}$$

has the property:

$$\mathcal{D}(\mathcal{T}\mathcal{S}) = \mathcal{D}(\mathcal{T})\mathcal{D}(\mathcal{S}).$$

Proof. This is proved as Lemma 60. □

We say what it means for spans to be 'equivalent' in Definition 25. Equivalent spans give the same linear operator: $\mathcal{S} \simeq \mathcal{T}$ implies $\mathcal{D}(\mathcal{S}) = \mathcal{D}(\mathcal{T})$. Spans of groupoids obey many of the basic laws of linear algebra—up to equivalence.

In fact, degroupoidification is a functor

$$\mathcal{D}\colon \mathrm{Span}(\mathrm{Grpd}) \to \mathrm{Vect}$$

where Vect is the 2-category of real vector spaces, linear operators, and identity 2-morphisms and Span(Grpd) is a bicategory with:

- groupoids as objects,
- spans as 1-morphisms,
- isomorphism classes of equivalences of spans as 2-morphisms

So, groupoidification is not merely a way of replacing linear algebraic structures with purely combinatorial structures. It is also a form of 'categorification', where we take structures defined in the category Vect and find analogues that live in the bicategory Span(Grpd).

We could go even further and think of Span as a tricategory (i.e., weak 3-category) with

- groupoids as objects,
- spans as morphisms,
- maps of spans as 2-morphisms,
- maps of maps of spans as 3-morphisms.

However, we have not yet found a use for this further structure.

Lastly we would like to say a few words about tensors. We can define the **tensor product** of groupoids $\mathcal{G}$ and $\mathcal{H}$ to be their cartesian product $\mathcal{G} \times \mathcal{H}$, and the **tensor product** of spans:

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & S' \arrow[dl, "q'"'] \arrow[dr, "p'"] & \\ \mathcal{H}' & & \mathcal{G}' \end{tikzcd}$$

to be the span

$$\begin{tikzcd} & S \times S' \arrow[dl, "q \times q'"'] \arrow[dr, "p \times p'"] & \\ \mathcal{H} \times \mathcal{H}' & & \mathcal{G} \times \mathcal{G}' \end{tikzcd}$$

Defining the tensor product of maps of spans in a similar way, the bicategory Span becomes a *monoidal* bicategory [20, 21].

Theorem. Span(Grpd) *is a monoidal bicategory.*

Proof. This is Theorem;35. ∎

Then degroupoidification is a 'monoidal functor', or homomorphism of monoidal bicategories, thanks to the natural isomorphism:

$$\mathbb{R}[\underline{\mathcal{G}}] \otimes \mathbb{R}[\underline{\mathcal{H}}] \cong \mathbb{R}[\underline{\mathcal{G} \times \mathcal{H}}].$$

Theorem. *Degroupoidification is a homomorphism of monoidal bicategories:*

$$\mathcal{D}\colon \mathrm{Span(Grpd)} \to \mathrm{Vect}.$$

Proof. This is Theorem 62. ∎

The important thing about the 'monoidal' aspect of degroupoidification is that it lets us mimic all the usual manipulations for tensors with groupoids replacing vector spaces. We have seen the simplest example: composition of spans via weak pullback is a generalization of matrix multiplication.

6.2 Degroupoidifying a Span

In the previous section, we described a process for turning a span of groupoids into a linear operator. In this section, we show this process is well-defined and give an explicit formula for the the operator coming from a span. As part of our work, we also show that equivalent spans give the same operator. This tells us that we can extend the degroupoidification functor [6] to a functor between bicategories.

6.2.1 Spans Give Operators

To prove that a span gives a well-defined operator, we begin with three lemmas that are of some interest in themselves. The previous chapter recalled the familiar concept of 'equivalence' of groupoids, which serves as a basis for this:

Definition 44. *Two groupoids over a fixed groupoid $\mathcal{G}$, say $v\colon \mathcal{V} \to \mathcal{G}$ and $u\colon \mathcal{U} \to \mathcal{G}$, are* **equivalent** *as groupoids over $\mathcal{G}$ if there is an equivalence $p\colon \mathcal{U} \to \mathcal{V}$ such that this diagram*

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "p"] \arrow[dr, "u"'] & & \mathcal{V} \arrow[dl, "v"] \\ & \mathcal{G} & \end{tikzcd}$$

commutes up to natural isomorphism.

Lemma 45. *Let $\mathcal{V}$ and $\mathcal{U}$ be equivalent groupoids over $\mathcal{G}$, then*

$$\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{U}).$$

Proof. This follows directly from Lemmas 39 and 50. ∎

Lemma 46. *Given groupoids $\mathcal{V}$ and $\mathcal{U}$ over $\mathcal{G}$,*

$$\mathcal{D}(\mathcal{V} + \mathcal{U}) = \mathcal{D}(\mathcal{V}) + \mathcal{D}(\mathcal{U}).$$

More generally, given any finite collection of groupoids $\mathcal{V}_i$ over $\mathcal{G}$, the coproduct $\sum_i \mathcal{V}_i$ is naturally a groupoid over $\mathcal{G}$, and

$$\mathcal{D}\left(\sum_i \mathcal{V}_i\right) = \sum_i \mathcal{D}(\mathcal{V})_i.$$

Proof. The full inverse image of any object $x \in \mathcal{G}$ in the coproduct $\sum_i \mathcal{V}_i$ is the coproduct of its full inverse images in each groupoid $\mathcal{V}_i$. Since groupoid cardinality is additive under coproduct, the result follows. □

Lemma 47. *Given a span of groupoids*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

we have

1. $\mathcal{S}(\sum_i \mathcal{V}_i) \simeq \sum_i \mathcal{S}\mathcal{V}_i$
2. $\mathcal{S}(\Lambda \times \mathcal{V}) \simeq \Lambda \times \mathcal{S}\mathcal{V}$

whenever $v_i \colon \mathcal{V}_i \to \mathcal{G}$ are groupoids over $\mathcal{G}$, $v \colon \mathcal{V} \to \mathcal{G}$ is a groupoid over $\mathcal{G}$, and Λ is a groupoid.

Proof. To prove 1, we need to describe a functor

$$F \colon \sum_i \mathcal{S}\mathcal{V}_i \to \mathcal{S}\left(\sum_i \mathcal{V}_i\right)$$

that will provide our equivalence. For this, we simply need to describe for each i a functor $F_i \colon \mathcal{S}\mathcal{V}_i \to \mathcal{S}\left(\sum_i \mathcal{V}_i\right)$. An object in $\mathcal{S}\mathcal{V}_i$ is a triple (s, z, α) where $s \in \mathcal{S}$, $z \in \mathcal{V}_i$ and $\alpha \colon p(s) \to v_i(z)$. F_i simply sends this triple to the same triple regarded as an object of $\mathcal{S}(\sum_i \mathcal{V}_i)$. One can check that F extends to a functor and that this functor extends to an equivalence of groupoids over $\mathcal{S}$.

To prove 2, we need to describe a functor $F \colon \mathcal{S}(\Lambda \times \mathcal{V}) \to \Lambda \times \mathcal{S}\mathcal{V}$. This functor simply re-orders the entries in the quadruples which define the objects in each groupoid. One can check that this functor extends to an equivalence of groupoids over $\mathcal{G}$. □

Finally we need the following lemma, which simplifies the computation of groupoid cardinality:

Lemma 48. *If $\mathcal{G}$ is a groupoid, then*

$$|\mathcal{G}| = \sum_{x \in \mathcal{G}} \frac{1}{|\mathrm{Mor}(x,-)|}$$

where $\mathrm{Mor}(x,-) = \bigcup_{y \in \mathcal{G}} \hom(x,y)$ is the set of morphisms whose source is the object $x \in \mathcal{G}$.

Proof. We check the following equalities:

$$\sum_{[x] \in \underline{\mathcal{G}}} \frac{1}{|\mathrm{Aut}(x)|} = \sum_{[x] \in \underline{\mathcal{G}}} \frac{|[x]|}{|\mathrm{Mor}(x,-)|} = \sum_{x \in \mathcal{G}} \frac{1}{|\mathrm{Mor}(x,-)|}.$$

Here $[x]$ is the set of objects isomorphic to x, and $|[x]|$ is the ordinary cardinality of this set. To check the above equations, we first choose an isomorphism $\gamma_y \colon x \to y$ for each object y isomorphic to x. This gives a bijection from $[x] \times \mathrm{Aut}(x)$ to $\mathrm{Mor}(x,-)$ that takes $(y, f \colon x \to x)$ to $\gamma_y f \colon x \to y$. Thus

$$|[x]|\,|\mathrm{Aut}(x)| = |\mathrm{Mor}(x,-)|,$$

and the first equality follows. We also get a bijection between $\mathrm{Mor}(y,-)$ and $\mathrm{Mor}(x,-)$ that takes $f \colon y \to z$ to $f\gamma_y \colon x \to z$. Thus, $|\mathrm{Mor}(y,-)| = |\mathrm{Mor}(x,-)|$ whenever y is isomorphic to x. The second equation follows from this. ☐

Now we are ready to prove the main theorem of this section:

Theorem 49. *Given a span of groupoids*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

there exists a unique linear operator $\mathcal{D}(S) \colon \mathbb{R}[\underline{\mathcal{G}}] \to \mathbb{R}[\underline{\mathcal{H}}]$ such that $\mathcal{D}(S)\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{S}\mathcal{V})$ for any vector $\underline{\mathcal{V}}$ obtained from a groupoid $\mathcal{V}$ over $\mathcal{G}$.

Proof. It is easy to see that these conditions uniquely determine $\mathcal{D}(S)$. Suppose $v \colon \underline{\mathcal{G}} \to \mathbb{R}$ is any nonnegative function. Then we can find a groupoid $\mathcal{V}$ over $\mathcal{G}$ such that $\mathcal{D}(\mathcal{V}) = v$. So, $\mathcal{D}(\mathcal{S})$ is determined on nonnegative functions by the condition that $\mathcal{D}(S)\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{S}\mathcal{V})$. Since every function is a difference of two nonnegative functions and $\mathcal{D}(\mathcal{S})$ is linear, this uniquely determines $\mathcal{D}(\mathcal{S})$.

The real work is proving that $\mathcal{D}(\mathcal{S})$ is well-defined. For this, assume we have a collection $\{v_i \colon \mathcal{V}_i \to \mathcal{G}\}_{i\in I}$ of groupoids over $\mathcal{G}$ and real numbers $\{\alpha_i \in \mathbb{R}\}_{i\in I}$ such that

$$\sum_i \alpha_i \underset{\sim}{\mathcal{V}_i} = 0. \tag{6.2}$$

We need to show that

$$\sum_i \alpha_i \underset{\sim}{\mathcal{S}\mathcal{V}_i} = 0. \tag{6.3}$$

We can simplify our task as follows. First, recall that a **skeletal** groupoid is one where isomorphic objects are equal. Every groupoid is equivalent to a skeletal one. Thanks to Lemmas 45 and the fact that a weak pullback is equivalent to the weak pullback of the skeletal cospan, proved in [6], we may therefore assume without loss of generality that $\mathcal{S}$, $\mathcal{G}$, $\mathcal{H}$ and all the groupoids $\mathcal{V}_i$ are skeletal.

Second, recall that a skeletal groupoid is a coproduct of groupoids with one object. By Lemma 46, degroupoidification converts coproducts of groupoids over $\mathcal{G}$ into sums of vectors. Also, by Lemma 47, the operation of taking weak pullback distributes over coproduct. As a result, we may assume without loss of generality that each groupoid $\mathcal{V}_i$ has one object. Write $*_i$ for the one object of $\mathcal{V}_i$.

With these simplifying assumptions, Equation 6.2 says that for any $x \in \mathcal{G}$,

$$0 = \sum_{i\in I} \alpha_i \, \mathcal{D}(\mathcal{V}_i)([x]) = \sum_{i\in I} \alpha_i \, |v_i^{-1}(x)| = \sum_{i\in J} \frac{\alpha_i}{|\mathrm{Aut}(*_i)|} \tag{6.4}$$

where J is the collection of $i \in I$ such that $v_i(*_i)$ is isomorphic to x. Since all groupoids in sight are now skeletal, this condition implies $v_i(*_i) = x$.

Now, to prove Equation 6.3, we need to show that

$$\sum_{i\in I} \alpha_i \underset{\sim}{\mathcal{S}\mathcal{V}_i}([y]) = 0$$

for any $y \in \mathcal{H}$. But since the set I is partitioned into sets J, one for each $x \in \mathcal{G}$, it suffices to show

$$\sum_{i\in J} \alpha_i \underset{\sim}{\mathcal{S}\mathcal{V}_i}([y]) = 0. \tag{6.5}$$

for any fixed $x \in \mathcal{G}$ and $y \in \mathcal{H}$.

To compute $\mathcal{D}(\mathcal{S}\mathcal{V}_i)$, we need to take this weak pullback:

$$\begin{tikzcd}
 & & \mathcal{S}\mathcal{V}_i \arrow[dl, "\pi_{\mathcal{S}}"'] \arrow[dr, "\pi_{\mathcal{V}_i}"] & \\
 & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & & \mathcal{V}_i \arrow[dl, "v_i"] \\
\mathcal{H} & & \mathcal{G} &
\end{tikzcd}$$

We then have

$$\mathcal{D}(\mathcal{S}\mathcal{V}_i)([y]) = |(q\pi_{\mathcal{S}})^{-1}(y)|, \tag{6.6}$$

so to prove Equation 6.5 it suffices to show

$$\sum_{i \in J} \alpha_i \, |(q\pi_{\mathcal{S}})^{-1}(y)| = 0. \tag{6.7}$$

Using the definition of weak pullback, and taking advantage of the fact that $\mathcal{V}_i$ has just one object, which maps down to x, we can see that an object of $\mathcal{S}\mathcal{V}_i$ consists of an object $s \in \mathcal{S}$ with $p(s) = x$ together with an isomorphism $\alpha\colon x \to x$. This object of $\mathcal{S}\mathcal{V}_i$ lies in $(q\pi_{\mathcal{S}})^{-1}(y)$ precisely when we also have $q(s) = y$.

So, we may briefly say that an object of $(q\pi_{\mathcal{S}})^{-1}(y)$ is a pair (s, α), where $s \in \mathcal{S}$ has $p(s) = x$, $q(s) = y$, and α is an element of $\mathrm{Aut}(x)$. Since $\mathcal{S}$ is skeletal, there is a morphism between two such pairs only if they have the same first entry. A morphism from (s, α) to (s, α') then consists of a morphism $f \in \mathrm{Aut}(s)$ and a morphism $g \in \mathrm{Aut}(*_i)$ such that

$$\begin{tikzcd}
x \arrow[r, "\alpha"] \arrow[d, "p(f)"'] & x \arrow[d, "v_i(g)"] \\
x \arrow[r, "\alpha'"'] & x
\end{tikzcd}$$

commutes.

A morphism out of (s, α) thus consists of an arbitrary pair $f \in \mathrm{Aut}(s)$, $g \in \mathrm{Aut}(*_i)$, since these determine the target (s, α'). This fact and Lemma 48 allow us to compute:

$$\begin{aligned}
|(q\pi_{\mathcal{S}})^{-1}(y)| \quad &= \sum_{(s,\alpha) \in (q\pi_{\mathcal{S}})^{-1}(y)} \frac{1}{|\mathrm{Mor}((s, \alpha), -)|} \\
&= \sum_{s \in p^{-1}(y) \cap q^{-1}(y)} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)||\mathrm{Aut}(*_i)|}\,.
\end{aligned}$$

So, to prove Equation 6.7, it suffices to show

$$\sum_{i\in J} \sum_{s\in p^{-1}(x)\cap q^{-1}(y)} \frac{\alpha_i|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)||\mathrm{Aut}(*_i)|} = 0\,. \tag{6.8}$$

But this easily follows from Equation 6.4. So, the operator $\mathcal{D}(\mathcal{S})$ is well defined. $\square$

In Definition 25 we recalled the natural concept of 'equivalence' for spans of groupoids. The following lemma is used to prove the next theorem, which says that our process of turning spans of groupoids into linear operators sends equivalent spans to the same operator:

Lemma 50. *Given a diagram of groupoids*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "F"] \arrow[dr, "p"'] & \arrow[d, phantom, "\Nearrow\alpha"] & \mathcal{T} \arrow[dl, "q"] \\ & \mathcal{G} & \end{tikzcd}$$

where F is an equivalence of groupoids, the restriction of F to the full inverse image $p^{-1}(x)$

$$F|_{p^{-1}(x)} \colon p^{-1}(x) \to q^{-1}(x)$$

is an equivalence of groupoids, for any object $x \in \mathcal{G}$.

Proof. It is sufficient to check that $F|_{p^{-1}(x)}$ is a full, faithful, and essentially surjective functor from $p^{-1}(x)$ to $q^{-1}(x)$. First we check that the image of $F|_{p^{-1}(x)}$ indeed lies in $q^{-1}(x)$. Given $x \in \mathcal{G}$ and $y \in p^{-1}(x)$, there is a morphism $\alpha_y \colon p(y) \to qF(y)$ in $\mathcal{G}$. Since $p(y) \in [x]$, then $qF(y) \in [x]$. It follows that $F(y) \in q^{-1}(x)$. Next we check that $F|_{p^{-1}(x)}$ is full and faithful. This follows from the fact that full inverse images are full subgroupoids. It is clear that a full and faithful functor restricted to a full subgroupoid will again be full and faithful. We are left to check only that $F|_{p^{-1}(x)}$ is essentially surjective. Let $z \in q^{-1}(x)$. Then, since F is essentially surjective, there exists $y \in \mathcal{S}$ such that $F(y) \in [z]$. Since $qF(y) \in [x]$ and there is an isomorphism $\alpha_y \colon p(y) \to qF(y)$, it follows that $y \in q^{-1}(x)$. So $F|_{p^{-1}(x)}$ is essentially surjective. We have shown that $F|_{p^{-1}(x)}$ is full, faithful, and essentially surjective, and, thus, is an equivalence of groupoids. $\square$

Theorem 51. *Given equivalent spans*

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & \\
\mathcal{H} & & \mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & \\
\mathcal{H} & & \mathcal{G}
\end{tikzcd}
$$

the linear operators $\mathcal{D}(\mathcal{S})$ and $\mathcal{D}(\mathcal{T})$ are equal.

Proof. Since the spans are equivalent, there is a functor providing an equivalence of groupoids $F\colon \mathcal{S} \to \mathcal{T}$ along with a pair of natural isomorphisms $\alpha\colon p_{\mathcal{S}} \Rightarrow p_{\mathcal{T}}F$ and $\beta\colon q_{\mathcal{S}} \Rightarrow q_{\mathcal{T}}F$. Thus, the diagrams:

$$
\begin{tikzcd}[column sep=small]
\mathcal{S} \arrow[dr] & & \mathcal{V} \arrow[dl] \\
 & \mathcal{G} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathcal{T} \arrow[dr] & & \mathcal{V} \arrow[dl] \\
 & \mathcal{G} &
\end{tikzcd}
$$

are equivalent pointwise. The weak pullbacks $\mathcal{SV}$ and $\mathcal{TV}$ are equivalent groupoids with the equivalence given by a functor $\tilde{F}\colon \mathcal{SV} \to \mathcal{TV}$. From the universal property of weak pullbacks, along with F, we obtain a natural transformation $\gamma\colon F\pi_{\mathcal{S}} \Rightarrow \pi_{\mathcal{T}}\tilde{F}$. We then have a triangle:

$$
\begin{tikzcd}
\mathcal{TV} \arrow[d, "\pi_{\mathcal{T}}"'] & \mathcal{SV} \arrow[l, "\tilde{F}"'] \arrow[d, "\pi_{\mathcal{S}}"] \arrow[dl, phantom, "\Leftarrow\gamma"] \\
\mathcal{T} \arrow[dr, "q_{\mathcal{T}}"'] & \mathcal{S} \arrow[l, "F"'] \arrow[d, "q_{\mathcal{S}}"] \arrow[dl, phantom, "\Leftarrow\beta"] \\
 & \mathcal{H}
\end{tikzcd}
$$

where the composite of γ and β is $(q_{\mathcal{T}} \cdot \gamma)^{-1}\beta\colon q_{\mathcal{S}}\pi_{\mathcal{S}} \Rightarrow q_{\mathcal{T}}\pi_{\mathcal{T}}\tilde{F}$.

We can now apply Lemma 50. Thus, for every $y \in \mathcal{H}$, the full inverse images $(q_{\mathcal{S}}\pi_{\mathcal{S}})^{-1}(y)$ and $(q_{\mathcal{T}}\pi_{\mathcal{T}})^{-1}(y)$ are equivalent. It follows from Lemma 39 that for each $y \in \mathcal{H}$, the groupoid cardinalities $|(q_{\mathcal{S}}\pi_{\mathcal{S}})^{-1}(y)|$ and $|(q_{\mathcal{T}}\pi_{\mathcal{T}})^{-1}(y)|$ are equal. Thus, the linear operators $\mathcal{D}(\mathcal{S})$ and $\mathcal{D}(\mathcal{T})$ are the same. ∎

Our calculations in the proof of Theorem 49 yield an explicit formula for the operator coming from a span:

Theorem 52. *Given a span of groupoids:*

$$
\begin{tikzcd}
& \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathcal{H} & & \mathcal{G}
\end{tikzcd}
$$

then, for any $f \in \mathcal{R}[\underline{\mathcal{G}}]$, we have:

$$(\mathcal{D}(\mathcal{S})f)([y]) = \sum_{[x]\in\underline{\mathcal{G}}} \; \sum_{[s]\in\underline{p^{-1}(x)}\bigcap\underline{q^{-1}(y)}} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)|} \, f([x])\,.$$

Proof. This is proved in greater generality in Section 5.2 of [6]. □

The previous theorem has many nice consequences. For example:

Proposition 53. *Suppose $\mathcal{S}$ and $\mathcal{T}$ are spans from a groupoid $\mathcal{G}$ to a groupoid $\mathcal{H}$. Then $\mathcal{D}(\mathcal{S}+\mathcal{T}) = \mathcal{D}(\mathcal{S}) + \mathcal{D}(\mathcal{T})$.*

Proof. This follows from the explicit formula given in Theorem52. □

6.3 Properties of Degroupoidification

In this section, we prove all the remaining results stated in Chapter 6 thus far. We start with results about scalar multiplication. Then we show that degroupoidification is a functor.

6.3.1 Scalar Multiplication

To prove facts about scalar multiplication, we use the following lemma:

Lemma 54. *Given a groupoid Λ and a functor between groupoids $p\colon \mathcal{G} \to \mathcal{H}$, then the functor $c \times p\colon \Lambda \times \mathcal{H} \to 1 \times \mathcal{G}$ (where $c\colon \Lambda \to 1$ is the unique morphism from Λ to the terminal groupoid 1) satisfies:*

$$|(c \times p)^{-1}(1, x)| = |\Lambda||p^{-1}(x)|$$

for all $x \in \mathcal{G}$.

Proof. Recall that by definition of the full inverse image

$$(c \times p)^{-1}(1, x) = \{(\lambda, y) \in \Lambda \times \mathcal{H} \mid \exists\gamma\colon (c \times p)(\lambda, y) \to (1, x)\}.$$

We notice that the element λ plays no real role in determining the morphism γ, and $(\lambda, y) \in (c \times p)^{-1}(1, x)$ for all λ if and only if $y \in p^{-1}(x)$. Now consider the groupoid cardinality of this groupoid. By definition we have

$$|(c \times p)^{-1}(1, x)| = \sum_{[(\lambda,y)]} \frac{1}{|\mathrm{Aut}(\lambda, y)|}$$

Since we are working over the product $\Lambda \times \mathcal{H}$, an automorphism of (λ, y) is automorphism of λ together with an automorphism of y. It follows that

$$|\mathrm{Aut}(\lambda, y)| = |\mathrm{Aut}(\lambda)||\mathrm{Aut}(y)|.$$

For a given $y \in p^{-1}(x)$ we can combine all the terms containing $|\mathrm{Aut}(y)|$ to obtain the sum

$$|(c \times p)^{-1}(1, x)| = \sum_{[y] \in p^{-1}(x)} \left(\sum_{[\lambda]} \frac{1}{|\mathrm{Aut}(\lambda)|} \right) \frac{1}{|\mathrm{Aut}(y)|}$$

which then after factoring is equal to $|\Lambda||p^{-1}(x)|$, as desired. $\square$

Proposition 55. *Given a groupoid* Λ *and a groupoid over* $\mathcal{G}$*, say* $v \colon \mathcal{V} \to \mathcal{G}$*, the groupoid* $\Lambda \times \mathcal{V}$ *over* $\mathcal{G}$ *satisfies*

$$\mathcal{D}(\Lambda \times \mathcal{V}) = |\Lambda| \mathcal{D}(\mathcal{V}).$$

Proof. This follows from Lemma 54. $\square$

Proposition 56. *Given a groupoid* Λ *and a span*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

then:

$$\mathcal{D}(\Lambda \times \mathcal{S}) = |\Lambda| \, \mathcal{D}(S).$$

Proof. This follows from Lemma 54. $\square$

6.3.2 Functoriality of Degroupoidification

In this section we prove that our process of turning groupoids into vector spaces and spans of groupoids into linear operators is indeed a functor. In the next section, we will prove a categorified version of this theorem. We first show that the process preserves identities, then show associativity of composition, from which many other things follow, including the preservation of composition. The lemmas in this section add up to a proof of the following theorem:

Theorem 57. *Degroupoidification is a functor from the category of groupoids and equivalence classes of spans to the category of real vector spaces and linear operators.*

Proof. As mentioned above, the proof follows from Lemmas 58 and 60. □

Lemma 58. *Degroupoidification preserves identities, i.e., given a groupoid $\mathcal{G}$, $\mathcal{D}(1_{\mathcal{G}}) = 1_{\mathcal{D}(\mathcal{G})}$, where $1_{\mathcal{G}}$ is the identity span from $\mathcal{G}$ to $\mathcal{G}$ and $1_{\mathcal{D}(\mathcal{G})}$ is the identity operator on $\mathcal{D}(\mathcal{G})$.*

Proof. This follows from the explicit formula given in Theorem 52. □

We now want to prove the associativity of composition of spans. Amongst the consequences of this proposition we can derive the preservation of composition under degroupoidification. Given a triple of composable spans:

$$\begin{tikzcd}
& \mathcal{T} \arrow[dl, "q_{\mathcal{T}}"'] \arrow[dr, "p_{\mathcal{T}}"] & & \mathcal{S} \arrow[dl, "q_{\mathcal{S}}"'] \arrow[dr, "p_{\mathcal{S}}"] & & \mathcal{R} \arrow[dl, "q_{\mathcal{R}}"'] \arrow[dr, "p_{\mathcal{R}}"] & \\
\mathcal{K} & & \mathcal{J} & & \mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

we want to show that composing in the two possible orders—$\mathcal{T}(\mathcal{S}\mathcal{R})$ or $(\mathcal{T}\mathcal{S})\mathcal{R}$—will provide equivalent spans of groupoids. In fact, since groupoids, spans of groupoids, and isomorphism classes of maps between spans of groupoids naturally form a bicategory, there exists a natural isomorphism called the **associator**. This tells us that the spans $\mathcal{T}(\mathcal{S}\mathcal{R})$ and $(\mathcal{T}\mathcal{S})\mathcal{R}$ are in fact equivalent. We give an explicit construction of the equivalence $\mathcal{T}(\mathcal{S}\mathcal{R}) \xrightarrow{\sim} (\mathcal{T}\mathcal{S})\mathcal{R}$.

Proposition 59. *Given a composable triple of spans, the operation of composition of spans by weak pullback is associative up to equivalence of spans of groupoids.*

Proof. We consider the above triple of spans in order to construct the aforementioned equivalence. The equivalence is simple to describe if we first take a close look at the groupoids $\mathcal{T}(\mathcal{S}\mathcal{R})$ and $(\mathcal{T}\mathcal{S})\mathcal{R}$. The composite $\mathcal{T}(\mathcal{S}\mathcal{R})$ has objects $(t,(s,r,\alpha),\beta)$ such that $r \in \mathcal{R}$, $s \in \mathcal{S}$, $t \in \mathcal{T}$, $\alpha\colon q_{\mathcal{R}}(r) \to p_{\mathcal{S}}(s)$, and $\beta\colon q_{\mathcal{S}}(s) \to p_{\mathcal{T}}(t)$, and morphisms

$f\colon (t,(s,r,\alpha),\beta) \to (t',(s',r',\alpha'),\beta')$, which consist of a map $g\colon (s,r,\alpha) \to (s',r',\alpha')$ in $\mathcal{SR}$ and a map $h\colon t \to t'$ such that the following diagram commutes:

$$\begin{tikzcd} q_{\mathcal{S}}\pi_s((s,r,\alpha)) \arrow[r, "\beta"] \arrow[d, "q_{\mathcal{S}}\pi_{\mathcal{S}}(g)"'] & p_{\mathcal{T}}(t) \arrow[d, "p_{\mathcal{T}}(h)"] \\ q_{\mathcal{S}}\pi_s((s',r',\alpha')) \arrow[r, "\beta'"'] & p_{\mathcal{T}}(t') \end{tikzcd}$$

where $\pi_{\mathcal{S}}$ maps the composite $\mathcal{SR}$ to $\mathcal{S}$. Further, g consists of a pair of maps $k\colon r \to r'$ and $j\colon s \to s'$ such that the following diagram commutes:

$$\begin{tikzcd} q_{\mathcal{R}}(r) \arrow[r, "\alpha"] \arrow[d, "q_{\mathcal{S}}(k)"'] & p_{\mathcal{S}}(s) \arrow[d, "p_{\mathcal{S}}(j)"] \\ q_{\mathcal{R}}(r') \arrow[r, "\alpha'"'] & p_{\mathcal{S}}(s') \end{tikzcd}$$

The groupoid $(\mathcal{TS})\mathcal{R}$ has objects $((t,s,\alpha),r,\beta)$ such that $r \in \mathcal{R}$, $s \in \mathcal{S}$, $t \in \mathcal{T}$, $\alpha\colon q_{\mathcal{S}}(s) \to p_{\mathcal{T}}(t)$, and $\beta\colon q_{\mathcal{R}}(r) \to p_{\mathcal{S}}(s)$, and morphisms $f\colon ((t,s,\alpha),r,\beta) \to ((t',s',\alpha'),r',\beta')$, which consist of a map $g\colon (t,s,\alpha) \to (t',s',\alpha')$ in $\mathcal{TS}$ and a map $h\colon r \to r'$ such that the following diagram commutes:

$$\begin{tikzcd} p_{\mathcal{R}}(r) \arrow[r, "\beta"] \arrow[d, "p_{\mathcal{R}}(h)"'] & p_{\mathcal{S}}\pi_{\mathcal{S}}((t,s,\alpha)) \arrow[d, "p_{\mathcal{S}}\pi_{\mathcal{S}}(g)"] \\ p_{\mathcal{R}}(r') \arrow[r, "\beta'"'] & p_{\mathcal{S}}\pi_{\mathcal{S}}((t',s',\alpha')) \end{tikzcd}$$

Further, g consists of a pair of maps $k\colon s \to s'$ and $j\colon t \to t'$ such that the following diagram commutes:

$$\begin{tikzcd} q_{\mathcal{S}}(s) \arrow[r, "\alpha"] \arrow[d, "q_{\mathcal{S}}(k)"'] & p_{\mathcal{T}}(t) \arrow[d, "p_{\mathcal{T}}(j)"] \\ q_{\mathcal{S}}(s') \arrow[r, "\alpha'"'] & p_{\mathcal{T}}(t') \end{tikzcd}$$

We can now write down a functor $F\colon \mathcal{T}(\mathcal{SR}) \to (\mathcal{TS})\mathcal{R}$:

$$(t,(s,r,\alpha),\beta) \mapsto ((t,s,\beta),r,\alpha)$$

Again, a morphism $f\colon (t,(s,r,\alpha),\beta) \to (t',(s',r',\alpha'),\beta')$ consists of maps $k\colon r \to r'$, $j\colon s \to s'$, and $h\colon t \to t'$. We need to define $F(f)\colon ((t,s,\beta),r,\alpha) \to ((t',s',\beta'),r',\alpha')$. The first component $g'\colon (t,s,\beta) \to (t',s',\beta')$ consists of the maps $j\colon s \to s'$ and $h\colon t \to t'$, and the following diagram commutes:

$$\begin{tikzcd}
q_{\mathcal{S}}(s) \arrow[r, "\beta"] \arrow[d, "q_{\mathcal{S}}(j)"'] & p_{\mathcal{T}}(t) \arrow[d, "p_{\mathcal{T}}(h)"] \\
q_{\mathcal{S}}(s') \arrow[r, "\beta'"'] & p_{\mathcal{T}}(t')
\end{tikzcd}$$

The other component map of $F(f)$ is $k\colon r \to r'$ and we see that the following diagram also commutes:

$$\begin{tikzcd}
p_{\mathcal{R}}(r) \arrow[r, "\alpha"] \arrow[d, "p_{\mathcal{R}}(k)"'] & p_{\mathcal{S}}\pi_s((t,s,\beta)) \arrow[d, "p_{\mathcal{S}}\pi_{\mathcal{S}}(g')"] \\
p_{\mathcal{R}}(r') \arrow[r, "\alpha'"'] & p_{\mathcal{S}}\pi_s((t',s',\beta'))
\end{tikzcd}$$

thus, defining a morphism in $(\mathcal{TS})\mathcal{R}$.

We now just need to check that F preserves identities and composition and that it is indeed an isomorphism. We will then have shown that the apexes of the two spans are isomorphic. First, given an identity morphism $1\colon (t,(s,r,\alpha),\beta) \to (t,(s,r,\alpha),\beta)$, then $F(1)$ is the identity morphism on $((t,s,\beta),r,\alpha)$. The components of the identity morphism are the respective identity morphisms on the objects r,s, and t. By the construction of F, it is clear that $F(1)$ will then be an identity morphism.

Given a pair of composable maps $f\colon (t,(s,r,\alpha),\beta) \to (t',(s',r',\alpha'),\beta')$ and $f'\colon (t',(s',r',\alpha'),\beta') \to (t'',(s'',r'',\alpha''),\beta'')$ in $\mathcal{T}(\mathcal{SR})$, the composite is a map $f'f$ with components $g'g\colon (s,r,\alpha) \to (s'',r'',\alpha'')$ and $h'h\colon t \to t''$. Further, $g'g$ has component morphisms $k'k\colon r \to r''$ and $j'j\colon s \to s'$. It is then easy to check that under the image of F this composition is preserved.

The construction of the inverse of F is implicit in the construction of F, and it is easy to verify that each composite FF^{-1} and $F^{-1}F$ is an identity functor. Further, the natural isomorphisms required for an equivalence of spans can each be taken to be the identity. □

It follows from the associativity of composition that degroupoidification preserves composition:

Lemma 60. *Degroupoidification preserves composition. That is, given a pair of composable spans:*

$$\begin{tikzcd}
& \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathcal{J} & & \mathcal{H} & & \mathcal{G}
\end{tikzcd}$$

we have

$$\mathcal{D}(\mathcal{T})\mathcal{D}(\mathcal{S}) = \mathcal{D}(\mathcal{T}\mathcal{S}).$$

Proof. Consider the composable pair of spans above along with a groupoid $\mathcal{V}$ over $\mathcal{G}$:

$$\begin{tikzcd}
 & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{V} \arrow[dl] \arrow[dr] & \\
\mathcal{J} & & \mathcal{H} & & \mathcal{G} & & 1
\end{tikzcd}$$

We can consider the groupoid over $\mathcal{G}$ as a span by taking the right leg to be the unique map to the terminal groupoid. We can compose this triple of spans in two ways; either $\mathcal{T}(\mathcal{S}\mathcal{V})$ or $(\mathcal{T}\mathcal{S})\mathcal{V}$. By the Proposition 59 stated above, these spans are equivalent. By Theorem 51, degroupoidification produces the same linear operators. Thus, composition is preserved. That is,

$$\mathcal{D}(\mathcal{T})\mathcal{D}(\mathcal{S})\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{T}\mathcal{S})\mathcal{D}(\mathcal{V}).$$

□

6.4 Degroupoidification as a Homomorphism of Bicategories

Theorem 61. *Degroupoidification is a homomorphism of bicategories* $\mathcal{D}\colon \mathrm{Span(Grpd)} \to \mathrm{Vect}$.

Proof. We again write Span for Span(Grpd). We describe the structure of $\mathcal{D}$, which consists of the following:

- a function assigning to each groupoid $\mathcal{G}$, vector space $\mathcal{D}(\mathcal{G}) = \mathcal{R}[\underline{G}]$,
- for each pair of groupoids $\mathcal{G}, \mathcal{H}$, a functor $\mathcal{D}_{\mathcal{G}\mathcal{H}}\colon \mathrm{Span}(\mathcal{G},\mathcal{H}) \to \mathrm{Vect}(\mathcal{D}(\mathcal{G}), \mathcal{D}(\mathcal{H}))$, which takes a span of groupoids to a linear operator and an equivalence of spans to the identity 2-morphism — that this defines a functor is straightforward,

- for each triple of groupoids $\mathcal{G}, \mathcal{H}, \mathcal{K}$, natural isomorphisms

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{G},\mathcal{H})\times\mathrm{Span}(\mathcal{H},\mathcal{K}) \arrow[rr, "c"] \arrow[d, "\mathcal{D}_{\mathcal{G}\mathcal{H}}\times\mathcal{D}_{\mathcal{H}\mathcal{K}}"'] & & \mathrm{Span}(\mathcal{G},\mathcal{H}) \arrow[d, "\mathcal{D}_{\mathcal{G}\mathcal{K}}"] \\
\mathrm{Vect}(\mathcal{D}(\mathcal{G}),\mathcal{D}(\mathcal{H}))\times\mathrm{Vect}(\mathcal{D}(\mathcal{H}),\mathcal{D}(\mathcal{K})) \arrow[rr, "c'"'] \arrow[urr, phantom, "\Nearrow\phi_{\mathcal{G}\mathcal{H}\mathcal{K}}"] & & \mathrm{Vect}(\mathcal{D}(\mathcal{G}),\mathcal{D}(\mathcal{K}))
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "I_{\mathcal{G}}"] \arrow[d, equal] & \mathrm{Span}(\mathcal{G},\mathcal{G}) \arrow[d, "\mathcal{D}_{\mathcal{G}\mathcal{G}}"] \\
1 \arrow[r, "I'_{\mathcal{D}(\mathcal{G})}"'] \arrow[ur, phantom, "\Nearrow\phi_{\mathcal{G}}"] & \mathrm{Vect}(\mathcal{D}(\mathcal{G}),\mathcal{D}(\mathcal{G}))
\end{tikzcd}$$

these are then invertible 2-cells $\phi_{\mathcal{T}\mathcal{S}}\colon \mathcal{D}(\mathcal{T}) \circ' \mathcal{D}(\mathcal{S}) \xrightarrow{\sim} \mathcal{D}(\mathcal{T} \circ \mathcal{S})$ and $\phi_{\mathcal{G}}\colon I'_{\mathcal{D}(\mathcal{G})} \xrightarrow{\sim} \mathcal{D}(I_{\mathcal{G}})$.

□

6.5 Degroupoidification as a Monoidal Functor

Theorem 62. *The degroupoidification functor* $\mathcal{D}\colon \mathrm{Span}(\mathrm{Grpd}) \to \mathrm{Vect}$ *is a homomorphism between monoidal bicategories.*

Proof. We first describe the monoidal data of $\mathcal{D}$. We again write Span for Span(Grpd). We use prime notation for the monoidal bicategory structure of Vect.

- a pseudo natural equivalence $\chi : \otimes' \circ (\mathcal{D} \times \mathcal{D}) \Rightarrow \mathcal{D} \circ \otimes$

$$\begin{tikzcd}
\mathrm{Span}\times\mathrm{Span} \arrow[r, "\mathcal{D}\times\mathcal{D}"] \arrow[d, "\otimes"'] & \mathrm{Vect}\times\mathrm{Vect} \arrow[d, "\otimes'"] \arrow[dl, phantom, "\swarrow\chi"] \\
\mathrm{Span} \arrow[r, "\mathcal{D}"'] & \mathrm{Vect}
\end{tikzcd}$$

in $\mathbf{Bicat}(\mathrm{Span} \times \mathrm{Span}, \mathrm{Vect})$ consisting of the following data:

  - for each pair of groupoids $\mathcal{G}, \mathcal{H}$, an isomorphism of vector spaces:

$$\chi_{\mathcal{G},\mathcal{H}}\colon \mathbb{R}[\underline{G}] \otimes' \mathbb{R}[\underline{H}] \to \mathbb{R}[\underline{\mathcal{G} \times \mathcal{H}}]$$

    which is defined in the obvious way;

  - since Vect has no non-trivial 2-morphisms, the second piece of data becomes an axiom — that is, for each pair of spans:

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathcal{H} & & \mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{S}' \arrow[dl] \arrow[dr] & \\
\mathcal{H}' & & \mathcal{G}'
\end{tikzcd}$$

there is an equation, or identity 2-morphism:

$$\mathcal{D}(\mathcal{S}\times\mathcal{S}')\circ\chi_{\mathcal{G},\mathcal{H}}=\chi_{\mathcal{G}',\mathcal{H}'}\circ\mathcal{D}(\mathcal{S})\otimes\mathcal{D}(S').$$

and we check following axioms:

–

- a pseudo natural equivalence $\iota : I' \Rightarrow \mathcal{D}\circ I$

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "I'"] \arrow[dr, "I"'] & \arrow[d, phantom, "\Downarrow\iota"] & \mathrm{Vect} \\
& \mathrm{Span} \arrow[ur, "\mathcal{D}"'] &
\end{tikzcd}$$

in $\mathbf{Bicat}(\mathbf{1},\mathrm{Vect})$ consisting of the following data:

– an the identity linear operator on the ground field:

$$\chi_1 \colon \mathbb{R}\to\mathbb{R}[\underline{1}];$$

– the second family of data again becomes an equation and is trivially satisfied;

and we check following axioms:

–

- an invertible modification as pictured below;

$$\begin{tikzcd}
& \mathrm{Span}^3 \arrow[rr, "\mathcal{D}\times\mathcal{D}\times\mathcal{D}"] \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & & \mathrm{Vect}^3 \arrow[dr, "\otimes'\times 1"] \arrow[d, phantom, "\Downarrow\chi\times 1"] & \\
\mathrm{Span}^2 \arrow[dr, "\otimes"'] \arrow[r, phantom, "\overset{a}{\Leftarrow}"] & & \mathrm{Span}^2 \arrow[rr, "\mathcal{D}\times\mathcal{D}"] \arrow[dl, "\otimes"] & \arrow[d, phantom, "\Downarrow\chi"] & \mathrm{Vect}^2 \arrow[dl, "\otimes'"] \\
& \mathrm{Span} \arrow[rr, "\mathcal{D}"'] & & \mathrm{Vect} &
\end{tikzcd}$$

$$\overset{\omega}{\Rrightarrow}$$

$$\begin{tikzcd}
& \mathrm{Span}^3 \arrow[rr, "\mathcal{D}\times\mathcal{D}\times\mathcal{D}"] \arrow[dl, "1\times\otimes"'] \arrow[d, phantom, "\Downarrow 1\times\chi"] & & \mathrm{Vect}^3 \arrow[dl, "1\times\otimes'"'] \arrow[dr, "\otimes'\times 1"] & \\
\mathrm{Span}^2 \arrow[rr, "\mathcal{D}\times\mathcal{D}"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\Downarrow\chi"] & \mathrm{Vect}^2 \arrow[dr, "\otimes'"] \arrow[rr, phantom, "\overset{a'}{\Leftarrow}"] & & \mathrm{Vect}^2 \arrow[dl, "\otimes'"] \\
& \mathrm{Span} \arrow[rr, "\mathcal{D}"'] & & \mathrm{Vect} &
\end{tikzcd}$$

which describes an equation that should hold in Vect since there are no non-trivial 2-morphisms.

- invertible modifications γ and δ as pictured below;

$$\begin{tikzcd}
& & \mathrm{Vect}^2 \arrow[drr, "\otimes'"] & & \\
\mathrm{Vect} \arrow[urr, "I'\times 1"] & \overset{\Rightarrow}{\iota\times 1} & & \overset{\Rightarrow}{\chi} & \mathrm{Vect} \\
& & \mathrm{Span}^2 \arrow[uu, "\mathcal{D}\times\mathcal{D}"'] \arrow[drr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] & & \\
\mathrm{Span} \arrow[uu, "\mathcal{D}"] \arrow[urr, "I\times 1"] \arrow[rrrr, "1"'] & & {} & & \mathrm{Span} \arrow[uu, "\mathcal{D}"']
\end{tikzcd}$$

$$\overset{\gamma}{\Rrightarrow}$$

$$\begin{tikzcd}
& \mathrm{Vect}^2 \arrow[dr, "\otimes'"] \arrow[d, phantom, "\Downarrow l'"] & \\
\mathrm{Vect} \arrow[ur, "I'\times 1"] \arrow[rr, "1"'] & {} & \mathrm{Vect} \\
& = & \\
\mathrm{Span} \arrow[uu, "\mathcal{D}"] \arrow[rr, "1"'] & & \mathrm{Span} \arrow[uu, "\mathcal{D}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Vect} \arrow[rr, "1"] & & \mathrm{Vect} \\
\mathrm{Span} \arrow[u, "\mathcal{D}"] \arrow[rr, "1"] \arrow[dr, "1\times I"'] & {} \arrow[u, phantom, "="] \arrow[d, phantom, "\Downarrow r"] & \mathrm{Span} \arrow[u, "\mathcal{D}"'] \\
& \mathrm{Span}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
\overset{\delta}{\Rrightarrow}
\begin{tikzcd}
\mathrm{Vect} \arrow[rr, "1"] \arrow[dr, "1\times I'"'] & \Downarrow r' & \mathrm{Vect} \\
\overset{\Rightarrow}{1\times\iota} & \mathrm{Vect}^2 \arrow[ur, "\otimes'"'] & \overset{\Rightarrow}{\chi} \\
\mathrm{Span} \arrow[uu, "\mathcal{D}"] \arrow[dr, "1\times I"'] & & \mathrm{Span} \arrow[uu, "\mathcal{D}"'] \\
& \mathrm{Span}^2 \arrow[uu, "\mathcal{D}\times\mathcal{D}"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

in **Bicat**(Span, Vect), which again describe equations that should hold in Vect.

Then we are left to check the axioms of a monoidal homomorphism.

- Checking the axioms is as usual a straightforward, yet tedious, exercise.

□

6.6 The Pull-Push Approach to Degroupoidification

Here we give another description of degroupoidification. In particular, we describe a different way of obtaining a linear operator from a span of groupoids. In this approach, any groupoid $\mathcal{G}$ gives a vector space $\mathbb{R}[\underline{\mathcal{G}}]$ as before, but we turn spans into operators with the help of the following operations:

Definition 63. *Let $p\colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids. Then we define the operator $p^*\colon \mathbb{R}[\underline{\mathcal{H}}] \to \mathbb{R}[\underline{\mathcal{G}}]$ by*

$$p^* f([x]) = \frac{|\mathrm{Aut}(p(x))|}{|\mathrm{Aut}(x)|}\, f([p(x)])$$

where $f \in \mathbb{R}[\underline{\mathcal{H}}]$ and $[x] \in \underline{\mathcal{G}}$.

Note that here we are using the ordinary cardinality of the *sets* $\mathrm{Aut}(x)$ and $\mathrm{Aut}(p(x))$, not groupoid cardinality.

Definition 64. *Let $p\colon \mathcal{G} \to \mathcal{H}$ be a functor between groupoids. Then we define the operator $p_*\colon \mathbb{R}[\underline{\mathcal{G}}] \to \mathbb{R}[\underline{\mathcal{H}}]$ by*

$$p_* f([y]) = \sum_{[x] \in \underline{p^{-1}(y)}} \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(x)|} f([x])$$

where $f \in \mathbb{R}[\underline{\mathcal{G}}]$ and $[y] \in \underline{\mathcal{H}}$.

Theorem 65. *There is a covariant functor* $(-)_*\colon \mathrm{Grpd} \to \mathrm{Vect}$*, which takes a functor* $p\colon \mathcal{G} \to \mathcal{H}$ *between groupoids to the linear operator* $p_*\colon \mathbb{R}[\underline{\mathcal{G}}] \to \mathbb{R}[\underline{H}]$*. Similarly, there is a contravariant functor* $(-)^*\colon \mathrm{Grpd} \to \mathrm{Vect}$*, which takes a functor* $p\colon \mathcal{G} \to \mathcal{H}$ *between groupoids to the linear operator* $p^*\colon \mathbb{R}[\underline{\mathcal{H}}] \to \mathbb{R}[\underline{\mathcal{G}}]$.

Proof. Checking that each of these processes preserve identities is an easy exercise in applying the definitions. Given and identity functor $1\colon \mathcal{G} \to \mathcal{G}$ and any $f \in \mathcal{R}[\underline{\mathcal{G}}]$, we first apply the lower star process to get:

$$1_* f[x] \;=\; \sum_{[v]\in \underline{1^{-1}(x)}} f([v]) \;=\; f([x]) \;=\; f([x]).$$

Similarly, for the upper star process, we get:

$$1^* f([x]) \;=\; \frac{|\mathrm{Aut}(1(x))|}{|\mathrm{Aut}(x)|} f([1(x)]) \;=\; \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(x)|} f([x]) \;=\; f([x]).$$

So both processes preserve identities.

To check that the lower star process preserves composition, we first write down the formula for $(qp)_*$.

$$(qp)_* f([z]) \;=\; \sum_{x\in \underline{(qp)^{-1}(z)}} f([x])$$

We then check what happens as we apply the operators separately.

$$\begin{aligned} q_* p_* f([z]) &= \sum_{y\in q^{-1}(z)} p_* f([y]) \\ &= \sum_{y\in q^{-1}(z)} \sum_{x\in p^{-1}(y)} f([x]) \end{aligned}$$

The second sums are over all $[x]$ such that $[x] \in \underline{p^{-1}(y)}$ and $[y] \in \underline{q^{-1}(z)}$. A quick consideration of simple set theory reveals that this is the same as $[x] \in \underline{(qp)_*(z)}$. Thus these two sums are equal.

Similarly we can check that composition is preserved by the upper star process, but this time as a contravariant functor. This is a straight forward calculation.

$$\begin{aligned} p^* q^* f([x]) &= \frac{|\mathrm{Aut}(p(x))|}{|\mathrm{Aut}(x)|} q^* f([p(x)]) \\ &= \frac{|\mathrm{Aut}(p(x))|}{|\mathrm{Aut}(x)|} \frac{|\mathrm{Aut}(q(p(x)))|}{|\mathrm{Aut}(p(x))|} f([q(p(x))]) \\ &= \frac{|\mathrm{Aut}(q(p(x)))|}{|\mathrm{Aut}(x)|} f([q(p(x))]) \\ &= (qp)^* f([x]) \end{aligned}$$

□

We can now state and prove the important theorem of this section which describes the process of turning a span into a linear operator.

Theorem 66. *Given a span of groupoids:*

$$\begin{tikzcd} & \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

there is a unique linear operator defined by the composite

$$q_* p^* \colon \mathbb{R}[\underline{\mathcal{G}}] \to \mathbb{R}[\underline{\mathcal{H}}]$$

and $q_* p^* = \mathcal{D}(\mathcal{S})$

Proof. The composition operator is well defined simply by the definitions the two functors $(-)_*$ and $(-)^*$.

The uniqueness of the operator is seen by calculating explicitly $q_* p^*(f)$ for any $f \in \mathbb{R}[\underline{\mathcal{G}}]$. This calculation is very straightforward, and yields to following formula:

$$q_* p^*(f)([y]) = \sum_{[x] \in \underline{\mathcal{G}}} \sum_{[s] \in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)|} f([x]) \,. \tag{6.9}$$

Finally, we can apply the functor $\mathcal{D}$. One then only needs to compare two simple equations to see that these two processes produce the same linear operator:

$$\mathcal{D}(\mathcal{S}) = q_* p^* .$$

□

Chapter 7

Enriched Bicategories

7.1 Enriched bicategories

From the point of view of category theory, enriched category theory is obviously attractive for many reasons. A very simple reason is that the definition of a category is very amenable to defining the concepts of monoids and groups — that is, by looking at categories with one object, where in the case of groups we add the extra requirement that all morphisms be invertible. By enriching over certain monoidal categories, such as the category of vector spaces Vect with the usual tensor product, we can define associative algebras as one object categories enriched over Vect. These are just fun toy examples, but this theory is, in fact, very rich.

The important point of which one should make note is that we can only enrich over a category if it has a monoidal structure. Why is this? Very simply, a monoidal structure is a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, which becomes the composition map in the enriched category. In other words, for objects x, y, z of a $\mathcal{V}$-enriched category, there is a composition map in $\mathcal{V}$ given by

$$\mathfrak{c}_{xyz}\colon \hom(x,y) \otimes \hom(y,z) \to \hom(x,z).$$

We have encountered several examples of monoidal categories at this point – that is, Span and FinVect. In fact, Span is an example of a monoidal bicategory. We shall treat both Span and FinVect as such structures, where FinVect is trivial at the top level, i.e., it has only identity 2-morphisms. So we are in need of a theory of "enriched bicategories". There are many possible approaches to such a theory, but we will present one such approach and give some evidence that it is the "correct" approach, in the sense that certain theorems hold.

Before giving the definition, we give an outline of the structure of enriched categories. Starting with a monoidal category $\mathcal{V}$, a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects $x, y, z, \ldots$, and for each pair of objects x, y, an object $\hom(x, y)$ of $\mathcal{V}$. Further, the structure maps of $\mathcal{C}$, which we will detail below are morphisms in $\mathcal{V}$.

It is useful to note that an enriched category is not necessarily a category, and an enriched bicategory is not necessarily a bicategory. However, there are certain examples of monoidal categories and bicategories for which the resulting enriched structures should be very familiar. We will provide some examples and state some theorems at the end of this section.

7.1.1 Definition: $\mathcal{V}$-categories

In this section we consider a monoidal category $\mathcal{V}$.

Definition 67 ((Enriched category))**.** *A* **$\mathcal{V}$-category** $\mathcal{C}$ *consists of the following data subject to the following axioms:*
Data:

- *A set* $\mathrm{Ob}(\mathcal{C})$ *of objects* $x, y, z, \ldots$*;*
- *for each pair of objects* x, y*, a* **hom-object** $\hom(x, y) \in \mathcal{V}$*, which we will often denote* (x, y)*;*
- *a morphism called* **composition**

$$c = c_{xyz} \colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

  for each triple of objects $x, y, z \in \mathcal{C}$*;*
- *an* **identity-assigning** *morphism*

$$i_x \colon I \to \hom(x, x)$$

  for each object $a \in \mathcal{C}$*;*

Axioms:

- $$\begin{tikzcd}
((w,x)\otimes(x,y))\otimes(y,z) \arrow[rr, "a"'] \arrow[d, "c\otimes 1"'] & & (w,x)\otimes((x,y)\otimes(y,z)) \arrow[d, "1\otimes c"] \\
(w,y)\otimes(y,z) \arrow[dr, "c"'] & & (w,x)\otimes(x,z) \arrow[dl, "c"] \\
& (w,z) &
\end{tikzcd}$$

  for each quadruple of objects $w, x, y, z \in \mathcal{B}$;

- $$\begin{tikzcd}
& (x,x)\otimes(x,y) \arrow[dl, "c_{xxy}"'] & (x,y)\otimes(y,y) \arrow[dr, "c_{xyy}"] & \\
(x,y) & I\otimes(x,y) \arrow[l, "r_{xy}"] \arrow[u, "i_x\otimes 1"'] & (x,y)\otimes I \arrow[u, "1\otimes i_y"] \arrow[r, "l_{xy}"'] & (x,y)
\end{tikzcd}$$

  for each pair of objects $x, y \in \mathcal{B}$;

7.1.2 Definition: $\mathcal{V}$-bicategories

Definition 68 ((Enriched bicategory))**.** *Let $\mathcal{V}$ be a monoidal bicategory. A* **$\mathcal{V}$-bicategory** *$\mathcal{B}$ consists of the following data subject to the following axioms:*
Data:

- *A set* $\mathrm{Ob}(\mathcal{B})$ *of objects $x, y, z, \ldots$;*

- *for every pair of objects x, y, a* **hom-object** $\hom(x, y) \in \mathcal{V}$*, which we will often denote (x, y), while suppressing the tensor product when necessary;*

- *a morphism called* **composition**

  $$c = c_{xyz} \colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

  for each triple of objects $x, y, z \in \mathcal{B}$;

- *an* **identity-assigning** *morphism*

  $$i_x \colon I \to \hom(x, x)$$

for each object $a \in \mathcal{B}$*;*

- *an invertible* 2*-morphism called the* **associator**

$$\begin{tikzcd}
((w,x)\otimes(x,y))\otimes(y,z) \arrow[rr, "a"'] \arrow[d, "c\otimes 1"'] & & (w,x)\otimes((x,y)\otimes(y,z)) \arrow[d, "1\otimes c"] \\
(w,y)\otimes(y,z) \arrow[dr, "c"'] \arrow[rr, phantom, "\overset{\alpha_{abcd}}{\Swarrow}"] & & (w,x)\otimes(x,z) \arrow[dl, "c"] \\
& (w,z) &
\end{tikzcd}$$

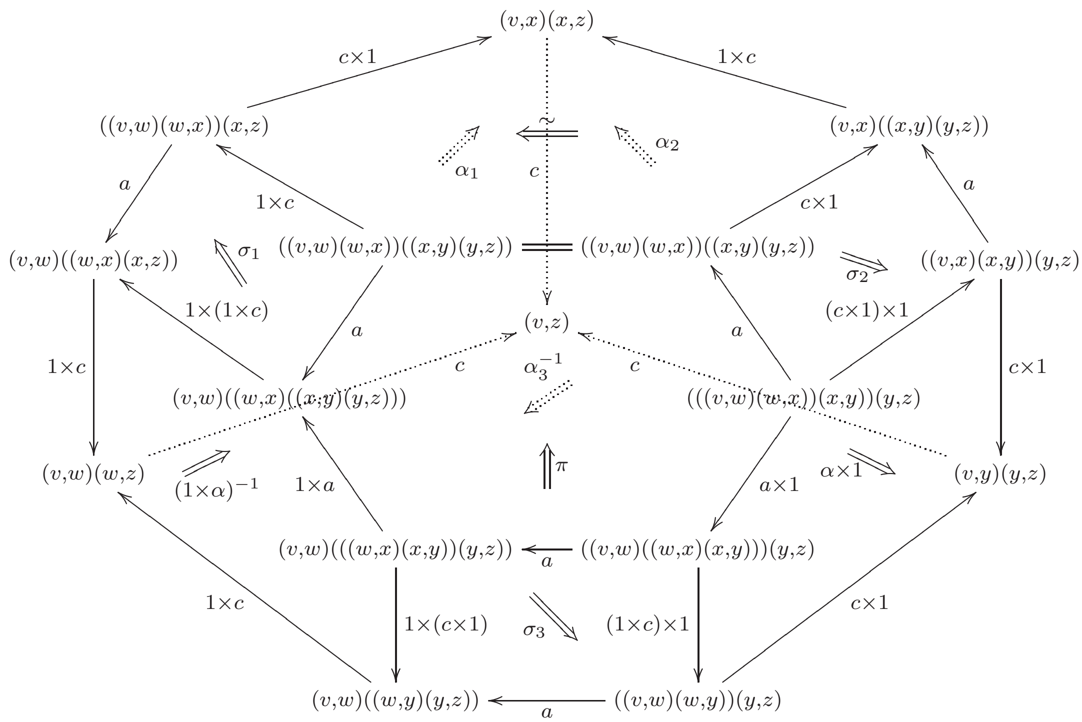
for each quadruple of objects $w, x, y, z \in \mathcal{B}$*;*

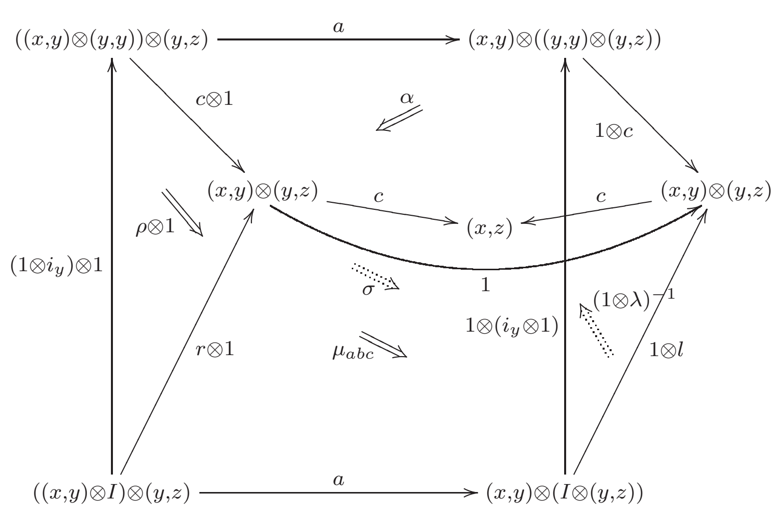
- *and invertible* 2*-morphisms called the* **right unitor** *and* **left unitor**

$$\begin{tikzcd}
& (x,x)\otimes(x,y) \arrow[dl, "c_{xxy}"'] \\
(x,y) & I\otimes(x,y) \arrow[l, "r_{xy}"] \arrow[u, "i_x\otimes 1"'] \arrow[ul, phantom, "\Nwarrow\rho_{xy}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(x,y)\otimes(y,y) \arrow[dr, "c_{xyy}"] & \\
(x,y)\otimes I \arrow[u, "1\otimes i_y"] \arrow[r, "l_{xy}"'] \arrow[ur, phantom, "\Nearrow\lambda_{xy}"] & (x,y)
\end{tikzcd}$$

for every pair of objects $x, y \in \mathcal{B}$*;*

Axioms:





σ is the component 2-cell expressing the pseudo naturality of the associator for the tensor product in the monoidal bicategory $\mathcal{V}$, and the arrow marked $\sim$ is just the associator natural isomorphism in the underlying bicategory of $\mathcal{V}$.

7.1.2.1 Unpacking the axioms

We can write the first axiom as an composite equal to the identity (note these are written in the usual functional ordering):

$$\left(\alpha_3^{-1} \cdot ((1_{vw} \times c_{wxy}) \times 1_{yz})(a_{vwxy} \times 1_{yz})\right):$$

$$c_{vyz}(c_{vwy} \times 1_{yz})((1_{vw} \times c_{wxy}) \times 1_{yz})(a_{vwxy} \times 1_{yz})$$

$$\Rightarrow c_{vwz}(1_{vw} \times c_{wyz})a_{vxyz}((1_{vw} \times c_{wxy}) \times 1_{yz})(a_{vwxy} \times 1_{yz})$$

$$\left(c_{vyz} \cdot (\alpha \times 1)^{-1}\right):$$

$$c_{vyz}(c_{vxy} \times 1_{yz})((c_{vwx} \times 1_{xy}) \times 1_{yz})$$

$$\Rightarrow c_{vyz}(c_{vwy} \times 1_{yz})((1_{vw} \times c_{wxy}) \times 1_{yz})$$

$$(\alpha_2 \cdot ((c_{vwx} \times 1_{xy}) \times 1_{yz})):$$

$$c_{vxz}(1_{vx} \times c_{xyz})a_{vxyz}((c_{vwx} \times 1_{xy}) \times 1_{yz})$$

$$\Rightarrow c_{vyz}(c_{vxy} \times 1_{yz})((c_{vwx} \times 1_{xy}) \times 1_{yz})$$

$$(c_{vxz}(1_{vx} \times c_{xyz})\sigma_2):$$

$$c_{vxz}(1_{vx} \times c_{xyz})(c_{vwx} \times 1_{xyz})a_{vxyz}$$

$$\Rightarrow c_{vxz}(1_{vx} \times c_{xyz})a_{vxyz}((c_{vwx} \times 1_{xy}) \times 1_{yz})$$

$$(c_{vxz} \cdot = \cdot a_{vxyz}):$$

$$c_{vxz}(c_{vwx} \times 1_{xz})(1_{vx} \times c_{xyz})1a_{vxyz}$$

$$\Rightarrow c_{vxz}(1_{vx} \times c_{xyz})(c_{vwx} \times 1_{xyz})a_{vxyz}$$

$$(\alpha_1 \cdot (1_{vx} \times c_{xyz})1a_{vxyz}):$$

$$c_{vwz}(1_{vw} \times c_{vwx})a_{vwxz}(1_{vx} \times c_{xyz})1a_{vxyz}$$

$$\Rightarrow c_{vxz}(c_{vwx} \times 1_{vx})(1_{vx} \times c_{vwx})1a_{vxyz}$$

$$(c_{vwz}(1_{vw} \times c_{wxz}) \cdot \sigma_1 \cdot 1a_{vxyz}):$$

$$c_{vwz}(1_{vw} \times c_{wxz})(1_{vw} \times (1_{wx} \times c_{xyz}))a_{vwxz}1a_{vxyz}$$

$$\Rightarrow c_{vwz}(1_{vw} \times c_{wxz})a_{vwxz}(1_{vx} \times c_{xyz})1a_{vxyz}$$

$$(c_{vwz}(1_{vw} \times c_{wxz})(1_{vw} \times (1_{wx} \times c_{xyz})) \cdot \pi):$$

$$c_{vwz}(1_{vw} \times c_{wxz})(1_{vw} \times (1_{wx} \times c_{xyz}))(1_{vw} \times a_{wxyz})a_{vwyz}(a_{vwxy} \times 1_{yz}) \Rightarrow$$

$$c_{vwz}(1_{vw} \times c_{wxz})(1_{vw} \times (1_{wx} \times c_{xyz}))a_{vwxz}1a_{vxyz}$$

$$\left(c_{vwz} \cdot (1 \times \alpha)^{-1} \cdot a_{vwyz}(a_{vwxy} \times 1_{yz})\right):$$

$$c_{vwz}(1_{vw} \times c_{wyz})(1_{vw} \times (c_{wxy} \times 1_{yz}))a_{vwyz}(a_{vwxy} \times 1_{yz})$$

$$\Rightarrow c_{vwz}(1_{vw} \times c_{wxz})(1_{vw} \times (1_{wx} \times c_{xyz}))(1_{vw} \times a_{wxyz})a_{vwyz}(a_{vwxy} \times 1_{yz})$$

$$(c_{vwz}(1_{vw} \times c_{wyz}) \cdot \sigma_3 \cdot (a_{vwxy} \times 1_{yz})):$$

$$c_{vwz}(1_{vw} \times x_{wyz})(1_{vw} \times (c_{wxy} \times 1_{yz}))a_{vwyz}(a_{vwxy} \times 1_{yz})$$

$$\Rightarrow c_{vwz}(1_{vw} \times c_{wyz})a_{vxyz}((1_{vw} \times c_{wxy}) \times 1_{yz})(a_{vwxy} \times 1_{yz})$$

We can similarly unpack the second axiom:

$$\alpha \cdot ((1_{xy} \times i_y) \times 1_{yz}):$$

$$c_{xyz}(1_{xy} \times c_{yyz})a_{xyyz}((1_{xy} \times i_y) \times 1_{yz})$$

$$\Rightarrow c_{xyz}(c_{xyy} \times 1_{yz})((1_{xy} \times i_y) \times 1_{yz})$$

$$c_{xyz} \cdot (\rho \times 1):$$

$$c_{xyz}(c_{xyy} \times 1_{yz})((1_{xy} \times i_y) \times 1_{yz})$$

$$c_{xyz}(r_{xy} \times 1_{yz})$$

$$= \cdot(r_{xy} \times 1_{yz}):$$

$$c_{xyz}(r_{xy} \times 1_{yz})$$

$$\Rightarrow c_{xyz}(r_{xy} \times 1_{yz})$$

$$c_{xyz} \cdot \mu\colon$$

$$c_{xyz}(r_{xy} \times 1_{yz})$$

$$\Rightarrow c_{xyz}(1_{xy} \times l_{yz})a_{xyIyz}$$

$$c_{xyz} \cdot (1 \times \lambda)^{-1} \cdot a_{xyIyz}\colon$$

$$c_{xyz}(1_{xy} \times l_{yz})a_{xyIyz}$$

$$\Rightarrow c_{xyz}(1_{xy} \times c_{yyz}(1_{xy} \times (i_y \times 1_{yz}))a_{xyIyz}$$

$$c_{xyz}(1_{xy} \times c_{yyz}) \cdot \sigma\colon$$

$$c_{xyz}(1_{xy} \times c_{yyz})(1_{xy} \times (i_y \times 1_{yz}))a_{xyIyz}$$

$$\Rightarrow c_{xyz}(1_{xy} \times c_{yyz})a_{xyyz}((1_{xy} \times i_y) \times 1_{yz})$$

7.1.2.2 On enriched bicategories

We write down some basic theorems concerning enriched bicategories.

Proposition 69. *Given a monoidal bicategory $\mathcal{V}$ which has only identity 2-morphisms, then every $\mathcal{V}$-bicategory is a $\mathcal{V}$-category in the obvious way.*

Proof. The proof is straightforward. □

Claim 70. *Prove that enrichment over symmetric monoidal bicategory gives monoidal structure.*

Claim 71. *Let* Cat *be the bicategory of (small) categories, functors and natural transformations. Then a* Cat*-bicategory is a bicategory.*

7.1.3 Change of base

We have developed a theory of enriched bicategories with the intention of writing down a theorem on *change of base* — that is, given a functor between monoidal bicategories $f\colon \mathcal{V} \to \mathcal{V}'$ and a $\mathcal{V}$-bicategory then can pass the structure along this map to obtain the structure of a $\mathcal{V}'$-bicategory. In the present setting we consider only the monoidal functor called degroupoidification from the monoidal bicategory Span(Grpd) to Vect. Since Vect is a monoidal category, but trivial as a monoidal bicategory, the change of base theorem has no content with respect to this functor — that is, a Span(Grpd)-bicategory is easily seen to yield a Vect-bicategory. Nonetheless, we prove the general result since it is quite straightforward and has interesting applications, which have been suggested in Hoffnung [22] and will be developed in Baez-Hoffnung [7].

Theorem 72. *Given a map of monoidal bicategories $f\colon \mathcal{V} \to \mathcal{V}'$ and a $\mathcal{V}$-bicategory $\mathcal{B}_{\mathcal{V}}$, then there is a $\mathcal{V}'$-bicategory $\mathcal{B}_{\mathcal{V}'}$.*

Proof. This proof consists of two parts. First, we give the data of the desired $\mathcal{V}'$-bicategory $\mathcal{B}_{\mathcal{V}'}$. In the second part, we show using the axioms of $\mathcal{B}_{\mathcal{V}}$ that the relevant axioms hold for $\mathcal{B}_{\mathcal{V}'}$.

$\mathcal{B}_{\mathcal{V}'}$ has as data:

- the same objects as $\mathcal{B}_{\mathcal{V}}$, i.e.,

$$\mathrm{Ob}(\mathcal{B}_{\mathcal{V}'}) := \mathrm{Ob}(\mathcal{B}_{\mathcal{V}});$$

- for each pair of objects a, b in $\mathcal{B}_{\mathcal{V}'}$, a hom-object

$$\hom_{\mathcal{B}_{\mathcal{V}'}}(a,b) := f(\hom_{\mathcal{B}_{\mathcal{V}}}(a,b));$$

- for each triple of objects a, b in $\mathcal{B}_{\mathcal{V}'}$, a morphism called composition

$$c' = c'_{abc} : f(\hom_{\mathcal{B}_{\mathcal{V}}}(a,b)) \otimes f(\hom_{\mathcal{B}_{\mathcal{V}}}(b,c)) \to f(\hom_{\mathcal{B}_{\mathcal{V}}}(a,c))$$

  defined by

$$c' = c'_{abc} := f(c_{abc}) \circ \chi_{\hom_{\mathcal{B}_{\mathcal{V}}}(a,b),\hom_{\mathcal{B}_{\mathcal{V}}}(b,c)},$$

  where χ is the structural pseudo natural equivalence of f expressing the preservation of the tensor product;

- for each object a in $\mathcal{B}_{\mathcal{V}'}$, an identity-assigning morphism

$$i'_a := f(i_a) \circ \iota \colon I' \to f(\hom_{\mathcal{B}_{\mathcal{V}}}(a,a)),$$

  where ι is the structural pseudo natural equivalence of f expressing the preservation of the unit of the tensor product;

- for each quadruple of objects a, b, c, d in $\mathcal{B}_{\mathcal{V}'}$, an associator given by the composite of the 2-cells in the following diagram:

$$\begin{tikzcd}
& (f(a,b)\otimes f(b,c))\otimes f((c,d)) \arrow[rrrr, "a_{\mathcal{V}'}"] \arrow[dr, "\chi\otimes 1"] \arrow[dddl, "c'\otimes 1"'] & & & & f((a,b))\otimes(f(b,c)\otimes f(c,d)) \arrow[dl, "1\otimes\chi"'] \arrow[dddr, "1\otimes c'"] & \\
& & f((a,b)\otimes(b,c))\otimes f(c,d) \arrow[rr, phantom, "\omega_{abcd}\Rightarrow"] \arrow[d, "\chi"] \arrow[ddll, "f(c)\otimes 1"] & & f((a,b))\otimes f((b,c)\otimes(c,d)) \arrow[d, "\chi"] \arrow[ddrr, "1\otimes f(c)"'] & & \\
& \arrow[r, phantom, "\chi_{c'1}\Rightarrow"] & f(((a,b)\otimes(b,c))\otimes(c,d)) \arrow[rr, "f(a_{\mathcal{V}})"'] \arrow[dd, "f(c\otimes 1)"] & & f((a,b)\otimes((b,c)\otimes(c,d))) \arrow[dd, "f(1\otimes c)"'] \arrow[r, phantom, "\Leftarrow\chi_{1c'}"] & \, & \\
f((a,b))\otimes f((b,d)) \arrow[drr, "\chi"'] \arrow[ddrrr, "c'"'] & & & & & & f((a,c))\otimes f((c,d)) \arrow[dll, "\chi"] \arrow[ddlll, "c'"] \\
& & f((a,b)\otimes(b,d)) \arrow[dr, "f(c)"] \arrow[rr, phantom, "\Rightarrow f(\alpha_{abcd})"] & & f((a,c)\otimes(c,d)) \arrow[dl, "f(c)"'] & & \\
& & & f((a,d)) & & &
\end{tikzcd}$$

  The 1- and 2-morphisms χ and ω are part of the structure of the monoidal homomorphism f.

- given a pair of objects a, b in $\mathcal{B}_{f(\mathcal{V})}$, a right unitor given by the composite of the 2-cells in the following diagram:

$$
\begin{tikzcd}
 & & & & f((a,a))\otimes f((a,b)) \arrow[ddddllll, "c'"'] \arrow[dll, "\chi"] \\
 & & f((a,a)\otimes(a,b)) \arrow[dddll, "f(c)"] \arrow[rr, phantom, "\Leftarrow \chi_{1i_a}"] & & \\
 & \arrow[r, phantom, "\Leftarrow f(\rho_{ab})"] & & f((a,b))\otimes f(I) \arrow[uur, "f(i_a)\otimes 1"'] \arrow[dl, "\chi"] & \\
 & & f((a,b)\otimes I) \arrow[uu, "f(i_a\otimes 1)"] \arrow[dll, "f(r_{ab})"'] \arrow[dr, phantom, "\Uparrow \delta_{(a,b)}"] & & \\
f((a,b)) & & & & f((a,b))\otimes I' \arrow[llll, "r'_{ab}"] \arrow[uul, "\iota\otimes 1"] \arrow[uuuu, "i'_a\otimes 1"']
\end{tikzcd}
$$

and a left unitor given by the composite of the 2-cells in the following diagram:

$$
\begin{tikzcd}
f((a,a))\otimes f((a,b)) \arrow[drr, "\chi"] \arrow[ddddrrrr, "c'"] & & & & \\
& \arrow[r, phantom, "\overset{\chi_{1 i_a}}{\Rightarrow}"] & f((a,a)\otimes(a,b)) \arrow[dddrr, "f(c)"] & & \\
& f(I)\otimes f((a,b)) \arrow[uul, "f(i_a)\otimes 1"] \arrow[dr, "\chi"'] & & \arrow[r, phantom, "f(\lambda_{ab})\Rightarrow"] & \\
& & f(I\otimes(a,b)) \arrow[uu, "f(1\otimes i_a)"'] \arrow[drr, "f(l_{ab})"] & & \\
I'\otimes f((a,b)) \arrow[uuuu, "i'_a\otimes 1"] \arrow[uur, "\iota\otimes 1"] \arrow[rrrr, "l'_{ab}"'] & \arrow[r, phantom, "\gamma_{(a,b)}\Rightarrow"] & & & f((a,b))
\end{tikzcd}
$$

- We are left to show that the data of $\mathcal{B}_{\mathcal{V}'}$ given above satisfy the two axioms of enriched bicategories. This is straightforward to check. The idea is to consider the image of each of the axioms in $\mathcal{B}_{\mathcal{V}}$. These will necessarily be commuting diagrams in $\mathcal{V}'$. Each vertex of these commuting surfaces is the image of a tensor product of hom-objects of $\mathcal{B}_{\mathcal{V}}$. These can be canonically extended to objects which are the tensor product of hom-objects of $\mathcal{B}_{\mathcal{V}'}$ with canonical fillings of the resulting surfaces by the structural 2-cells of $\mathcal{V}'$. In particular, each surface of the image of the axioms yields a new surface in $\mathcal{V}'$ and together these are the closed surfaces which are the axioms of $\mathcal{B}_{\mathcal{V}'}$. The fact that these surfaces compose immediately implies that they commute. For clarity, we give one example of the new surface obtained from the face $f(\mu)$.

$$
\begin{tikzcd}
f(a,b)\otimes f(b,c) \arrow[rrrrr, "1"] \arrow[drr, "\chi"] & & & & & f(a,b)\otimes f(b,c) \arrow[dll, "\chi"'] \\
& & f((a,b)\otimes(b,c)) \arrow[r, "f(1)"'] \arrow[r, phantom, shift left=5ex, "\Downarrow\chi"] & f((a,b)\otimes(b,c)) & & \\
& & (f(a,b)\otimes I')\otimes f(b,c) \arrow[u, "f(r\otimes 1)"'] \arrow[r, "f(a)"'] \arrow[ur, phantom, "\Uparrow f(\mu)"] & f(a,b)\otimes(I'\otimes f(b,c)) \arrow[u, "f(1\otimes l)"] & & \\
& f((a,b)\otimes I)\otimes f(b,c) \arrow[uuul, "f(r)\otimes 1"] \arrow[ur, "\chi"'] \arrow[uur, phantom, "\chi\Downarrow"] \arrow[rrr, phantom, shift right=4ex, "\Leftarrow\omega"] & & & f(a,b)\otimes f(I\otimes(b,c)) \arrow[uuur, "1\otimes f(l)"'] \arrow[ul, "\chi"] \arrow[uul, phantom, "\chi\Downarrow"] & \\
& (f(a,b)\otimes f(I))\otimes f(b,c) \arrow[u, "\chi\otimes 1"'] \arrow[rrr, "a'"] \arrow[to=4-1, phantom, "\delta\otimes 1\,\Leftarrow"] & & & f(a,b)\otimes(f(I)\otimes f(b,c)) \arrow[u, "1\otimes\chi"] \arrow[to=4-6, phantom, "\Rightarrow 1\otimes\gamma"] & \\
(f(a,b)\otimes I')\otimes f(b,c) \arrow[uuuuu, "r'\otimes 1"] \arrow[ur, "(1\otimes\iota)\otimes 1"'] \arrow[rrrrr, "a'"'] & & & & & f(a,b)\otimes(I'\otimes f(b,c)) \arrow[uuuuu, "1\otimes l'"'] \arrow[ul, "1\otimes(\iota\otimes 1)"]
\end{tikzcd}
$$

$\square$

Chapter 8

Proof of Fundamental Theorem

In this chapter, we prove, up to coherence, the main results of this work. That is, we give the main structure of the higher categories which appear, but leave as easy exercises in category theory the checking of coherence axioms.

8.1 The Hecke Bicategory

In Section 3.4, we claimed that there exists a Span(Grpd)-enriched bicategory for every finite group G, and we hinted at such a structure. Here we give a more complete account of this structure.

Theorem 73. *Given a finite group G, there is a* Span(Grpd)*-enriched bicategory called* Hecke(G) *consisting of:*

- *finite G-sets as objects,*
- *for each pair of finite G-sets, a* hom*-groupoid*

$$\mathrm{hom}(X, Y) = (X \times Y)//G \in \mathrm{Span}(\mathrm{Grpd}),$$

- *for each triple of finite G-sets X, Y, Z, a span of groupoids called composition given by:*

$$\begin{tikzcd} & (X \times Y \times Z)//G \arrow[dl, "p_X \times p_Z"'] \arrow[dr, "(p_X \times p_Y) \times (p_Y \times p_Z)"] & \\ (X \times Z)//G & & (X \times Y)//G \times (Y \times Z)//G \end{tikzcd}$$

- *for each finite groupoid X, an identity-assigning span of groupoids:*

$$\begin{tikzcd} & (X\times X)//G \arrow[dl, "Id"'] \arrow[dr] & \\ (X\times X)//G & & 1 \end{tikzcd}$$

In the following diagrams, we will shorten the notation of action groupoids from $(S)//G$ to S, where (S) is a finite G-set. To avoid confusion we will use square brackets to denote associativity of groupoids.

- *for each quadruple of finite G-sets W, X, Y, Z, an isomorphism class of equivalences of spans of groupoids, called the associator, between the composites of the spans:*

$$\begin{tikzcd} & (W{\times}X{\times}Z) \arrow[dl] \arrow[dr] & & (W{\times}X){\times}(X{\times}Y{\times}Z) \arrow[dl] \arrow[dr] & & {[(W{\times}X){\times}(X{\times}Y)]{\times}(Y{\times}Z)} \arrow[dl] \arrow[dr] & \\ (W{\times}Z) & & (W{\times}X){\times}(X{\times}Z) & & (W{\times}X){\times}[(X{\times}Y){\times}(Y{\times}Z)] & & {[(W{\times}X){\times}(X{\times}Y)]{\times}(Y{\times}Z)} \end{tikzcd}$$

with associativity of composites of spans given by $((\cdot,\cdot),\cdot)$ and:

$$\begin{tikzcd} & (W{\times}Y{\times}Z) \arrow[dl] \arrow[dr] & & (W{\times}X{\times}Y){\times}(Y{\times}Z) \arrow[dl] \arrow[dr] & \\ (W{\times}Z) & & (W{\times}Y){\times}(Y{\times}Z) & & {[(W{\times}X){\times}(X{\times}Y)]{\times}(Y{\times}Z)} \end{tikzcd}$$

- *and isomorphism classes of equivalences of spans of groupoids, called the right unitor, from the composite of the spans:*

$$\begin{tikzcd} & (X{\times}X{\times}Y) \arrow[dl] \arrow[dr] & & (X{\times}X){\times}(X{\times}Y) \arrow[dl] \arrow[dr] & \\ (X{\times}Y) & & (X{\times}X){\times}(X{\times}Y) & & 1{\times}(X{\times}Y) \end{tikzcd}$$

to the span:

$$\begin{tikzcd} & (X{\times}X){\times}(X{\times}Y) \arrow[dl] \arrow[dr] & \\ (X{\times}Y) & & 1{\times}(X{\times}Y) \end{tikzcd}$$

and called the left unitor, from the composite of the spans:

$$\begin{tikzcd}[column sep=small]
 & (X{\times}Y{\times}Y) \arrow[dl] \arrow[dr] & & (X{\times}Y){\times}(Y{\times}Y) \arrow[dl] \arrow[dr] & \\
(X{\times}Y) & & (X{\times}Y){\times}(Y{\times}Y) & & (X{\times}Y){\times}1
\end{tikzcd}$$

to the span:

$$\begin{tikzcd}[column sep=small]
 & (X{\times}Y){\times}(Y{\times}Y) \arrow[dl] \arrow[dr] & \\
(X{\times}Y) & & (X{\times}Y){\times}1
\end{tikzcd}$$

Proof. Defining the maps between the spans in the structure of the Hecke bicategory is straightforward in each case. It is equally simple to check that each map is an equivalence of spans. Then one is just left to check the commutativity of the closed surface axioms in the definition of enriched bicategories. This is a simple, yet messy in its explicit form, equation of isomorphism classes of equivalences of spans of groupoids. $\square$

8.2 Fundamental Theorem of Hecke Operators

There is not a lot to show here. In particular, for each finite group G we want to show an equivalence of algebroids, or Vect-enriched categories, between the degroupoidification of the Hecke bicategory $\bar{\mathcal{D}}(\mathrm{Hecke}(G))$ and the category of permutation representations $\mathrm{PermRep}(G)$.

Theorem 74. *For each finite group G, there exists a full, faithful and essentially surjective* Vect*-enriched functor:*

$$T\colon \mathrm{PermRep}(G) \to \bar{\mathcal{D}}(\mathrm{Hecke}(G)).$$

Proof. The functor T assigns to a permutation representation V_X the finite G-set X, which is the chosen basis of V_X fixed under the action of G. We need to show that T locally consists of maps of vector spaces that respect composition and identities.

We first define T on hom-spaces for each pair of finite G-sets X, Y:

$$T\colon \hom(V_X, V_Y) \to \mathbb{R}[(X \times Y)/\!/G]$$

Recall that everything in sight is finite, and thus we can switch freely from functions to linear combinations. A linear operator $f\colon V_X \to V_Y$ can be identified with a function $f\colon \mathbb{R}^X \to \mathbb{R}^Y$ and, thus, a matrix $f\colon X \times Y \to \mathbb{R}$ Since f is an intertwiner, the matrix is constant on orbits. It follows that there exists a function $\widetilde{f}$ as shown here:

$$\begin{tikzcd}
X \times Y \arrow[d] \arrow[dr, "\mathbf{f}"] & \\
\underline{(X \times Y)//G} \arrow[r, "\widetilde{f}"'] & \mathbb{R}
\end{tikzcd}$$

where the vertical arrow sends pairs (x, y) to their orbit under the diagonal action of G. Then we define:

$$T(f) := \widetilde{f}.$$

It is clear that T is an isomorphism of vector spaces, and it follows that T is an equivalence of Vect-categories if it preserves the enriched structure. This involves checking two commutative diagrams.

We first show the following diagram commutes:

$$\begin{tikzcd}
\hom(V_X, V_Y) \otimes \hom(V_Y, V_Z) \arrow[r, "\circ"] \arrow[d, "T \times T"'] & \hom(V_X, V_Z) \arrow[d, "T"] \\
\mathbb{R}[\underline{(X \times Y)//G}] \otimes \mathbb{R}[\underline{(Y \times Z)//G}] \arrow[r, "\mathcal{D}(\bar{\circ})"'] & \mathbb{R}[\underline{(X \times Z)//G}]
\end{tikzcd}$$

We consider the diagram of spans of groupoids:

$$\begin{tikzcd}
 & (X \times Y \times Z)//G \arrow[dl, "q'"'] \arrow[dr, "p'"] & \\
(X \times Z)//G & & (X \times Y)//G \times (Y \times Z)//G \\
 & X \times Y \times Z \arrow[uu, "i_2"] \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X \times Z \arrow[uu, "i_3"] & & (X \times Y) \times (Y \times Z) \arrow[uu, "i_1"']
\end{tikzcd}$$

where the bottom span is a span of sets, i.e., discrete groupoids, and the vertical arrows are inclusion functors. It is not difficult to check that the functions on such a span of sets can be pulled and pushed through the span yielding matrix multiplication. Thus, we consider the diagram:

$$
\begin{tikzcd}
 & \mathbb{R}[\underline{(X\times Y\times Z)/\!/G}] \arrow[dl, "q'_*"'] \arrow[dd, "i_2^*"] & \\
\mathbb{R}[\underline{(X\times Z)/\!/G}] \arrow[dd, "i_3^*"'] & & \mathbb{R}[\underline{(X\times Y)/\!/G}]\otimes\mathbb{R}[\underline{(Y\times Z)/\!/G}] \arrow[ul, "p'^*"'] \arrow[dd, "i_1^*"] \\
 & \mathbb{R}[X\times Y\times Z] \arrow[dl, "q_*"'] & \\
\mathbb{R}[X\times Z] & & \mathbb{R}[X\times Y]\otimes\mathbb{R}[Y\times Z] \arrow[ul, "p^*"]
\end{tikzcd}
$$

where the composite $q'_* p'^* = \mathcal{D}(\bar{\circ})$. The square on the left commutes as the image under the pullback functor. We are left to check the square on the right.

We prove this in a slightly more general setting. Suppose S and T are G-sets and $f\colon S\to T$ is equivariant. Then the following diagram commutes:

$$
\begin{tikzcd}
S/\!/G \arrow[r, "r'"] & T/\!/G \\
S \arrow[u, "i_S"] \arrow[r, "r"'] & T \arrow[u, "i_T"']
\end{tikzcd}
$$

and we need to show the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{R}[\underline{S/\!/G}] \arrow[r, "r'_*"] \arrow[d, "i_S^*"'] & \mathbb{R}[\underline{T/\!/G}] \arrow[d, "i_T^*"] \\
\mathbb{R}[S] \arrow[r, "r_*"'] & \mathbb{R}[T]
\end{tikzcd}
$$

This is just a simple verification of an equality using the pull-push formulas given in Section **??**.

Finally, in a similar yet simpler way, one can prove that the functor T preserves identities. $\square$

8.3 Categorified Hecke Algebras

A corollary of the previous two sections is that the Hecke bicategory locally categorifies the Hecke algebras for finite G-sets X, which come from the appropriate representation theoretic data. This was discussed in Section 4.2.

Given Dynkin diagram Γ and a prime power q, there is an algebraic group $G = G(\Gamma, q)$ over the finite field $\mathbb{F}_q$. We can choose a Borel subgroup $B\subset G$ and form a finite G-set $X = G/B$ called the flag variety. We can view this as an object of Hecke(G) and look at the hom-groupoid Hecke$(G)(X, X)$. From the Fundamental Theorem of Hecke Operators in the previous section we have the following result:

Theorem 75. *Given a Dynkin diagram Γ and a prime power q, the resulting algebraic group $G = G(\Gamma, q)$ over $\mathbb{F}_q$ is finite and thus there is a corresponding Hecke bicategory* Hecke(G)*, and $\mathcal{D}((X \times X)//G)$ is the Hecke algebra.*

Proof. This follows from the isomorphism between the Hecke algebra as a deformation of the group algebra of the corresponding Coxeter group and the algebra of intertwining operators from a particular induced representation from a Borel to the full group G to itself. This happens to be a permutation representation and thus the space of endo-operators is a hom-space in PermRep(G).

$\square$

Part III

Third Part: Appendix and References

Chapter 9

Appendix

This appendix gives the definition of bicategory, maps between bicategories, monoidal bicategory, and homomorphism between monoidal bicategories. The definitions of bicategory and maps between bicategory are taken from Tom Leinster's *Basic Bicategories* [31], and the definitions of monoidal bicategory and homomorphism are taken from Nick Gurski's thesis [21] and adapted for our purposes. This appendix is meant to make the present work self-contained.

9.1 Bicategory Definitions

9.1.1 Bicategories

Definition 76. *A* **bicategory** B *has the data*

a class of objects $a, b, c, \ldots$

for each pair a, b *of objects, a small category* $B(a, b)$ *with arrows* $f, g, h, \ldots$ *as objects and arrows* $F, G, H, \ldots$ *between them as morphisms. Composition in* $B(a, b)$ *is denoted by* $\cdot$ *and for a morphism* $f\colon a \to b$ *the identity on* f *is denoted* $1_f\colon f \Rightarrow f$.

for each triple a, b, c *of objects, a composition law given by a functor*

$$\circ_{a,b,c}\colon B(a, b) \times B(b, c) \to B(a, c)$$

for each object a*, an identity functor*

$$I_a\colon 1 \to B(a, a)$$

where 1 *stands for the final object in the category* Cat *of small categories. In particular,* I_A *is a* 1*-morphism in* B*.*

for each a, b, c, d*, natural isomorphisms*

$$\alpha_{a,b,c,d} \colon \circ_{a,b,d} \bar{\circ}(Id \times \circ_{b,c,d}) \Longrightarrow \circ_{a,c,d} \bar{\circ}(\circ_{a,b,c} \times Id)$$

$$\rho_{a,b} \colon \circ_{a,a,b} \bar{\circ}(I_a \times Id) \Longrightarrow Id$$

$$\lambda_{a,b} \colon \circ_{a,a,b} \bar{\circ}(Id \times I_b) \Longrightarrow Id$$

given morphisms $f \colon a \to b$*,* $g \colon b \to c$*,* $h \colon c \to d$*, the components of the isomorphisms are*

$$\alpha_{f,g,h} \colon \circ(f, \circ(g,h)) \overset{\cong}{\Longrightarrow} \circ(\circ(f,g),h)$$

$$\rho_f \colon \circ(I_a, f) \overset{\cong}{\Longrightarrow} f$$

$$\lambda_f \colon \circ(f, I_b) \overset{\cong}{\Longrightarrow} f$$

subject to the following coherence axioms:

$$\begin{tikzcd}
\circ(\circ(f,\circ(g,\circ(h,k)))) \arrow[rr, "\circ(1_f,\alpha)"] \arrow[d, "\alpha"'] & & \circ(f,\circ(\circ(g,h),k)) \arrow[d, "\alpha"] \\
\circ(\circ(f,g),\circ(h,k)) \arrow[dr, "\alpha"'] & & \circ(\circ(f,\circ(g,h)),k) \arrow[dl, "\circ(\alpha,1_k)"] \\
& \circ(\circ(\circ(f,g),h),k) &
\end{tikzcd}$$

with $k \colon d \to e$*, and*

$$\begin{tikzcd}
\circ(\circ(f,I_b),g) \arrow[rr, "\alpha"] \arrow[dr, "\rho_f \times Id"'] & & \circ(f,\circ(I_b,g)) \arrow[dl, "Id \times \lambda_g"] \\
& \circ(f,g) &
\end{tikzcd}$$

9.1.2 Homomorphisms

Definition 77. *A* **homomorphism** *of bicategories* $F = (F, \phi) \colon B \to B'$ *consists of the following data subject to the following axioms.*

Data:

A function $F\colon obB \to obB'$;

for $(a,b) \in obB \times obB$, a functor $F_{ab}\colon B(a,b) \to B'(Fa,Fb)$;

for $(a,b,c) \in obB \times obB \times obB$, natural isomorphisms

$$
\begin{tikzcd}
B(a,b)\times B(b,c) \arrow[r, "c"] \arrow[d, "F_{ab}\times F_{bc}"'] \arrow[dr, phantom, "\Nearrow\phi_{abc}"] & B(a,c) \arrow[d, "F_{ac}"] \\
B'(Fa,Fb)\times B'(Fb,Fc) \arrow[r, "c'"'] & B'(Fa,Fc)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "I_a"] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow\phi_a"] & B(a,a) \arrow[d, "F_{aa}"] \\
1 \arrow[r, "I'_{Fa}"'] & B'(Fa,Fa)
\end{tikzcd}
$$

these are then invertible 2-cells $\phi_{gf}\colon Fg \circ Ff \xrightarrow{\sim} F(g \circ f)$ and $\phi_a\colon I'_{fa} \xrightarrow{\sim} FI_a$.

Axioms:

The following commute:

$$
\begin{tikzcd}
(Fh\circ Fg)\circ Ff \arrow[r, "\phi * 1"] \arrow[d, "a'"] & F(h\circ g)\circ Ff \arrow[r, "\phi"] & F((h\circ g)\circ f) \arrow[d, "Fa"] \\
Fh\circ(Fg\circ Ff) \arrow[r, "1*\phi"'] & Fh\circ F(g\circ f) \arrow[r, "\phi"] & F(h\circ(g\circ f))
\end{tikzcd}
$$

$$
\begin{tikzcd}
Ff\circ I'_{Fa} \arrow[r, "1*\phi"] \arrow[d, "r'"'] & Ff\circ FI_a \arrow[r, "\phi"] & F(f\circ I_a) \arrow[d, "Fr"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}
\qquad
\begin{tikzcd}
I'_{Fb}\circ Ff \arrow[r, "\phi*1"] \arrow[d, "l'"'] & FI_b\circ Ff \arrow[r, "\phi"] & F(I_b\circ f) \arrow[d, "Fl"] \\
Ff \arrow[rr, equal] & & Ff
\end{tikzcd}
$$

Remark 78. *If ϕ_{abc} and ϕ_a are all identities, so that $Fg \circ Ff = F(g \circ f)$ and $FI = I'$, then F is called a* **strict homomorphism***.*

9.1.3 Transformations

Definition 79. *A* **transformation** *$\sigma\colon F \to G$, where $F = (F, \phi)$ and $G = (G, \psi)$ are morphisms from $\mathcal{B}$ to $\mathcal{B}'$, is defined by the following data and axioms. Below we use* *the notation* $h_*\colon \mathcal{B}(c,d) \to \mathcal{B}(c,e)$ *for the functor induced by a* 1*-cell* $h\colon d \to e$ *of a bicategory* $\mathcal{B}$*, and similarly* $h^*\colon \mathcal{B}(e,c) \to \mathcal{B}(d,c)$*.*

Data:

1*-cells* $\sigma_a\colon Fa \to Ga$

Natural transformations

$$\begin{tikzcd} B(a,b) \arrow[r, "F_{ab}"] \arrow[d, "G_{ab}"'] \arrow[dr, phantom, "\nearrow\sigma_{ab}"] & B'(Fa,Fb) \arrow[d, "(\sigma_b)_*"] \\ B'(Ga,Gb) \arrow[r, "(\sigma_a)^*"'] & B'(Fa,Gb) \end{tikzcd}$$

these are then 2*-cells* $\sigma_f\colon Gf \circ \sigma_a \to \sigma_b \circ Ff$*.*

Axioms:

The following commute:

$$\begin{tikzcd} (Gg\circ Gf)\circ\sigma_a \arrow[r, "a'"'] \arrow[d, "\psi*1"'] & Gg\circ(Gf\circ\sigma_a) \arrow[r, "1*\sigma_f"] & Gg\circ(\sigma_b\circ Ff) \arrow[r, "a'^{-1}"] & (Gg\circ\sigma_b)\circ Ff \arrow[r, "\sigma_g*1"] & (\sigma_c\circ Fg)\circ Ff \arrow[r, "a'"] & \sigma_c\circ(Fg\circ Ff) \arrow[d, "1*\phi"] \\ G(g\circ f)\circ\sigma_a \arrow[rrrrr, "\sigma_{gf}"'] & & & & & \sigma_c\circ F(g\circ f) \end{tikzcd}$$

$$\begin{tikzcd} I'_{Ga}\circ\sigma_a \arrow[r, "l'"] \arrow[d, "\psi*1"'] & \sigma_a \arrow[r, "r'^{-1}"] & \sigma_a\circ I'_{Fa} \arrow[d, "1*\phi"] \\ GI_a\sigma_a \arrow[rr, "\sigma_{I_a}"'] & & \sigma_a\circ FI_a \end{tikzcd}$$

Remark 80. *If* σ_{ab} *are all natural isomorphisms then* σ *is called a* **pseudo natural (or strong) transformation***. If* σ_{ab} *are all identities then* σ *is called a* **strict transformation***.*

9.1.4 Modifications

Definition 81. *A* **modification** $\Gamma\colon \sigma \to \tilde{\sigma}$, *where* $\sigma, \tilde{\sigma}\colon F \to G$ *are transformations and* $F = (F, \phi), G = (G, \psi)$ *are morphisms from* $\mathcal{B}$ *to* $\mathcal{B}'$, *is defined by the following data and axioms.*

Data:

2-*cells* $\Gamma_a\colon \sigma_a \to \tilde{\sigma}_a$

Axioms:

The following commute:

$$\begin{tikzcd}
Gf \circ \sigma_a \arrow[r, "1 * \Gamma_a"] \arrow[d, "\sigma_f"'] & Gf \circ \tilde{\sigma}_a \arrow[d, "\tilde{\sigma}_f"] \\
\sigma_b \circ Ff \arrow[r, "\Gamma_b * 1"'] & \tilde{\sigma}_b \circ Ff
\end{tikzcd}$$

9.2 Monoidal Bicategories

9.2.1 Monoidal Bicategories

Definition 82. *A* **monoidal bicategory** $\mathcal{B}$ *consists of the following data subject to the following axioms.*

DATA:

- *an underlying bicategory which we also denote* $\mathcal{B}$*;*
- *a homomorphism* $\otimes\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ *called composition;*
- *a homomorphism* $I : 1 \to \mathcal{B}$, *where* 1 *denotes the unit bicategory;*
- *a pseudo natural equivalence* $\mathbf{a}$

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \times \mathcal{B} \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] \arrow[dr, phantom, "\Downarrow \mathbf{a}"] & \mathcal{B} \times \mathcal{B} \arrow[d, "\otimes"] \\
\mathcal{B} \times \mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B}
\end{tikzcd}$$

in $\mathbf{Bicat}(\mathcal{B}\times\mathcal{B}\times\mathcal{B},\mathcal{B})$*;*

- *pseudo natural equivalences* $\mathbf{l}$ *and* $\mathbf{r}$

$$\begin{tikzcd}
& \mathcal{B}\times\mathcal{B} \arrow[dr, "\otimes"] & \\
\mathcal{B} \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & \arrow[u, phantom, "\Downarrow \mathbf{l}"] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{B}\times\mathcal{B} \arrow[dr, "\otimes"] & \\
\mathcal{B} \arrow[ur, "1\times I"] \arrow[rr, "1"'] & \arrow[u, phantom, "\Uparrow \mathbf{r}"] & \mathcal{B}
\end{tikzcd}$$

in $\mathbf{Bicat}(\mathcal{B},\mathcal{B})$*;*

- *an invertible modification* π

$$\begin{tikzcd}
& \mathcal{B}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, "1\times\otimes\times1"] & & \mathcal{B}^3 \arrow[dr, "\otimes\times1"] \arrow[dl, phantom, "\Downarrow a\times 1"] & \\
\mathcal{B}^3 \arrow[dr, "1\times\otimes"'] \arrow[rr, phantom, "\overset{1\times a}{\Leftarrow}"] & & \mathcal{B}^3 \arrow[rr, "\otimes\times1"] \arrow[dl, "1\times\otimes"] & & \mathcal{B}^2 \arrow[dl, "\otimes"] \\
& \mathcal{B}^2 \arrow[rr, "\otimes"'] & & \mathcal{B} \arrow[u, phantom, "\Downarrow a"] &
\end{tikzcd}
\overset{\pi}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{B}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, phantom, "="] & & \mathcal{B}^3 \arrow[dr, "\otimes\times1"] \arrow[dl, "1\times\otimes"'] & \\
\mathcal{B}^3 \arrow[dr, "1\times\otimes"'] \arrow[rr, "\otimes\times1"] & & \mathcal{B}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow a"] \arrow[rr, phantom, "\overset{a}{\Leftarrow}"] & & \mathcal{B}^2 \arrow[dl, "\otimes"] \\
& \mathcal{B}^2 \arrow[rr, "\otimes"'] & {} & \mathcal{B} &
\end{tikzcd}$$

in the bicategory $\mathbf{Bicat}(\mathcal{B}^4,\mathcal{B})$*, for example;*

- *Invertible modifications*

$$\begin{tikzcd}
\mathcal{B}^2 \arrow[dr, "1\times I\times 1"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & {} & \\
& \mathcal{B}^3 \arrow[r, "\otimes\times1"'] \arrow[d, "1\times\otimes"] \arrow[u, phantom, "\Downarrow r\times 1"] \arrow[l, phantom, "\Leftarrow 1\times l"] \arrow[dr, phantom, "\Downarrow a"] & \mathcal{B}^2 \arrow[d, "\otimes"] \\
& \mathcal{B}^2 \arrow[r, "\otimes"'] & \mathcal{B}
\end{tikzcd}
\overset{\mu}{\Rrightarrow}
\begin{tikzcd}
\mathcal{B}^2 \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & {} \arrow[dl, Rightarrow, "1"'] \\
& {} & \mathcal{B}^2 \arrow[d, "\otimes"] \\
& \mathcal{B}^2 \arrow[r, "\otimes"'] & \mathcal{B}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{B}^3 \arrow[dr, "\otimes\times1"] \arrow[d, phantom, "\Downarrow l\times 1"] & \\
\mathcal{B}^2 \arrow[ur, "I\times1\times1"] \arrow[rr, "1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "="] & {} & \mathcal{B}^2 \arrow[d, "\otimes"] \\
\mathcal{B} \arrow[rr, "1"'] & & \mathcal{B}
\end{tikzcd}
\overset{\lambda}{\Rrightarrow}
\begin{tikzcd}
& \mathcal{B}^3 \arrow[dr, "\otimes\times1"] \arrow[d, "1\times\otimes"] & \\
\mathcal{B}^2 \arrow[ur, "I\times1\times1"] \arrow[d, "\otimes"'] \arrow[r, phantom, "="] & \mathcal{B}^2 \arrow[dr, "\otimes"] \arrow[r, phantom, "\Downarrow a"] \arrow[d, phantom, "\Downarrow l"] & \mathcal{B}^2 \arrow[d, "\otimes"] \\
\mathcal{B} \arrow[ur, "I\times1"] \arrow[rr, "1"'] & {} & \mathcal{B}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "1"] \arrow[dr, phantom, "="] & & \mathcal{B} \\
\mathcal{B}^2 \arrow[u, "\otimes"] \arrow[rr, "1"] \arrow[dr, "1\times1\times I"'] & {} & \mathcal{B}^2 \arrow[u, "\otimes"'] \\
& \mathcal{B}^3 \arrow[ur, "1\times\otimes"'] \arrow[u, phantom, "\Downarrow 1\times r"] &
\end{tikzcd}
\overset{\rho}{\Rrightarrow}
\begin{tikzcd}
\mathcal{B} \arrow[rr, "1"] \arrow[dr, "1\times I"'] & {} & \mathcal{B} \\
\mathcal{B}^2 \arrow[u, "\otimes"] \arrow[dr, "1\times1\times I"'] \arrow[r, phantom, "="] & \mathcal{B}^2 \arrow[ur, "\otimes"] \arrow[u, phantom, "\Downarrow r"] \arrow[r, phantom, "\Rightarrow a"] & \mathcal{B}^2 \arrow[u, "\otimes"'] \\
& \mathcal{B}^3 \arrow[u, "\otimes\times1"] \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}$$

AXIOMS:

- *The following equation of 2-cells holds in the bicategory $\mathcal{B}$, where we have used parentheses instead of $\otimes$ for compactness and the unmarked isomorphisms are naturality isomorphisms for a.*

$$\begin{tikzcd}
 & & (k(j(hg)))f \arrow[dr, "a"] & & \\
 & (k((jh)g))f \arrow[ur, "(1a)1"] \arrow[dr, "a"] \arrow[rr, phantom, "\cong"] & & k((j(hg))f) \arrow[dr, "1a"] & \\
((k(jh))g)f \arrow[ur, "a1"] \arrow[dr, "a"] \arrow[rr, phantom, "\Downarrow\pi"] & & k(((jh)g)f) \arrow[ur, "1(a1)"] \arrow[dr, "1a"] \arrow[rr, phantom, "\Downarrow 1\pi"] & & k(j((hg)f)) \arrow[d, "1(1a)"] \\
(((kj)h)g)f \arrow[u, "(a1)1"] \arrow[dr, "a"] \arrow[r, phantom, "\cong"] & (k(jh))(gf) \arrow[rr, "a"] & & k((jh)(gf)) \arrow[r, "1a"] & k(j(h(gf))) \\
 & ((kj)h)(gf) \arrow[u, "a(11)"] \arrow[rr, "a"] \arrow[urr, phantom, "\Downarrow\pi"] & & (kj)(h(gf)) \arrow[ur, "a"] &
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
 & & (k(j(hg)))f \arrow[dr, "a"] & & \\
 & (k((jh)g))f \arrow[ur, "(1a)1"] & & k((j(hg))f) \arrow[dr, "1a"] & \\
((k(jh))g)f \arrow[ur, "a1"] \arrow[r, phantom, "\Downarrow\pi 1"] & {} & & {} \arrow[l, phantom, "\Downarrow\pi"] & k(j((hg)f)) \arrow[dd, "1(1a)"] \\
 & ((kj)(hg))f \arrow[uuur, "a1"] \arrow[drr, "a"] & & & \\
(((kj)h)g)f \arrow[uu, "(a1)1"] \arrow[ur, "a1"] \arrow[dr, "a"] \arrow[rr, phantom, "\Downarrow\pi"] & & & (kj)((hg)f) \arrow[uur, "a"] \arrow[d, "(11)a"] \arrow[r, phantom, "\cong"] & k(j(h(gf))) \\
 & ((kj)h)(gf) \arrow[rr, "a"] & & (kj)(h(gf)) \arrow[ur, "a"] &
\end{tikzcd}$$

- *The following equation of 2-cells holds in the bicategory $\mathcal{B}$, where the unmarked isomorphisms are either naturality isomorphisms for a or unique coherence isomorphisms from $\mathcal{B}$.*

$$\begin{tikzcd}
 & & (h(Ig))f \arrow[d, "a"] \arrow[dr, "(1l)1"] & \\
 & ((hI)g)f \arrow[ur, "a1"] \arrow[d, "a"] \arrow[r, phantom, "\Downarrow\pi"] & h((Ig)f) \arrow[dd, "1a"] \arrow[ddr, "1(l1)"] \arrow[r, phantom, "\cong"] & (hg)f \arrow[dd, "a"] \\
(hg)f \arrow[ur, "(r1)1"] \arrow[dr, "a"] \arrow[r, phantom, "\cong"] & (hI)(gf) \arrow[dr, "a"] & {} \arrow[r, phantom, "\Downarrow 1\lambda"] & {} \\
 & h(gf) \arrow[u, "r(11)"] \arrow[rr, bend right, "1"'] \arrow[r, phantom, "\Downarrow\mu"] & h(I(gf)) \arrow[r, "1l"] & h(gf)
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
 & & (h(Ig))f \arrow[dr, "(1l)1"] & \\
 & ((hI)g)f \arrow[ur, "a1"] \arrow[r, phantom, "\Downarrow\mu 1"] & {} & (hg)f \arrow[d, "a"] \\
(hg)f \arrow[ur, "(r1)1"] \arrow[urrr, "11"'] \arrow[r, "a"] & h(gf) \arrow[rr, "1"] \arrow[r, phantom, "\cong"] & & h(gf)
\end{tikzcd}$$

- *The following equation of 2-cells holds in the bicategory $\mathcal{B}$.*

$$
\begin{tikzcd}
 & h((gI)f) \arrow[dr, "1a"] & & \\
h(gf) \arrow[ur, "1(r1)"] \arrow[r, phantom, "\cong"] & (h(gI))f \arrow[u, "a"] \arrow[r, phantom, "\Downarrow\pi"] & h(g(If)) \arrow[dr, "1(1l)"] & \\
 & & (hg)(If) \arrow[u, "a"] \arrow[dd, "(11)l"] \arrow[r, phantom, "\cong"] & h(gf) \\
 & ((hg)I)f \arrow[uu, "a1"] \arrow[ur, "a"] \arrow[d, phantom, "\Downarrow\mu"] & & \\
(hg)f \arrow[uuu, "a"] \arrow[uuur, "(1)1"] \arrow[ur, "r1"'] \arrow[rr, bend right, "1"'] \arrow[u, phantom, "\Downarrow\rho 1"] & & (hg)f \arrow[uur, "a"'] &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & h((gI)f) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow 1\mu"] & & \\
h(gf) \arrow[ur, "1(r1)"] \arrow[drrr, "11"] & {} & h(g(If)) \arrow[dr, "1(1l)"] & \\
 & & \cong & h(gf) \\
(hg)f \arrow[uu, "a"] \arrow[rr, "1"'] & & (hg)f \arrow[ur, "a"'] &
\end{tikzcd}
$$

9.2.2 Monoidal Functors

In [20], the maps defined between tricategories, and thus monoidal bicategories, are lax functors. A *homomorphism* satisfies slightly stronger properties. To avoid excess terminology, we will define homomorphisms between monoidal bicategories straightaway, as all our examples will be of this form.

Definition 83. *Let $\mathcal{B}$ and $\mathcal{B}'$ be monoidal bicategories. A* **homomorphism** *of monoidal bicategories $F : \mathcal{B} \to \mathcal{B}'$ consists of the following data subject to the following axioms.*
DATA:

- *A homomorphism of bicategories $F : \mathcal{B} \to \mathcal{B}'$;*

- *a pseudo natural equivalence $\chi : \otimes' \circ (F \times F) \Rightarrow F \circ \otimes$*

$$
\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[r, "F \times F"] \arrow[d, "\otimes"'] & \mathcal{B}' \times \mathcal{B}' \arrow[d, "\otimes'"] \arrow[dl, phantom, "\overset{\chi}{\Leftarrow}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{B}'
\end{tikzcd}
$$

in $\mathbf{Bicat}(\mathcal{B} \times \mathcal{B}, \mathcal{B}')$*;*

- *a pseudo natural equivalence $\iota : I' \Rightarrow F \circ I$*

$$
\begin{tikzcd}
\mathbf{1} \arrow[rr, "I'"] \arrow[dr, "I"'] & \Downarrow\iota & \mathcal{B}' \\
 & \mathcal{B} \arrow[ur, "F"'] &
\end{tikzcd}
$$

in $\mathbf{Bicat}(\mathbf{1},\mathcal{B}')$;

- *an invertible modification as pictured below;*

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{B}^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] & & \mathcal{B}'^3 \arrow[dr, "\otimes'\times 1"] \arrow[d, phantom, "\Downarrow\chi\times 1"] & \\
\mathcal{B}^2 \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{a}{\Leftarrow}"] & & \mathcal{B}^2 \arrow[rr, "F\times F"] \arrow[dl, "\otimes"] & {} & \mathcal{B}'^2 \arrow[dl, "\otimes'"] \\
 & \mathcal{B} \arrow[rr, "F"'] & & \mathcal{B}' \arrow[u, phantom, "\Downarrow\chi"] &
\end{tikzcd}
\overset{\omega}{\Rrightarrow}
\begin{tikzcd}[column sep=small]
 & \mathcal{B}^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow 1\times\chi"] & & \mathcal{B}'^3 \arrow[dl, "1\times\otimes'"'] \arrow[dr, "\otimes'\times 1"] & \\
\mathcal{B}^2 \arrow[rr, "F\times F"] \arrow[dr, "\otimes"'] \arrow[drr, phantom, "\Downarrow\chi"] & & \mathcal{B}'^2 \arrow[dr, "\otimes'"] \arrow[rr, phantom, "\overset{a'}{\Leftarrow}"] & & \mathcal{B}'^2 \arrow[dl, "\otimes'"] \\
 & \mathcal{B} \arrow[rr, "F"'] & & \mathcal{B}' &
\end{tikzcd}
$$

- *invertible modifications* γ *and* δ *as pictured below;*

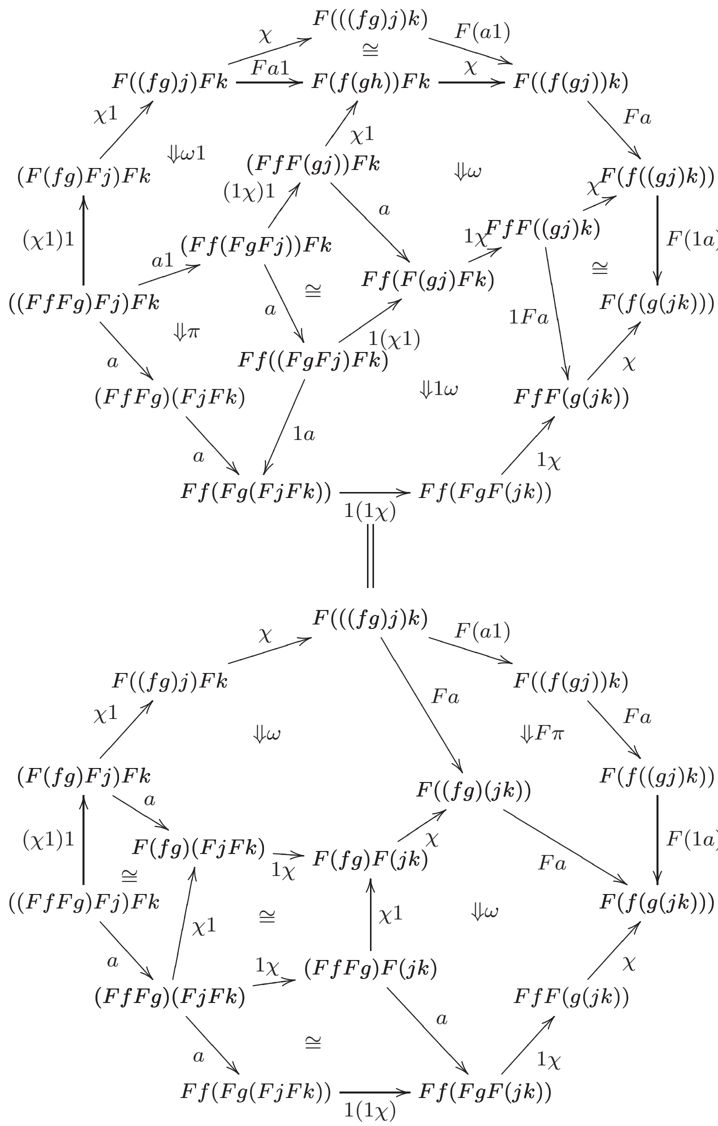
$$
\begin{tikzcd}
 & \mathcal{B}'^2 \arrow[dr, "\otimes'"] & \\
\mathcal{B}' \arrow[ur, "I'\times 1"] \arrow[r, phantom, "\overset{\Rightarrow}{\iota\times 1}"] & {} & \mathcal{B}' \arrow[l, phantom, "\overset{\Rightarrow}{\chi}"] \\
 & \mathcal{B}^2 \arrow[uu, "F\times F"] \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] & \\
\mathcal{B} \arrow[uu, "F"] \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & {} & \mathcal{B} \arrow[uu, "F"']
\end{tikzcd}
\overset{\gamma}{\Rrightarrow}
\begin{tikzcd}
 & \mathcal{B}'^2 \arrow[dr, "\otimes'"] \arrow[d, phantom, "\Downarrow l'"] & \\
\mathcal{B}' \arrow[ur, "I'\times 1"] \arrow[rr, "1"'] & {} & \mathcal{B}' \\
 & = & \\
\mathcal{B} \arrow[uu, "F"] \arrow[rr, "1"'] & & \mathcal{B} \arrow[uu, "F"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{B}' \arrow[rr, "1"] & & \mathcal{B}' \\
 & = & \\
\mathcal{B} \arrow[uu, "F"] \arrow[rr, "1"] \arrow[dr, "1\times I"'] & {} \arrow[d, phantom, "\Downarrow r"] & \mathcal{B} \arrow[uu, "F"'] \\
 & \mathcal{B}^2 \arrow[ur, "\otimes"'] &
\end{tikzcd}
\overset{\delta}{\Rrightarrow}
\begin{tikzcd}
\mathcal{B}' \arrow[rr, "1"] \arrow[dr, "1\times I'"'] & {} \arrow[d, phantom, "\Downarrow r'"] & \mathcal{B}' \\
 & \mathcal{B}'^2 \arrow[ur, "\otimes'"'] & \\
\mathcal{B} \arrow[uu, "F"] \arrow[dr, "1\times I"'] \arrow[ur, phantom, "\overset{\Rightarrow}{1\times\iota}"] & & \mathcal{B} \arrow[uu, "F"'] \arrow[ul, phantom, "\overset{\Rightarrow}{\chi}"] \\
 & \mathcal{B}^2 \arrow[uu, "F\times F"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

in $\mathbf{Bicat}(\mathcal{B},\mathcal{B}')$;

AXIOMS:

- *For all 1-cells* $(x,y,z,w)\in\mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B}$*, the following equation of modifications* *holds;*



$F(((fg)j)k)$
χ
$F(a1)$
$\cong$
$F((fg)j)Fk$
$Fa1$
$F(f(gh))Fk$
χ
$F((f(gj))k)$
$\chi 1$
Fa
$\Downarrow\omega 1$
$\chi 1$
$(F(fg)Fj)Fk$
$(FfF(gj))Fk$
$\Downarrow\omega$
χ
$F(f((gj)k))$
$(1\chi)1$
a
1χ
$FfF((gj)k)$
$(\chi 1)1$
$(Ff(FgFj))Fk$
$F(1a)$
$a1$
$\cong$
$Ff(F(gj)Fk)$
$\cong$
$((FfFg)Fj)Fk$
a
$F(f(g(jk)))$
$1Fa$
$\Downarrow\pi$
$1(\chi 1)$
a
$Ff((FgFj)Fk)$
χ
$\Downarrow 1\omega$
$(FfFg)(FjFk)$
$FfF(g(jk))$
$1a$
a
1χ
$Ff(Fg(FjFk))$
$Ff(FgF(jk))$
$1(1\chi)$
$F(((fg)j)k)$
χ
$F(a1)$
$F((fg)j)Fk$
$F((f(gj))k)$
Fa
$\chi 1$
Fa
$\Downarrow\omega$
$\Downarrow F\pi$
$(F(fg)Fj)Fk$
$F((fg)(jk))$
$F(f((gj)k))$
a
χ
$(\chi 1)1$
$F(fg)(FjFk)$
$F(fg)F(jk)$
Fa
$F(1a)$
1χ
$\cong$
$((FfFg)Fj)Fk$
$\chi 1$
$\cong$
$\chi 1$
$\Downarrow\omega$
$F(f(g(jk)))$
a
1χ
$(FfFg)F(jk)$
χ
$(FfFg)(FjFk)$
$FfF(g(jk))$
a
a
$\cong$
1χ
$Ff(Fg(FjFk))$
$Ff(FgF(jk))$
$1(1\chi)$

- *For all 1-cells $(x, y) \in \mathcal{B} \times \mathcal{B}$, the following equation of modifications holds.*

$$
\begin{tikzcd}
 & & F((fI)g) \arrow[rr, "Fa"] \arrow[dr, phantom, "\Downarrow\omega"] & & F(f(Ig)) \arrow[ddrr, "F(1l)"] & & \\
 & & & & FfF(Ig) \arrow[u, "\chi"'] \arrow[ddddrr, "1Fl"] \arrow[r, phantom, "\cong"] & & \\
F(fg) \arrow[uurr, "F(r1)"] \arrow[r, phantom, "\cong"] & & & Ff(FIFg) \arrow[ur, "1\chi"] & & & F(fg) \\
 & F(fI)Fg \arrow[uuur, "\chi"] & (FfFI)Fg \arrow[l, "\chi 1"'] \arrow[ur, "a"] \arrow[dr, phantom, "\cong"] & & \Downarrow 1\gamma & & \\
 & \overset{\delta 1}{\Rightarrow} & (FfI)Fg \arrow[u, "(1\iota)1"] \arrow[r, "a"] \arrow[d, phantom, "\Downarrow\mu"] & Ff(IFg) \arrow[uu, "1(\iota 1)"'] \arrow[drrr, "1l"] & & & \\
FfFg \arrow[uuu, "\chi"] \arrow[uur, "Fr1"] \arrow[urr, "r1"'] \arrow[rrrrrr, "1"'] & & & & & & FfFg \arrow[uuu, "\chi"']
\end{tikzcd}
$$

$$
=
$$

$$
\begin{tikzcd}
 & F((fI)g) \arrow[r, "Fa"] \arrow[d, phantom, "\Downarrow F\mu"] & F(f(Ig)) \arrow[dr, "F(1l)"] & \\
F(fg) \arrow[ur, "F(r1)"] \arrow[rrr, "1"'] & & & F(fg) \\
FfFg \arrow[u, "\chi"] \arrow[rrr, "1"'] \arrow[rrru, phantom, "\cong"] & & & FfFg \arrow[u, "\chi"']
\end{tikzcd}
$$

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