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# Extended TQFT's and Quantum Gravity

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Jeffrey Colin Morton

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Dissertation Committee:

Dr. John Baez, Chairperson
Dr. Michel Lapidus
Dr. Marta Asaeda

The Dissertation of Jeffrey Colin Morton is approved:

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Committee Chairperson

University of California, Riverside

"Just because They aren't *real*, doesn't mean They don't *exist*." Dedicated to Them.

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ABSTRACT OF THE DISSERTATION

Extended TQFT's and Quantum Gravity

by

Jeffrey Colin Morton

Doctor of Philosophy, Graduate Program in Mathematics
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Dr. John Baez, Chairperson

This thesis gives a definition of an extended topological quantum field theory (TQFT) as a weak 2-functor $Z : \mathbf{nCob}_2 \to \mathbf{2Vect}$, by analogy with the description of a TQFT as a functor $Z : \mathbf{nCob} \to \mathbf{Vect}$. We also show how to obtain such a theory from any finite group $G$. This theory is related to a topological gauge theory, the Dijkgraaf-Witten model. To give this definition rigorously, we first define a bicategory of cobordisms between cobordisms. We also give some explicit description of a higher-categorical version of **Vect**, denoted **2Vect**, a bicategory of *2-vector spaces*. Along the way, we prove several results showing how to construct 2-vector spaces of **Vect**-*valued presheaves* on certain kinds of groupoids. In particular, we use the case when these are groupoids whose objects are connections, and whose morphisms are gauge transformations, on the manifolds on which the extended TQFT is to be defined. On cobordisms between these manifolds, we show how a construction of "pullback and pushforward" of presheaves gives both the morphisms and 2-morphisms in **2Vect** for the extended TQFT, and that these satisfy the axioms for a weak 2-functor. Finally, we discuss the motivation for this research in terms of Quantum Gravity. If the results can be extended

from a finite group $G$ to a Lie group, then for some choices of $G$ this theory will recover an existing theory of Euclidean quantum gravity in 3 dimensions. We suggest extensions of these ideas which may be useful to further this connection and apply it in higher dimensions.

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# Chapter 1

# Introduction

In this thesis, I will describe a connection between the ideas of extended topological quantum field theory and topological gauge theory. This is motivated by consideration of a possible application to quantum gravity, and in particular in 3 dimensions–a situation which is simpler than the more realistic 4D case but has many of the essential features. Here, we consider this example as related to one interesting case of a general formulation of "Extended" TQFT's. This is described in terms of higher category theory.

The idea that category theory could play a role in clarifying problems in quantum gravity seems to have been first expressed by Louis Crane [24], who coined the term "categorification" . Categorification is a process of replacing set-based concepts by category-based concepts. Categories are structures which have not only elements (called *objects*), but also arrows, or *morphisms* between objects as logically primitive concepts. In many examples of categories, the morphisms are

functions or relations between the objects, though this is not always the case. Categorification therefore is the reverse of a process of *decategorification* which involves discarding the structure encoded in morphisms. A standard example is the semiring of natural numbers $\mathbb{N}$, which can be seen as a decategorification of the category of finite sets with set functions as arrows, since each natural number can be thought of as an isomorphism class of finite sets. The sum and product on $\mathbb{N}$ correspond to the categorical operations of coproduct (disjoint union) and product (cartesian product), which have purely arrow-based descriptions. For some further background on the concept of categorification, see work by Crane and Yetter [26], or Baez and Dolan [9].

So what we study here are *categorified* TQFT's. The program of applying categorical notions to field theories was apparently first described by Dan Freed [37], where these are called "higher algebraic" structures. What we will show here is that this broader framework allows us to make a link not just to the vacuum version of this 3D quantum gravity, but to a form in which spacetime contains matter. To do this, we use the fact that concepts of a topological quantum field theory can be described in the language of category theory. Specifically, that a TQFT is a *functor* from a category of *cobordisms*—which is topological in character— into the category of Hilbert spaces. To "categorify" this means to construct an analogous theory in the language of higher categories—in particular,

2-categories. One of the obstacles to doing this is that one needs to have a suitable 2-category analogous to the category of cobordisms.

A cobordism between manifolds $S$ and $S'$ is a manifold with boundary $M$ such that $\partial M$ is the disjoint union of $S$ and $S'$, which we think of as an arrow $M : S \to S'$. One can define composition of cobordisms, by gluing along components of the boundary, leading to the definition of a category **nCob** of $n$-dimensional cobordisms between $(n-1)$-dimensional manifolds.

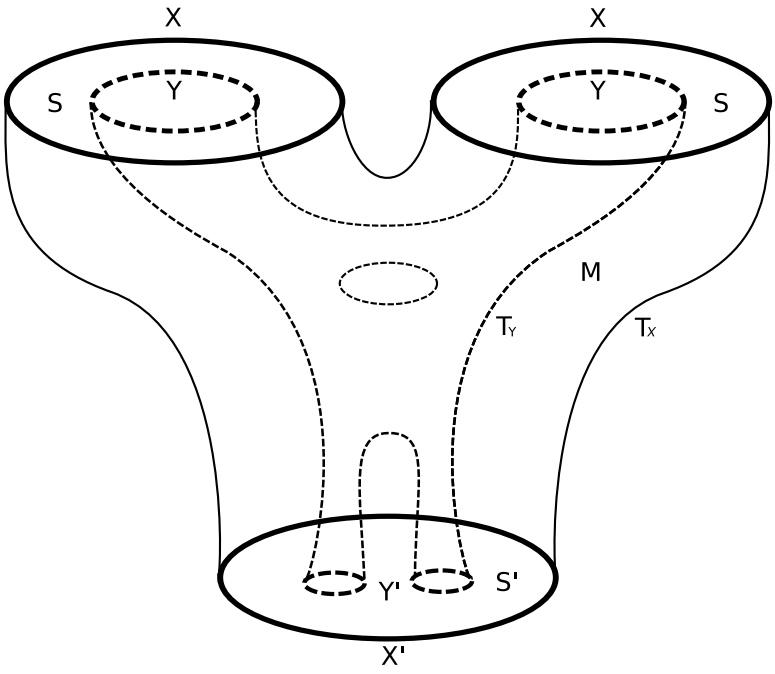



**Figure 1.1:** A Cobordism With Corners

In Figure 1.1 we see a 3-manifold with corners which illustrates these points and provides some motivating intuition. This can be seen a cobordism from the

pair of annuli at the top to the two-punctured disc at the bottom. These in turn can be thought of, respectively, as cobordisms from one pair of circles to another, and from one circle to two circles. The large cobordism has other boundary components: the outside boundary is itself a cobordism from two circles to one circle; the inside boundary (in dotted lines) is a cobordism from one pair of circles to another pair. We could "compose" this with another such cobordism with corners by gluing along any of the four boundary components: top or bottom, inside or outside. This involves attaching another such cobordism having a corresponding boundary component diffeomorphic to any of these. By "corresponding" is meant that to glue along two boundary components, one must be the source, and one the target, in the same direction—horizontal or vertical. These components are manifolds with boundary, and "gluing" is accomplished by specifying a diffeomorphism between them, which fixes the boundary.

We want to define an "extended TQFT", which assigns categorified algebraic data to the manifolds, cobordisms, and cobordisms with corners in this setting. One necessary preliminary for the example we are interested in is a description of topological quantum field theories in the usual sense. This is reviewed in Chapter 2, beginning in Section 2.1. Atiyah's axiomatic description of TQFTs [2], reviewed in Section 2.2), can be interpreted as defining TQFT's as functors from a category of *cobordisms* into **Vect**. A TQFT assigns a space of states to

each manifold, and a linear transformation between states to cobordisms. This is a functor from the category **nCob**, which has $(n-1)$-dimensional manifolds its objects and $n$-dimensional cobordisms as its morphisms.

Section 2.3 discusses a construction due to Fukuma, Hosono, and Kawai [43] for constructing a TQFT explicitly in dimension $n = 2$ starting from any finite group $G$. The FHK construction is an example of how a quantum theory intimately involves a relation between smooth geometric structures, and discretized geometric structures. Specifically, this topological theory can be thought of as coming from structures built on manifolds and cobordisms via a triangulation—a decomposition of the manifold into simplices. It turns out that there is a close connection between the ideas of a theory having "no local degrees of freedom" in the discrete and continuum setting. In the continuum setting, this means that the theory is *topological*—the vector spaces and linear operators it assigns depending only on the isomorphism class of the manifold or cobordism. In the discrete setting of a triangulated manifold, it means that the theory is *triangulation independent*

An important feature of this TQFT is that it assigns to a closed, connected 1-manifold (i.e. a circle) just some element of the centre of the group algebra of $G$, which we denoted $Z(\mathbb{C}[G])$. A standard interpretation of such a space in quantum theory would hold that this is a quantization of a classical space of states. The classical space would then simply be $Z(\mathbb{C}[G])$, so that quantum states are (com-

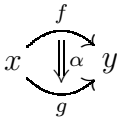
plex) linear combinations of classical states. An assignment of a group element to a circle, or loop, can be interpreted as a *connection* on the circle. Then $\mathbb{C}[G]$ consists of complex-valued linear combinations ("superpositions") of such connections. The centre, $Z(\mathbb{C}[G])$, consists of such superpositions which commute with any element of $G$ (and hence of $\mathbb{C}[G]$). These are thus invariant under conjugation by any element of $G$. Such a conjugation is a "gauge transformation" of a connection - so these elements are *gauge invariant superpositions* of connections.

These interpretations turn out to be useful when we aim to produce *extended TQFT's*. This notion was described by Ruth Lawrence [61], and denotes are theories similar to TQFT's, for which the theory is defined not on cobordisms, but on manifolds with corners. One setting where this arises is if we consider the possibility of manifolds *with boundary* connected by a cobordism. In particular, we are interested in the case where $S : X \to Y$ and $S' : X' \to Y'$ are already themselves cobordisms. These cobordisms between cobordisms, then, are manifolds with corners. Here we shall present a formalism for describing the ways such cobordisms can be glued together. Louis Crane has written a number of papers on this issue, including one with David Yetter [27] which gives a *bicategory* of such cobordisms. We want to define a structure $\mathbf{nCob}_2$, whose objects are $(n-2)$-manifolds, whose morphisms are $(n-1)$-cobordisms, and whose 2-morphisms are $n$-cobordisms with corners. Just as a TQFT assigns a space of states to a manifold and a linear map to

a cobordisms, an extended TQFT will assign some such algebraic data to $(n-2)$-manifolds, $(n-1)$-manifolds with boundary, and $n$-dimensional manifolds with corners. This data should have an interpretation similar to that for a TQFT.

To clarify how to do this, we need to consider more carefully what kind of structure $\mathbf{nCob}_2$ must be. So we consider some background on higher category theory. This field of study is still developing, but has been effectively introduced by Leinster [64] and by Cheng and Lauda [23]. The essential idea of higher category theory is However, for our purposes here, we only need to consider higher categories with morphisms represented by at most 2-dimensional cells. Chapter 3 discusses bicategories and double categories, which we will generalize later, and briefly describes some standard examples of these from homotopy theory.

Whereas a category has objects and morphisms between objects, a bicategory will have an extra layer of structure: objects, morphisms between objects, and 2-morphisms between morphisms:




(1.1)

The "strict" form of a bicategory is a 2-category, which are reviewed by Kelly and Street [53], but we are really interested in the weak forms—here, all the axioms which must be satisfied by a category hold only "up to" certain higher-dimensional morphisms. That is, what had been equations are replaced by speci-

fied 2-isomorphisms, which then must themselves satisfy certain coherence conditions. Such coherence conditions have been a persistent theme of category theory since its inception by MacLane and Eilenberg (see, for instance, [66]), and are important features of higher categorical structures.

Double categories, introduced by Ehresmann [32] [33], may be seen as "internal" categories in **Cat**. That is, a double category is a structure with a category of objects and a category of morphisms. Less abstractly, it has objects, horizontal and vertical morphisms which can be represented diagrammatically as edges, and squares. These can be composed in geometrically obvious ways to give diagrams analogous to those in ordinary category theory. Our example of cobordisms with corners appears to be an example of a double category: the objects are the manifolds, the morphisms are the cobordisms, and the 2-cells are the cobordisms with corners. In fact, as we shall see, this is too strict for our needs.

We note here that relations between TQFT's and extended TQFT's, and higher categories, are several. The categorical features of standard TQFT's are described in some detail by Bruce Bartlett [15]. Crane and Yetter [27] describe the algebraic structure of TQFT's and extended TQFT's, showing how certain algebraic and higher-algebraic structures appear from the definition of a TQFT. Examples include the well known equivalence between 2D TQFT's and Frobenius algebras; connections between 3D TQFT's and either suitable braided monoidal categories,

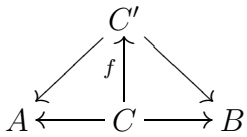
or Hopf algebras; and the appearance of "Hopf categories" in 4D TQFT's. These illustrate the move to higher-categorical structures in higher-dimensional field theories. Baez and Dolan [8] summarize the connection between TQFT's and higher category theory, in the form of the *Extended TQFT Hypothesis*, suggesting that all extended TQFT's can be viewed as representations of a certain kind of "free $n$-category".

The kind of $n$-category we are interested in here is a common generalization of a double category and a bicategory. Double categories are too strict to be really natural for our purpose, however—composition in a double category must be strictly associative, and in order to achieve this, one only considers equivalence classes of cobordisms, not cobordisms themselves, as morphisms. So we consider a weakening of this structure, in the sense that axioms for a double category giving equations (such as associativity) will be true only up to specified 2-morphisms. This allows us to take morphisms to actually be cobordisms themselves, and the diffeomorphisms between them as 2-morphisms. This is analogous to the way in which the idea of a bicategory is a weakening of the idea of a category.

Bicategories, however, are not really what we want either, since we want to describe systems with changing boundary conditions, and the most natural way to do this is by allowing both initial and final states, and these changing conditions, as part of the boundary. We call the structure which accomplishes this a Verity

double bicategory, referring to Dominic Verity, who introduced them and called them simply double bicategories. On the other hand, we show in Theorem 1 that Verity double bicategories satisfying certain conditions give rise to bicategories. In fact $\mathbf{nCob}_2$ is an example of this. The structure we use to describe such compositions is the one we call a *Verity double bicategory*. These we describe in Chapter 4. In these examples, the composition laws of double categories are weakened. That is, the associativity of composition, and unit laws, of the horizontal and vertical categories apply only up to specified higher morphisms.

In Section 4.2 we prove that a Verity double bicategory satisfying certain conditions gives a bicategory. To finish Section 4, we describe a general class of examples of Verity double bicategories, analogous to the result that Span($\mathbf{C}$) is a bicategory. A "span" is a diagram of the form $A \leftarrow C \rightarrow B$, in which one object $C$ has maps into two other objects $A$ and $B$. Given two spans $A \leftarrow C \rightarrow B$ and $A \leftarrow C' \rightarrow B$, a span map is a morphism $f : C \rightarrow C'$ such that the diagram:




(1.2)

commutes. A cospan is defined in the same way, but with the arrows reversed. It is a classical result of Bénabou [16] that for any category $C$ which has all limits, there is a bicategory Span($\mathbf{C}$) whose objects are objects of $C$, whose morphisms are spans in $C$, and whose 2-morphisms are span maps. The composition

of morphisms is by pullback - a universal construction. In Section 4.3, a similar concept in 2 dimensions is introduced, namely "double spans" and "double cospans". These give a broad class of examples of Verity double bicategories, and in particular, we can use them to derive the fact that there is a double bicategory of cobordisms with corners.

In Appendix A we return to prove some lemmas which were needed in the proof of Theorem 3. These extent some results about bicategories and double categories, namely that a double category can be seen as an *internal* category in **Cat**, and that *spans* in a category $\mathbf{C}$ with pullbacks constitute the morphisms of a bicategory, Span($\mathbf{C}$). We show a way to describe double bicategories, internal bicategories in **Bicat**, and that Verity double bicategories are simply examples of these which satisfy certain special conditions. We also show that *double spans* most naturally form an example of a double bicategory, but that they can be reduced by taking isomorphism classes in order to obtain a Verity double bicategory.

We describe more specifically the geometric framework for cobordisms with corners in Chapter 5. Gerd Laures [60] discusses the general theory of cobordisms of manifolds with corners. In the terminology used there, introduced by Jänich [48], what we primarily discuss in this work are $\langle 2 \rangle$-manifolds: in particular, the *codimension* of the manifold is 2. That is, the manifold $M$ (whose dimension is $\dim(M) = n$) will have a boundary $\partial M$, which will in turn be composed of faces

which are manifolds with boundary, of dimension $(n-1)$. However, the boundaries of these faces will be closed manifolds: they are manifolds of dimension $(n-2)$. This separates into *faces*. For us, the faces decompose into components, which are the source and the target in both horizontal and vertical directions. The corners, faces of codimension 2, are the source and target of these. We call the resulting structure $\mathbf{nCob}_2$, and in Section 5.3 we prove the main result about $\mathbf{nCob}_2$, Theorem 3, that this indeed forms a Verity double bicategory.

In Chapter 6 we turn to the next essential element of an extended TQFT, the 2-category **2Vect** of 2-vector spaces. This is the categorified equivalent of the category **Vect** of vector spaces. There are several alternative notions of what **2Vect** should be—this is a common feature of categorification, since the same structure may have arisen by discarding structure in more than one way. The view adopted here is that a 2-vector space is a certain kind of $\mathbb{C}$-linear additive category. The properties of being $\mathbb{C}$-*linear* and *additive* give equivalents of the linear structure of a vector space at both the object and morphism levels. $\mathbb{C}$-linearity means that the set of morphisms are complex vector spaces. We should remark that these properties mean that 2-vector spaces are closely related to abelian categories (introduced by Freyd [42], and studied extensively as the general setting for homological algebra) have a structure on objects which is similar to addition for vectors. In particular, we are interested in the equivalent of "finite

dimensional" vector spaces, so 2-vector spaces also need to be *finitely semi-simple*, so every object is a finite sum of simple ones.

Section 6.1 describes Kapranov-Voevodsky (KV) 2-vector spaces—the kind described above. Each of these is equivalent to the category $\mathbf{Vect}^n$ for some $n$—a higher analog of complex vector spaces, which are all equivalent to some $\mathbb{C}^n$. In fact, categories with both $\mathbb{C}$-linearity and additiveness naturally have a kind of "scalar" multiplication by vector spaces. So in the categorified setting, the category **Vect** itself plays the role of $\mathbb{C}$ for complex vector spaces. So Yetter's [88] alternative definition of a 2-vector space as a **Vect**-module turns out to be equivalent to a KV vector space in the case where it is finitely semisimple.

We describe the morphisms between KV 2-vector spaces—*2-linear maps*. A 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ can be represented as matrices of vector spaces:

$$\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{l,1} & \cdots & T_{l,k} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \tag{1.3}$$

which act on 2-vectors by matrix multiplication, using the tensor product $\otimes$ in the role of multiplication, and the direct sum $\oplus$ in the role of addition. All 2-morphisms between two such 2-linear maps can be represented as matrices of linear transformations which act componentwise.

We also show that the concept of an *adjoint* functor can be described in terms of matrix representations of 2-linear maps in much the same way that the description of the adjoint of a linear map relates to its matrix representation. So the two notions of "adjoint" turn out to be closely connected in 2-vector spaces.

A special example of a 2-vector spaces—a *group 2-algebra*—is described. This turns out to be the starting point to describe what 2-vector space a 3-dimensional extended TQFT assigns to a circle.

This example leads to discussion, in Section 6.2, of how to build 2-vector spaces from groupoids. We introduce the concept of "**Vect**-presheaves" on $\mathbf{X}$. These are just functors from $\mathbf{X}^{op}$ to **Vect** (or, since $\mathbf{X} \cong \mathbf{X}^{op}$ for a groupoid, just from $\mathbf{X}$ to **Vect**). The totality of these functors forms a category, which we call $[\mathbf{X}, \mathbf{Vect}]$, whose objects are functors from $\mathbf{X}$ to **Vect**, and whose morphisms are natural transformations between functors. One important result, Lemma 5, says that for any finite groupoid $\mathbf{X}$ (or one which is "essentially" finite, in a precise sense) the category $[\mathbf{X}, \mathbf{Vect}]$ is a KV 2-vector space.

Studying these **Vect**-presheaves on groupoids is of interest, partly because it opens up the possibility of a categorified version of quantizing a system by taking $L^2$ of its classical configuration space. This is a Hilbert space of complex-valued functions on that space—so considering a 2-vector space of $V$-valued functions is a categorified analog.

On the other hand, **Set**-valued presheaves on certain kinds of categories are generic examples of toposes, about which much is known (see, for example, Johnstone [49], [50]). Some results about these can be shown for **Vect**-valued presheaves also, although there are significant differences resulting from the fact that **Vect** is an additive category.

One of the theorems for **Vect**-valued presheaves which resembles one for **Set** is that functors between groupoids give rise to "pullback" and "pushforward" 2-linear maps between these 2-vector spaces of presheaves. Given a functor $f : \mathbf{X} \to \mathbf{Y}$, we get the "pullback" $f^* : [\mathbf{Y}, \mathbf{Vect}] \to [\mathbf{X}, \mathbf{Vect}]$, and the "pushforward" $f_* : [\mathbf{X}, \mathbf{Vect}] \to [\mathbf{Y}, \mathbf{Vect}]$. The pullback is straightforward: a functor on $\mathbf{Y}$ becomes a functor on $\mathbf{X}$ by composition with $f$. But the pushforward depends on the structure of **Vect**: as described in Definition 13, given a presheaf $V \in [\mathbf{X}, \mathbf{Vect}]$, the pushforward $f_*V$ gives a presheaf in $[\mathbf{Y}, \mathbf{Vect}]$ which gives, at any object in $\mathbf{Y}$, the colimit of a certain diagram. This depends critically on the ability to take finite colimits in **Vect**.

Both the pullback and pushforward maps carry presheaves on one groupoid to presheaves on another. For a given $f$, the two 2-linear maps $f^*$ and $f_*$ form an *ambidextrous adjunction*. That is, $f_*$ is both a left and a right adjoint to $f$, meaning that for any presheaves $V \in [\mathbf{X}, \mathbf{Vect}]$ and $W \in [\mathbf{Y}, \mathbf{Vect}]$, we have both $\hom(V, f^*W) \cong \hom(f_*V, W)$ and $\hom(f^*W, V) \cong \hom(W, f_*V)$. We then say

that they are adjoint 2-linear maps—this is an example of the connection between adjointness of functors and adjointness of linear maps.

This pair of adjoint maps, the pullback and pushforward, turns out to be essential to the constructions used to develop the extended TQFT's we are interested in. The reason is related to the fact that we described the cobordisms on which they are defined in terms of *cospans*, as we will see shortly.

In Section 6.3, we fill out some of the details of what a 2-Hilbert space should be, including a definition of the inner product, and an extension to infinite dimension. Not all of this will be used for our main theorem, but it is helpful to put the rest in perspective, and will be referred to in Chapter 8 when we discuss proposed extensions of our main results to quantum gravity.

In Chapter 7 we discuss how to construct an extended TQFT based on a double bicategory of cobordisms with corners, by means of the interpretation of a TQFT in terms of a *connection* on the manifolds involved. This is related to the *Dijkgraaf-Witten model*, a topological gauge theory. Our aim is to give a construction of an extended TQFT $Z_G$ as a weak 2-functor, starting from any finite gauge group $G$ (in a way which suggests how to extend the theory to an infinite gauge group).

Section 7.1 describes how to get a KV 2-vectorspace from a manifold. Given a manifold $B$, one first takes the fundamental groupoid $\Pi_1(B)$, whose objects are

the points in $B$ and whose morphisms are homotopy classes of paths in $B$. Then a connection on the cobordism (or one of the manifolds on the boundary) is a functor $A : \Pi_1(B) \to G$ where the gauge group $G$ is thought of as a category (in fact a groupoid) with one object.

These connections are naturally organized into a functor category $\hom(Pi_1(B), G)$, or just $[\Pi_1(B), G]$ for short. This category is a groupoid, and since manifolds have finitely generated fundamental groups. The gauge transformations are natural transformations between the functors into $G$. This functor category now plays the role of the "configuration space" of the theory.

We then want to *quantize* this configuration space $[\Pi_1(B), G]$. In ordinary quantum mechanics, quantization might involve taking an $L^2$ space of (certain) functions from a configuration space into $\mathbb{C}$. In the categorified setting, we take the category functors into **Vect**—what we have called **Vect**-presheaves—and get a 2-vector space. We will be considering only the case $G$ is finite, and as remarked, $\Pi_1(B)$ finitely generated. So then $[\Pi_1(B), G]$ is an essentially finite groupoid, and $Z_G(B) = \big[[\Pi_1(B), G], \mathbf{Vect}\big]$ will be a KV 2-vector space.

Then the question becomes one of how to find 2-linear maps from cobordisms. But a cobordism $S : B \to B'$ can be interpreted as a special cospan

$$\begin{array}{ccccc} & & S & & \\ & \overset{i}{\nearrow} & & \overset{i'}{\nwarrow} & \\ B & & & & B' \end{array} \tag{1.4}$$

with two inclusion maps. Since the operation $[\Pi_1(-), G]$ is a contravariant functor, applying it results in a span of the resulting groupoids, where the inclusions are replaced with restriction maps:

$$\begin{array}{ccccc} & & [\Pi_1(S), G] & & \\ & \overset{p}{\swarrow} & & \overset{p'}{\searrow} & \\ [\Pi_1(B), G] & & & & [\Pi_1(B'), G] \end{array} \tag{1.5}$$

This is a span where we have a groupoid of all "histories" in the middle, and of "configurations" at the ends, with projection maps from the histories to the configurations. These are source and target maps, when we think of this as a cobordism in **nCob**. This groupoid represents configurations of some system whose individual states are flat $G$-bundles. Thinking of spaces in terms of their path groupoids forces us to categorify the gauge group. The DW model accords with this if we think of $G$ as a one-object groupoid (though one might generalize to a 2-group, for example, as discussed by Martins and Porter [70]) and get a different theory.

After taking **Vect**-presheaves, we have a cospan (because the functor $[-, \mathbf{Vect}]$ is contravariant again), which looks like:

$$\begin{array}{ccccc} & & \big[[\Pi_1(S), G], \mathbf{Vect}\big] & & \\ & \overset{p^*}{\nearrow} & & \overset{(p')^*}{\nwarrow} & \\ \big[[\Pi_1(B), G], \mathbf{Vect}\big] & & & & \big[[\Pi_1(B'), G], \mathbf{Vect}\big] \end{array} \tag{1.6}$$

where the most evident choices for 2-linear maps between these KV 2-vector spaces are the pullbacks along the restriction maps. The functor $\big[[\Pi_1(-), G], \mathbf{Vect}\big]$ which gives 2-vector spaces for manifolds, and indeed topological spaces (as long as the fundamental group is finitely generated). We want to use it to yield some 2-functor $Z_G : \mathbf{nCob}_2 \to \mathbf{2Vect}$. Objects in $\mathbf{nCob}_2$ are objects in $\mathbf{C}$, but we then would like to get a 2-linear map from a cobordism. However, is given as a cospan, so we have two pullback maps in the above diagram, both of which have the adjoints discussed above. Since $S$ is a cobordism with source $B$ and target $B'$, we can replace $(p')^*$ with its adjoint, $(p')_*$, to get a 2-linear map:

$$(p')_* \circ (p)^* : Z_G(B) \to Z_G(B') \tag{1.7}$$

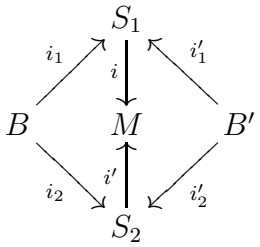
This will be $Z_G(S)$. We can describe this as a "pull-push" process. It consitsts of two stages. The first stage is a "pull", which gives a **Vect**-presheaf $p^*F$ on connections on the cobordism $S$ from $F$ on the manifold $B$. This is done by assigning to each connection $A$ on $S$ the vector space assigned by $F$ to the restriction of $A$ to $B$ (and acts on gauge transformations in a compatible way). The second stage

is a "push", which gives a **Vect**-presheaf on $B'$ from this $p^*F$ on the groupoid of connections on $S$. This assigns to each connection $A'$ on $B'$ a vector space $(p')_* \circ p^*(F)$, which is a colimit over all the connections on $S$ which restrict to $A'$. The colimit should be thought of as a direct sum over the equivalence classes of such components. The terms of the sum are, not the vector spaces assigned by $p^*F$, but quotients of these which arise from the fact that some connections may have nontrivial automorphisms.

The "pull-push" process is related to the idea of a "sum over histories". Recall that we can think of the 2-vector space of **Vect**-presheaves $Z_G(B)$ as a categorified equivalent of the Hilbert space $L^2(X)$ we get when quantizing a classical system with configuration space $X$. So what is a component in the matrix representation of the 2-linear transformation $Z_G(S)$? It is indexed by configurations (i.e. connections) on the initial and final spaces. The vector space can be interpreted as a categorified "amplitude" to get from the initial configuration to the final configuration.

A similar procedure, discussed in Section 7.3, is used to get a 2-morphism from a cobordism between cobordisms. That is, given a cobordism with corners,

$M : S_1 \to S_2$, between two cobordisms $S_1, S_2 : B \to B'$, we have :




(1.8)

To construct a natural transformation $Z_G(M) : Z_G(S_1) \to Z_G(S_2)$, a very similar process of "pull-push" The difference is that instead of pulling and pushing **Vect**-presheaves—that is, 2-vectors—one is pulling and pushing vectors. These vectors can be interpreted as $\mathbb{C}$-valued functions on a basis of the vector spaces which form the components of the 2-linear maps $Z_G(S_1)$ or $Z_G(S_2)$. Such a basis consists of equivalence classes of connections on $S_1$ and $S_2$ respectively. This basis, again, consists of configurations (connections) on $S_1$ or $S_2$ respectively. Choosing particular components (that is, fixing equivalence classes connections $A$ and $A'$ on $B$ and $B'$), one then builds a linear transformation

$$Z_G(M)_{[A],[A']} : Z_G(S_1)_{[A],[A']} \to Z_G(S_2)_{[A],[A']} \tag{1.9}$$

by a "pull-push". The "pull" phase of this process simply pulls $\mathbb{C}$-valued functors along the restriction map taking connections on $M$ to connections on $S_1$. The "push" phase here, as at the previous level, assigns to a connection $A_2$ on $S_2$ a sum over all connections on $M$ restricting to $A_2$. And again, the sum is not just of these components, but of a "quotient" which arises from the automorphism

group of each such connection on $M$. This is related to the concept of "groupoid cardinality", and this is discussed in Section 7.3.

So we have described a construction of an assignment $Z_G$ which gives a KV 2-vector space for any manifold, a 2-linear map for any cobordism of manifolds, and a natural transformation of 2-linear maps for any cobordism between cobordisms. The main theorem of the thesis, the focus of Section 7.4, is that this $Z_G$ indeed forms a weak 2-functor from $\mathbf{nCob}_2$ to **2Vect**. Along the way we will have proved most of the properties needed, and it remains to verify some technical conditions about the 2-morphisms which accomplish the weak preservation of composites and units.

Finally, Chapter 8 describes some of the motivation for this work coming from quantum gravity, and particularly 3-dimensional quantum gravity. To really apply these results to that subject, one would need to extend them. Most immediately, one would need to show that a construction like the one described will still give a weak 2-functor even when $G$ is not a finite group, but an inifinite Lie group.

To do this would presumably require the use of the infinite-dimensional variant of KV 2-vector spaces which Crane and Yetter [27] call measurable categories. This, and some of the categorified equivalent of the structure of Hilbert spaces is discussed in Section 6.3, and in Section 8.1 we discuss how it might be used to generalize the results above. In particular, we discuss the fact that we may

not have infinite colimits available to perform the "push" part of our "pull-push" construction. This means there would have to be some other way to apply the idea of a "sum over histories" in the categorified setting. Our proposal is that this should be the "direct integral" in the Crane-Yetter measurable categories mentioned above.

Section 8.2, considers the special case when $G = SU(2)$, which is the relevant gauge group for 3D quantum gravity. The particular case of interest is a 3D extended TQFT, where manifolds are 1-dimensional, joined by 2D cobordisms, which are in turn joined by 3D cobordisms with corners. We discuss how to interpret the theory as quantum gravity coupled to matter. The basic idea is that the manifolds represent boundaries of regions in space. A circle describes the boundary left when a point (up to homotopy) is removed from 2-dimensional space. The 2D cobordisms in our double bicategory can then represent the ambient space it is removed from. Alternatively the cobordisms can describe the "world-line" of such a point particle". The cobordisms of cobordisms then represent the whole "spacetime", in a general sense, in which this situation is set.

The cobordism with corners in Figure 1.1 would then be interpreted (reading top-to-bottom) as depicting a space in which two regions bounded by the outside circles merge together into a single reason over time. Inside each region at the beginning there is a single puncture. After the regions merge, the two punctures—

now in the same region—merge and split apart twice. At the "end" (i.e. the bottom of the picture), there is a single region containing two punctures. The physical intuition is that a "puncture", or equivalently the circular boundary around it, describes a point particle. The 2-vector space of states which the extended TQFT assigns to the circle is then the 2-vector space of states for a particle.

This 2-vector space consists of **Vect**-presheaves on $[\Pi_1(S^1), G]$. Example 7 shows for *finite* groups $G$ that this is generated by a finite set of objects, each of which corresponds to a pair $([g], \rho)$, where $[g]$ is a conjugacy class in $G$, and $\rho$ is a linear representation of $G$. There is an obstacle to an analogous fact in infinite dimensional 2-vector space, since these may not have a basis of simple objects. This fact is precisely analogous to the fact that an infinite dimensional Hilbert space need not have a countable basis, since it follows from the fact that not every object will be *finitely* generated from some set of simple objects - and we do not have infinite sums available in **Vect**. However, even in an infinite dimensional 2-vector space, it does make sense to speak of simple objects, and we expect these to be of the form described.

So then for $G = SU(2)$, we then have the simple **Vect**-presheaves classified by a conjugacy class in $SU(2)$, which is just an "angle" in $[0, 4\pi)$, and a representation of $SU(2)$, which are classified by integer "spins". These are precisely the same

data which label particles in 3D quantum gravity - the "angle" is a mass, which has a maximum value in 3D gravity, since mass causes a "conical defect" in the geometry of space, which has a maximum possible angle. The "spin" is related to angular momentum.

So this theory allows us to describe a space filled with world-lines of "particles" labelled by (bounded) mass and spin. This is exactly the setup of the Ponzano-Regge model of 3D quantum gravity. Our expectation is that this model can be recovered from an extended TQFT based on $SU(2)$. This is related to a program, on which more details can be found in a paper of Lee Smolin [78], which seeks to study 3D quantum gravity by means of its relation to a 3D TQFT associated to $SU(2)$ Chern-Simons theory.

Finally, in Section 8.3, we briefly suggest a possilble direction to look for links between the theory given here, and spin-foam models for BF theory, based on a categorification of the FHK state sum approach to defining an ordinary TQFT. We also suggest two more directions in which one might generalize the theory described in this thesis in the same style as the passage from finite groups to infinite Lie groups. Two others are to pass from groups to categorical groups, and to pass from groups to quantum groups.

We can think of a group as a kind of category with one object and all morphisms invertible. A categorical group will have a group of objects and a group

of morphisms, satisfying certain conditions. Replacing our gauge group $G$ with a categorical group gives a theory based not on *connections*, but on *2-connections*. There is extensive work on this topic, but a good overview is the discussion by Baez and Schreiber [12] (see also the definition of 2-bundles by Bartels [14]). The extension of the Dijkgraaf-Witten model to categorical groups is discussed in a somewhat different framework by Martins and Porter [70]. An extension of these ideas to quantum groups is less well studied, but the hope is to recover the connection between $q$-deformed $SU(2)$ and the Turaev-Viro model, just as using $SU(2)$ as gauge group recovers the Ponzano-Regge model, for quantum gravity.

In all these directions, and possibly more, the expression of an extended TQFT in functorial terms seems to provide a window on a variety of potentially useful applications and generalizations.

# Part I

# Prerequisites

# Chapter 2

# Topological Quantum Field Theories

## 2.1 The Category nCob

In this section, we review the structure of the symmetric monoidal category **2Cob** which we generalize in this thesis. Cobordism theory goes back to the work of René Thom [82], who showed that it is closely related to homotopy theory. In particular, Thom showed that *cobordism groups*, whose elements are *cobordism classes* of certain spaces, can be computed as homotopy groups in a certain complex. However, this goes beyond what we wish to examine here: a good introductory discussion suitable for our needs is found, e.g. in Hirsch [47]. There is substantial research on many questions in, and applications of, cobordism theory: a brief survey of some has been given by Michael Atiyah [3]. Some further examples related to our motivation here include Khovanov homology [55] (also

discussed in [13] and [52]), and Turaev's recent work on cobordism of knots on surfaces [84].

Two manifolds $S_1, S_2$ are *cobordant* if there is a compact manifold with boundary, $M$, such that $\partial M$ is isomorphic to the disjoint union of $S_1$ and $S_2$. This $M$ is called a *cobordism* between $S_1$ and $S_2$. We note that there is some similarity between this concept and that of homotopy of paths, except that such homotopies are understood as embedded in an ambient space. We will return to this in Section 3.5. Our aim here is to describe a generalization of categories of cobordisms. To begin with, we recall some of the structure of **nCob**, and particularly **2Cob**, to recall why this is of interest.

**Definition 1 2Cob** *is the category with:*

- *Objects: one-dimensional compact oriented manifolds*
- *Morphisms: diffeomorphism classes of two-dimensional compact oriented cobordisms between such manifolds.*

That is, the objects are collections of circles, and the morphisms are (diffeomorphism classes of) manifolds with boundary, whose boundaries are broken into two parts, which we consider their source and target. We think of the cobordism as "joining" two manifolds, rather as a relation joins two sets, in the category of sets and relations (this analogy will be made more precise when we discuss spans

and cospans). More generally, **nCob** is the category whose objects are (compact, oriented) $(n-1)$-dimensional manifolds, and whose morphisms are diffeomorphism classes of compact oriented $n$-dimensional cobordisms.

It has been known for some time that **2Cob** can be seen as the free symmetric monoidal category on a commutative Frobenius object. (This is shown in the good development by Joachim Kock [56].) This is a categorical formulation of the fact, shown by Abrams [1], that **2Cob** is generated from four generators, called the **unit**, **counit**, **multiplication**, **comultiplication**, subject to some relations. The generating cobordisms are the following: taking the empty set to the circle (the unit); taking two circles to one circle (the multiplication); adjoints of each of these (counit and comultiplication respectively).

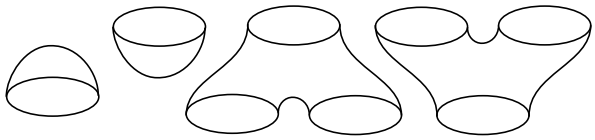


**Figure 2.1:** Generators of **2Cob**

The "commutative Frobenius object" here is the circle, equipped with these morphisms, as illustrated in Figure 2.1. The relations which these are subject to include associativity, coassociativity, and relations for the unit and counit. The most interesting is the Frobenius relation, illustrated in Figure 2.2.

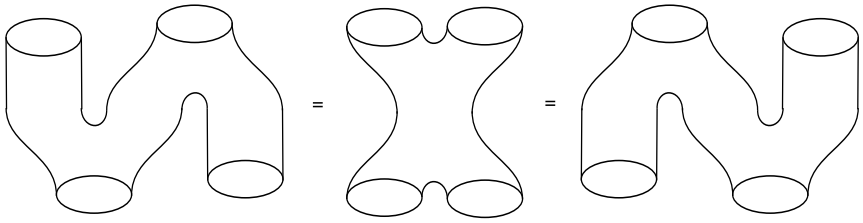


**Figure 2.2:** The Frobenius Relation

Diffeomorphism classes of cobordisms automatically satisfy these relations, since they identify composites of cobordisms which are, in fact, diffeomorphic.

Moreover, as a *monoidal* category, **2Cob** must have a tensor product operation. For objects, this is just the disjoint union: given objects $\mathbf{m}, \mathbf{n} \in \mathbf{2Cob}$, consisting of collections of $m$ and $n$ circles respectively, the object $\mathbf{m} \otimes \mathbf{n}$ is the disjoint union of $\mathbf{m}$ and $\mathbf{n}$: a collection of $m + n$ circles. The tensor product of two cobordisms $\mathbf{C_1} : \mathbf{m_1} \to \mathbf{n_1}$ and $\mathbf{C_2} : \mathbf{m_2} \to \mathbf{n_2}$ is likewise the disjoint union of the two cobordisms, giving $\mathbf{C_1} \otimes \mathbf{C_2} : \mathbf{m_1} \otimes \mathbf{m_2} \to \mathbf{n_1} \otimes \mathbf{n_2}$.

This monoidal operation has a *symmetry*, so in particular **2Cob** also includes the **switch** cobordism, exchanging the order of two circles by two cylinders (this gives the symmetry for the monoidal operation). These are required to exist by the assumption that **2Cob** is a free symmetric monoidal category. They are illustrated in Figure 2.3 (along with the identity, which is, of course, also required).

Two proofs can be given for the fact than **2Cob** is generated by these cobordisms. Each proof relies on some special conditions satisfied by 2D cobordisms.

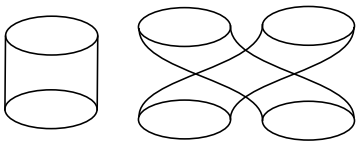


**Figure 2.3:** Morphisms Required for **2Cob** to be a Symmetric Monoidal Category

The first is that 2-dimensional manifolds with boundary can be completely classified up to diffeomorphism class by genus and number of punctures. The second is that we can use the results of Morse theory to decompose any such surface, equipped with a smooth Morse function into $[0, 1]$, into a composite of pieces, in the sense of composition of morphisms in **2Cob**. In each piece, there is just one "topology change" (a value in $[0, 1]$ where the preimage changes topology). We will return to this point when we discuss the question of how to present $\mathbf{nCob}_2$ in terms of generators.

So far, we have described the presentation of **2Cob** in terms of generators and relations, but not yet how the composition operation for morphisms works. The main idea is that we compose cobordisms by identifying their boundaries. However, since the morphisms in **2Cob** are *diffeomorphism* classes of manifolds with boundary, some extra considerations are needed to ensure that the composite is equipped with a differentiable structure.

In particular, the *collaring theorem* means that any manifold with boundary, $M$ can be equipped with a "collar": an injection $\phi : \partial M \times [0, 1] \to M$ such that

$\phi(x, 0) = x, \forall x \in \partial M$. The idea is that, while we can compose *topological* cobordisms along their boundaries, we should compose *smooth* cobordisms $M_1$ and $M_2$ along collars. This ensures that every point—including points on the boundary of $M_i$—will have a neighborhood with a smooth coordinate chart. Section 5.1 describes this in detail for a more general setting.

The category **2Cob** is particularly interesting in the study of topological quantum field theories (TQFT's), as formalized by Michael Atiyah [2]. Atiyah's axiomatic formulation of a TQFT amounts to saying that it is a symmetric monoidal functor $F : \mathbf{2Cob} \to \mathbf{Vect}$. The presentation of **2Cob** means that this immediately defines an algebraic structure with a unit, counit, multiplication, comultiplication, and identity, which satisfy the same relations as the corresponding cobordisms. This, together with the fact that $F$ preserves the symmetric monoidal structure of **2Cob** means that this structure satisfies the axioms of a commutative Frobenius algebra. A similar presentation has not been found for **nCob** for general $n$.

One may wish to describe an "extended topological quantum field theory" in the same format. These are topological field theories which are defined not just on manifolds with boundary, but also on manifolds with corners. This idea is described by Ruth Lawrence in [61]. In particular, what we are interested in here is that, instead of using a category of cobordisms between manifolds, we would

want to use some structure of cobordisms between *cobordisms between* manifolds, which we tentatively call $\mathbf{nCob}_2$. However, to do this, we must use a structure with more elaborate than a mere category.

Later, we will describe such a structure—a *Verity double bicategory*, and show how the putative $\mathbf{nCob}_2$ is an example, and indeed a special case of a wider class of examples.

## 2.2 TQFT's as Functors

Atiyah's formulation of the axioms for a TQFT can be summarized as follows:

**Definition 2** *A Topological Quantum Field Theory is a (symmetric) monoidal functor*

$$Z : \mathbf{2Cob} \to \mathbf{Vect} \tag{2.1}$$

*where* **2Cob** *is as described in Section 2.1, and* **Vect** *is the category whose objects are vector spaces and whose arrows are linear transformations.*

We note that **Vect** is naturally made into a *monoidal* category with the tensor product $\otimes$, where $V_1 \otimes V_2$ is generated by objects of the form $v_1 \otimes v_2$, modulo relations imposing bilinearity. Moreover, **2Cob** is a monoidal category as well, whose monoidal product on objects and morphisms is just the disjoint union of manifolds and cobordisms, respectively.

In fact, a quantum field theory should give a *Hilbert* space of states. However, **Hilb**, the category of Hilbert spaces and bounded linear maps, is a subcategory of **Vect**, so the above is still true.

What, however, does this definition mean?

A TQFT should give a *Hilbert space of states* for any manifold representing "space", and a map from one space of states to another for any cobordism representing "spacetime" connecting two space slices. Figure 2.4 shows an example in the case where space is 1-dimensional and spacetime is 2-dimensional:

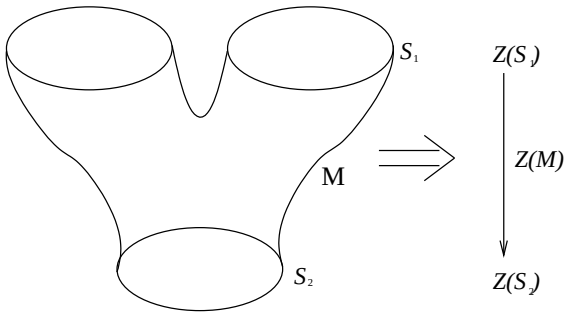



**Figure 2.4:** Effect of a TQFT

The TQFT should have the following properties:

- The Hilbert space assigned to a disjoint union of spaces $S_1 \amalg S_2$ will be the tensor product of the spaces assigned to each, $Z(S_1) \otimes Z(S_2)$, and therefore also $Z(\varnothing) = \mathbb{C}$ (a basic feature of quantum theories)

- The linear maps assigned to cobordisms respect "composition" of space-times, so $M_1$ followed by $M_2$ is assigned the map $Z(M_2) \circ Z(M_1)$, where "followed by" means the ending space of $M_1$ is the beginning space of $M_2$.

As remarked in Section 2.1, **2Cob** is a free symmetric monoidal category on a Frobenius object. In **Vect**, such an object is called a *Frobenius algebra*: in fact, a 2D TQFT $Z$ is equivalent to a choice of Frobenius algebra, namely the image of the circle uvder $Z$.

In general higher dimensions, no equally straightforward description of an $n$-dimensional TQFT is known. To provide one would require a presentation of **nCob** in terms of generators and relations (for both objects and morphisms).

Lauda and Pfeiffer [59] do provide such a presentation a similar, though more complicated, characterization of 2-dimensional *open-closed* TQFT's. In these, we do not assume that the manifolds representing spaces have no boundary. Lauda's doctoral thesis [58] develops this further.

## 2.3 The Fukuma-Hosono-Kawai Construction and Connections

Frobenius algebras are *semisimple* algebras $A$ (direct sums of simple algebras). These are characterized by having a nondegenerate linear pairing:

$$g : A \otimes A \to \mathbb{C} \tag{2.2}$$

If $A$ is a matrix algebra, then such a $g$ is given by the Killing form, or *trace*: $g(a, b) = \mathrm{tr}(L_a L_b)$. The nondegeneracy of this pairing means that it gives an isomorphism between $A$ and $A^*$.

Each algebra $A$ of this kind gives a TQFT whose effects can be described in an explicit and combinatorial way. This is the construction of Fukuma, Honoso, and Kawai [43]. We will be particularly interested in the case where the semisimple algebra $A$ is the group algebra $\mathbb{C}[G]$ for some finite group $G$.

Now we want to see how to get a TQFT $Z : \mathbf{2Cob} \to \mathbf{Vect}$ from any such algebra $A$, keeping in mind the example $A = \mathbb{C}[G]$. To do this, we first construct a map $\hat{Z} : \mathbf{\Delta 2Cob} \to \mathbf{Vect}$, where $\mathbf{\Delta 2Cob}$ is the category of *triangulated* manifolds and cobordisms, then show it is independent of the choice of triangulation.

To begin with, given a triangulated cobordism $M$ from $S_1$ to $S_2$, (so $M$, $S_1$ and $S_2$ are all triangulated), label the dual graph with copies of $A$.

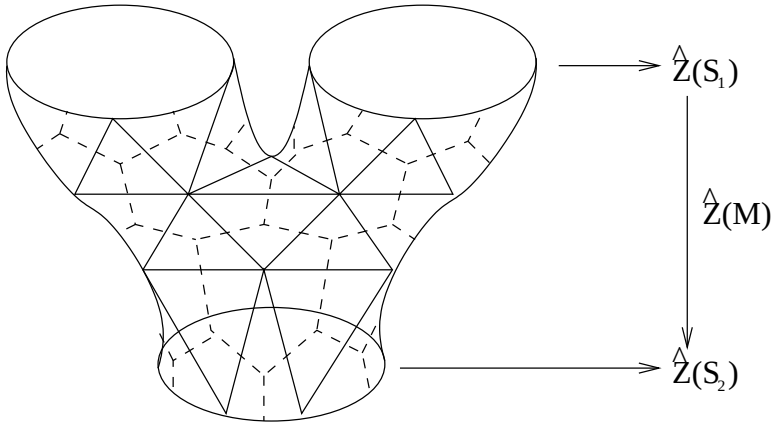



**Figure 2.5:** The Fukuma-Honoso-Kawai Construction

So each edge of a triangle (hence of the dual graph) is labelled by $A$ and each face of a triangle (hence each vertex of the dual graph) by an operator $m$.

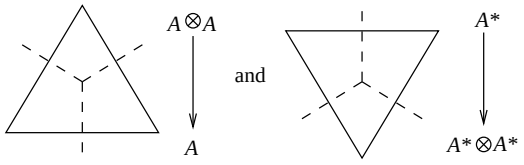



**Figure 2.6:** Multiplication Operators Assigned to Triangles

In the case where the semisimple algebra is $\mathbb{C}[G]$, we can write choices of vector in a basis consisting of group elements. So labellings of the dual edges can be described in terms of a basis where the dual edges are labelled with group elements.

## 2.4 Pachner Moves in 2D

How does $\hat{Z}$, acting on $\Delta$**2Cob**, give a TQFT acting on **2Cob**? First, notice that it depends only on the topology of $M$, and the triangulation on the boundary, not in the interior.

This is because **Alexander's Theorem** says that to pass between any two triangulations of the same compact 2-manifold, it is enough to repeatedly apply the two *Pachner moves*—the **2-2 move** and the **1-3 move** (and their inverses):

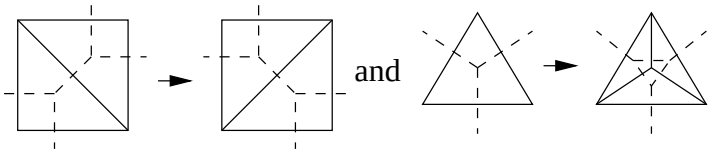



**Figure 2.7:** Pachner Moves

This will prove that the linear map we construct is independent of the triangulation chosen. In particular, the 2-2 move does not affect the outcome of composition, on applying $\hat{Z}$, since it passes from

$$V \otimes V \otimes V \xrightarrow{1 \otimes m} V \otimes V \xrightarrow{m} V \tag{2.3}$$

to

$$V \otimes V \otimes V \xrightarrow{m \otimes 1} V \otimes V \xrightarrow{m} V \tag{2.4}$$

These are the same by associativity.

The 1-3 move has no effect precisely when $(A, \eta, m)$ is semisimple. This comes from associativity and the "bubble move":

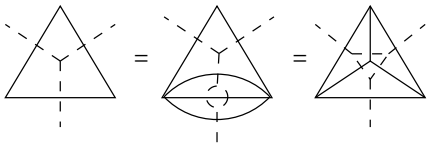


**Figure 2.8:** The Bubble Move

We can think of the Pachner moves as coming from tetrahedrons. Given a triangulation, attach a tetrahedron along one, two, or three triangular faces. The move consists of replacing the attached faces with the remaning faces of the tetrahedron. We can think of this as "evolving the triangulation by" that tetrahedron:

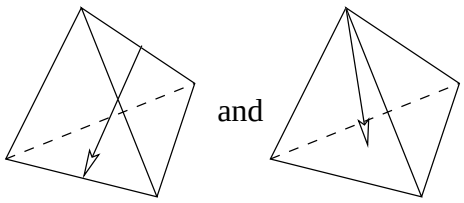



**Figure 2.9:** Pachner Moves as Tetrahedra

Any two triangulations are homologous—can be connected by a series of such moves since there is no nontrivial third homology of a 2D surface: any change in triangulation we want will be the boundary of some collection of tetrahedra.

(A triangulation of a 2-dimensional cobordism is a combination of 0, 1, and 2-chains—Pachner moves correspond to 3-chains).

Now, we know that a TQFT is determined by its effect on the generators of **2Cob**, so we want to know the space of states on $S^1$, which is a generator for objects. One observation is that the image of the generator $S^1 \times [0,1]$ is id, the identity map on $Z(S^1)$.

Consider the following triangulation of $S^1 \times [0,1]$:

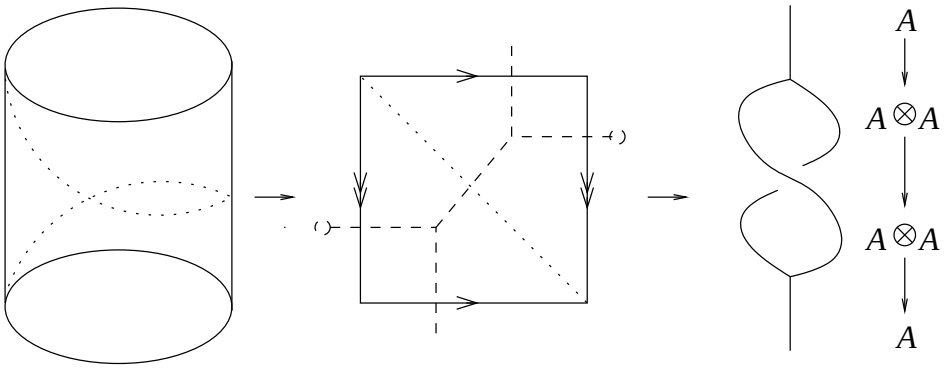



**Figure 2.10:** Identity or Projection Operator?

$\hat{Z}$ assigns $A$ to the top and bottom circles, but says that we should have

$$m \circ B \circ m^\dagger = \mathrm{id} \tag{2.5}$$

on $Z(S) \subset A$. This means that $Z(S)$ is a subset of the *centre* of $A$.

We know that the identity map in $A$ must come from the cylinder, so define

$$Z(S^1) = \mathrm{Ran}(\hat{Z}(S^1 \times [0,1])) \tag{2.6}$$

To get a TQFT $Z$, we restrict $\hat{Z}$ to $Z(S^1)$. This is a projection operator, and its range is in $Z(A)$. Project the space of states for a triangulated circle onto this to get the space of states for the circle under $Z$ (note that there is only one way to do this, independent of which triangulation of the cylinder we use to get the projection operator).

So it is well-defined to say:

$$Z(M) = \hat{Z}(M)|_{Z(S^1)} \tag{2.7}$$

since we always have $\hat{Z}(M)(Z(S_1)) = Z(S_2)$. (One can retriangulate $M$ to compose with the projection before and after, without changing the result.)

Then one can show that this $Z$ defines a *symmetric monoidal functor* from **2Cob** to **Vect**, namely a TQFT.

## 2.5 TQFT's and Connections

The FHK construction of a TQFT has a feature which may not at first be obvious. To the circle, $Z$ assigns a Hilbert space, but in a way that has a canonical choice of basis. This is $Z(S^1)$, the centre of the group algebra $\mathbb{C}[G]$, or simply $\mathbb{C}[\mathrm{Cent}(G)]$, the vector space spanned by the centre of the group $G$. So a basis for the space of states is just the set of ways of assigning to the circle an element of the group $G$ which happens to be in the centre of $G$.

One way to think of this is as a *G-connection* on the circle - so that the space of states is a free vector space on the set of $G$-connections on $S^1$. This way of thinking of what $Z$ produces is good because it will hold up even when we consider manifolds $B$ of higher dimension (and codimension). In particular, if a TQFT gives a space of states from the set of connections on $B$, given a map from the circle into $B$, any connection assigns to this loop a group element, or *holonomy*, up to conjugation.

So in order to look at extended TQFT's as examples of a categorification of the concet of a TQFT, it is useful to take this point of view relating the TQFT to connections. We point out, however, that there is a categorified analog of the FHK construction more or less directly. We expect that this would provide a "state-sum" point of view on the theory of a connection on a manifold which our extended TQFT will in fact involve. In fact, this is understood to a considerable degree, but this point of view is awkward because it involves the categorified versions of associativity - Stasheff's associahedra [79]. These play the role of Pachner moves in higher dimensions. We could proceed with this categorified version of the construction, when $G$ is a finite group.

It turns out that a natural generalization of the FHK construction gives a theory equivalent to the (untwisted) Dijkgraaf-Witten model [30]. This is a topological gauge theory, which crucially involves a (flat) connection on a manifold. We

will discuss this in more detail in Section 7.1, and explore how an extended TQFT can be constructed by taking a categorifed version of the (quantized) theory of a flat connection on manifolds and cobordisms.

# Chapter 3

# Bicategories and Double Categories

We will want to give a description of a *Verity double bicategory*, which is a weakened version of the concept of a double category, in order to describe cobordisms with corners. Weakening a concept $X$ in category theory generally involves creating a new concept in which equations in the original concept are replaced by isomorphisms. Thus, we say that the old equations hold only "up to" isomorphism in the weak version of $X$, and say that when they hold with equality, we have a "strict $X$". Thus, before describing our newly weakened concept, it makes sense to recall how this process works, and examine the strict form of the concept we want to weaken. We also want to see what the weakening process entails. So we begin by reviewing bicategories and double categories.

## 3.1 2-Categories

A category **E** is **enriched over** a category **C** (which must have products) when for $x, y \in \mathbf{E}$ we have $\hom(x, y) \in \mathbf{C}$. A special case of this occurs in "closed" categories, which are enriched over themselves—examples include **Set** (since there is a set of maps between any two sets) and **Vect** (since the linear operators between two vector spaces form a vector space).

A **2-category** is a category enriched over **Cat**. That is, if $\mathbf{C_2}$ is a 2-category, and $x, y \in \mathbf{C_2}$), then $\hom(x, y) \in \mathbf{Cat}$. Thus, there are sets of objects and morphisms in $\hom(x, y)$ itself, satisfying the usual category axioms. We describe a 2-category as having **objects**, **morphisms** between objects, and **2-morphisms** between morphisms. The morphisms of $\mathbf{C_2}$ are the objects of the hom-categories, and the 2-morphisms of $\mathbf{C_2}$ are the morphisms of the hom-categories. We depict these as in Diagram (1.1). There is a composition operation for morphisms in these hom categories, which we think of as "vertical" composition, denoted $\cdot$, between 2-morphisms. Furthermore, for all $x, y, z \in \mathbf{C_2}$, the composition operation

$$\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z) \tag{3.1}$$

must be a functor between hom-categories. So in particular this operation applies to both objects and morphisms in hom categories, and we think of these as "horizontal" composition for both morphisms and 2-morphisms. The requirement that

this be a functor means that the **interchange law** holds:

$$(\alpha \circ \beta) \cdot (\alpha' \circ \beta') = (\alpha \cdot \alpha') \circ (\beta \cdot \beta') \tag{3.2}$$

Now, in a 2-category, the associative law holds strictly: that is, for morphisms $f \in \hom(w, x)$, $g \in \hom(x, y)$, and $h \in \hom(y, z)$, we have the two possible triple-compositions in $\hom(w, z)$ the same, namely $f \circ (g \circ h) = (f \circ g) \circ h$. This is one of the axioms for a category—that is, a category enriched over **Set**. Since a 2-category is enriched over **Cat**, however, a weaker version of this rule is possible, since $\hom(w, z)$ is no longer a set in which elements can only be equal or unequal: it is a category, where it is possible to speak of isomorphic objects. This fact leads to the notion of bicategories.

## 3.2 Bicategories

Once we have the concept of a 2-category, we can *weaken* this concept, giving the idea of a **bicategory**. The definition is similar to that for a 2-category, but we only insist that the usual equations should be natural isomorphisms (satisfying some equations). That is, the following diagrams should commute up to natural

isomorphisms:

$$\begin{array}{ccc} \hom(w,x)\times\hom(x,y)\times\hom(y,z) & \xrightarrow{1\times\circ} & \hom(w,x)\times\hom(x,z) \\ \downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle \circ} \\ \hom(w,y)\times\hom(y,z) & \xrightarrow[\circ]{} & \hom(w,z) \end{array} \tag{3.3}$$

and

$$\begin{array}{ccc} \hom(x,y)\times\mathbf{1} & & \\ \downarrow{\scriptstyle \mathrm{id}\times !} & \searrow^{\pi_1} & \\ \hom(x,y)\times\hom(x,x) & \longrightarrow & \hom(x,y) \end{array} \tag{3.4}$$

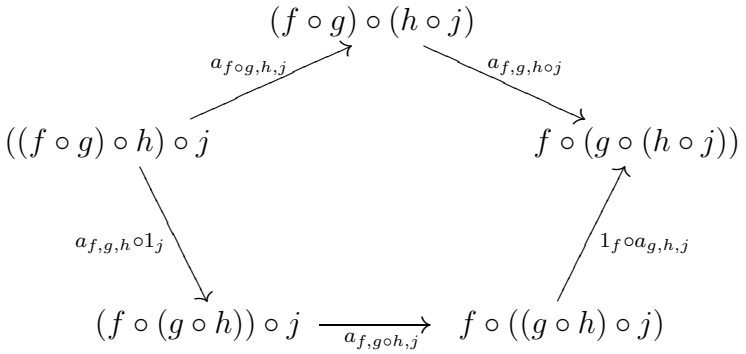
and

$$\begin{array}{ccc} \mathbf{1}\times\hom(x,y) & & \\ \downarrow{\scriptstyle !\times\mathrm{id}} & \searrow^{\pi_2} & \\ \hom(y,y)\times\hom(x,y) & \longrightarrow & \hom(x,y) \end{array} \tag{3.5}$$

That is: given $(f,g,h) \in \hom(w,x)\times\hom(x,y)\times\hom(y,z)$, there should be an isomorphism $a_{f,g,h} \in \hom(w,z)$ with $a_{f,g,h} : (f\circ g)\circ h \to f\circ(g\circ h)$; and isomorphisms $r_f : f\circ 1_x$, $l_f : 1_y \circ f$. The equations these satisfy are *coherence laws*. MacLane's Coherence Theorem shows that all such equations follow from two generating equations: the pentagon identity, and the unitor law:

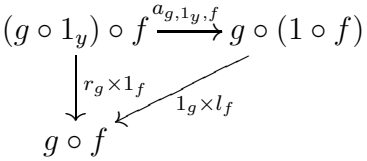
In a category, the associativity property stated that two composition operations can be performed in either order and the results should be equal: equality is the only sensible relation between a pair of morphisms in a category. There is an analogous statement for the associator 2-morphism: two different ways of composing it should yield the same result (since equality is the only sensible relation

between a pair of 2-morphisms in a bicategory). This property is the pentagon identity:




(3.6)

Similarly, the unit laws satisfy the property that the following commutes:




(3.7)

This last change is the sort of weakening we want to apply to the concept of a double category. Following the same pattern, we will first describe the strict notion in Section 3.4, before considering how to weaken it, in Chapter 4. First, however, we will recall a standard, quite general, example of bicategory, which we will generalize to give examples of double bicategories in Section 4.3.

## 3.3 Bicategories of Spans

Jean Bénabou [16] introduced bicategories in a 1967 paper, and one broad class of examples introduced there comes from the notion of a *span*.

**Definition 3** *(**Bénabou**) Given any category* $\mathbf{C}$*, a* ***span*** $(S, \pi_1, \pi_2)$ *between objects* $X_1, X_2 \in \mathbf{C}$ *is a diagram in* $\mathbf{C}$ *of the form*

$$P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2 \tag{3.8}$$

*Given two spans* $(S, s, t)$ *and* $(S', s', t')$ *between* $X_1$ *and* $X_2$ *between a* ***morphism of spans*** *is a morphism* $g : S \to S'$ *making the following diagram commute:*

$$\begin{array}{ccccc} & & S & & \\ & \swarrow^{\pi_1} & \downarrow g & \searrow^{\pi_2} & \\ X_1 & \xleftarrow[\pi_1']{} & S' & \xrightarrow[\pi_2']{} & X_2 \end{array} \tag{3.9}$$

*Composition of spans* $S$ *from* $X_1$ *to* $X_2$ *and* $S'$ *from* $X_2$ *to* $X_3$ *is given by a pullback: that is, an object* $R$ *with maps* $f_1$ *and* $f_2$ *making the following diagram commute:*

$$\begin{array}{ccccc} & & R & & \\ & \swarrow^{f_1} & & \searrow^{f_2} & \\ & S & & S' & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \quad \swarrow^{\pi_2'} & & \searrow^{\pi_3'} \\ X_1 & & X_2 & & X_3 \end{array} \tag{3.10}$$

*which is terminal among all such objects. That is, given any other* $Q$ *with maps* $g_1$ *and* $g_2$ *which make the analogous diagram commute, these maps factor through a unique map* $Q \to R$*. R becomes a span from* $X_1$ *to* $X_3$ *with the maps* $\pi_1 \circ f_1$ *and* $\pi_2 \circ f_2$*.*

The span construction has a dual concept:

**Definition 4** *A **cospan** in* $\mathbf{C}$ *is a span in* $\mathbf{C}^{\mathrm{op}}$*, morphisms of cospans are morphisms of spans in* $\mathbf{C}^{\mathrm{op}}$*, and composition of cospans is given by pullback in* $\mathbf{C}^{\mathrm{op}}$*. That is, by a pushout in* $\mathbf{C}$*.*

One fact about (co)spans which is important for our purposes is that any category $\mathbf{C}$ with limits (colimits, respectively) gives rise to a bicategory of spans (or cospans). This relies in part on the fact that the pullback is a universal construction (universal properties of Span($\mathbf{C}$) are discussed by Dawson, Paré and Pronk [29]).

**Remark 1 [16], ex. 2.6** Given any category $\mathbf{C}$ with all limits, there is a bicategory Span($\mathbf{C}$), whose objects are the objects of $\mathbf{C}$, whose *hom*-sets of morphisms Span($\mathbf{C}$)($X_1, X_2$) consist of all spans between $X_1$ and $X_2$ with composition as defined above, and whose 2-morphisms are morphisms of spans. Span($\mathbf{C}$) as defined above forms a bicategory (Cosp($\mathbf{C}$), of cospans similarly forms a bicategory).

This is a standard result, first shown by Jean Bénabou [16], as one of the first examples of a bicategory. We briefly describe the proof:

The identity for $X$ is $X \overset{id}{\leftarrow} X \overset{id}{\rightarrow} X$, which is easy to check.

The associator arises from the fact that the pullback is a *universal* construction. Given morphisms in Span($\mathbf{C}$) $f : X \rightarrow Y$, $g : Y \rightarrow Z$, $h : Z \rightarrow W$, the

composites $((f \circ g) \circ h)$ and $(f \circ (g \circ h))$ are pullbacks consisting of objects $O_1$ and $O_2$ with maps into $X$ and $W$. The universal property of pullbacks gives an isomorphism between $O_1$ and $O_2$. These isomorphisms satisfy the pentagon identity since they are unique (in particular, both sides of the pentagon give the same isomorphism).

It is easy to check that $\hom(X_1, X_2)$ is a category, since it inherits all the usual properties from $\mathbf{C}$.

## 3.4 Double Categories

The idea of a double category extends that of a category into two dimensions in a different way than does the concept of bicategory. A double category consists of:

- a set $O$ of objects
- *horizontal* and *vertical* categories, whose sets of objects are both $O$
- for any diagram of the form

$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[\phi']{} & y' \end{array} \tag{3.11}$$

a collection of square *2-cells*, having horizontal source and target $f$ and $f'$, and vertical source and target $\phi$ and $\phi'$

The 2-cells can be composed either horizontally or vertically in the obvious way. We denote a 2-cell filling the above diagram like this:

$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ {\scriptstyle f}\downarrow & \swarrow S & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[\phi']{} & y' \end{array} \tag{3.12}$$

and think of the composition of 2-cells in terms of pasting these squares together along an edge. The resulting 2-cell fills a square whose boundaries are the corresponding composites of the morphisms along its edges.

Moskaliuk and Vlassov [73] discuss the application of double categories to mathematical physics, particularly TQFT's, and dynamical systems with changing boundary conditions—that is, with inputs and outputs. Kerler and Lyubashenko [54] describe extended TQFT's as "double pseudofunctors" between double categories. This formulation involves, among other things, a double category of cobordisms with corners—we return to a weakening of this idea in Section 5.3

A double category can be thought of as an internal category in **Cat**. That is, it is a model of the theory of categories, denoted **Th(Cat)**, in **Cat**. This **Th(Cat)** consists of a category containing all finite limits, and having two distinguished

objects called Obj and Mor with morphisms of the form:

$$\mathrm{Mor} \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Obj} \tag{3.13}$$

and

$$\mathbf{Obj} \xrightarrow{id} \mathrm{Mor} \tag{3.14}$$

subject to some axioms. In particular, the composition operation is a partially defined operation on pairs of morphisms. In particular, there is a collection of composable pairs of morphisms, namely the fibre product $\mathbf{Pairs} = \mathrm{Mor} \times_{\mathbf{Obj}} \mathrm{Mor}$, which is a pullback of the two arrows from Mor to **Obj**. So **Pairs** is an equalizer in the following diagram:

$$\begin{array}{ccccc} & & \mathrm{Mor} & & \\ & \overset{\pi_1}{\nearrow} & & \overset{t}{\searrow} & \\ \mathbf{Pairs} \xrightarrow{i} & \mathrm{Mor}^2 & & & \mathrm{Obj} \\ & \underset{\pi_2}{\searrow} & & \overset{s}{\nearrow} & \\ & & \mathrm{Mor} & & \end{array} \tag{3.15}$$

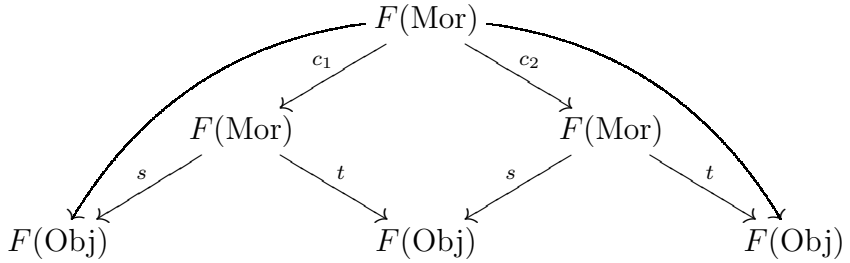
(Note that we assume the existence of pullbacks, here - in fact, **Th(Cat)** is a *finite limits theory.*) The composition map $\circ : \mathbf{Pairs} \to \mathrm{Mor}$ satisfies the usual properties for composition.

There is also an identity for each object: there is a map $\mathrm{Obj} \xrightarrow{1} \mathrm{Mor}$, such that for any morphism $f \in \mathrm{Mor}$, we have $1_{s(f)}$ and $1_{t(f)}$ are composable with $f$, and the composite is $f$ itself.

A model of **Th(Cat)** in **Cat** is a (limit-preserving) functor

$$F : \mathbf{Th(Cat)} \to \mathbf{Cat}$$

This gives a structure having a category **Ob** of objects and a category **Mor** of morphisms, with two functors $s$ ("source") and $t$ ("target") satisfying the usual category axioms. We can describe composition as a pullback construction in this category, which makes sense since the functor preserves finite limits (including pullbacks):




(3.16)

A category is a model of the theory **Th(Cat)** in **Set**, and we understand this to mean that when two morphisms $f$ and $g$ have the target of $f$ the same as the source of $g$, there is a composite morphism from the source of $f$ to the target of $g$. In the case of a double category, we have a model of **Th(Cat)** in **Cat**, so that $F(\text{Obj})$ and $F(\text{Mor})$ are categories and $F(s)$ and $F(t)$ are functors, we have the same condition for both objects and morphisms, subject to the compatibility conditions for these two maps which any functor must satisfy.

We thus have sets of objects and morphisms in **Ob**, which of course must satisfy the usual axioms. The same is true for **Mor**. The category axioms for the

double category are imposed in addition to these properties, for the composition and identity functors. Functoriality implies compatibility conditions between the category axioms in the two directions. The result is that we can think of both the objects in **Mor** and the morphisms in **Ob** as acting like morphisms between the objects in **Ob**, in a way compatible with the source and target maps. A double category can be, and often is, thought of as including the morphisms of two (potentially) different categories on the same collection of objects. These are the *horizontal* and *vertical* morphisms, intuitively capturing the picture:

$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[\hat{\phi}]{} & y' \end{array} \tag{3.17}$$

Here, the objects in the diagram can be thought of as objects in $F(\mathrm{Obj})$, the vertical morphisms $f$ and $f'$ can be thought of as morphisms in $F(\mathrm{Obj})$ and the horizontal morphisms $\phi$ and $\hat{\phi}$ as objects in $F(\mathrm{Mor})$. (In fact, there is enough symmetry in the axioms for an internal category in **Cat** that we can adopt either convention for distinguishing horizontal and vertical morphisms). However, we also have morphisms in Mor. We represent these as two-cells, or *squares*, like the 2-cell $S$ represented in (3.12).

The fact that the composition map $\circ$ is a functor means that horizontal and vertical composition of squares commutes.

## 3.5 Topological Examples

We can illustrate simple examples of bicategories and double categories in a topological setting, namely homotopy theory. This was the source of much of the original motivation for higher-dimensional category theory. Moreover, as we have already remarked in Section 2.1, there are close connections between cobordism and homotopy. These examples will turn out to suggest how to describe Verity double bicategories of cobordisms.

Our first example is one of the original motivations for bicategories.

**Example 1** Given a space $S$ in the category **Top** of topological spaces, we might wish to define a category Path($S$) whose objects are points of $X$, and whose morphisms are paths in $S$. That is, $Path(S)$ has:

- objects: points in $S$
- morphisms: paths $\gamma : [m, n] \to S$

Where such a path is thought of as a morphism from $\gamma(m)$ to $\gamma(n)$. These are parametrized paths: so suppose we are given two paths in Path($S$), say $\gamma_1 : [m_1, n_1] \to S$ from $a$ to $b$ and $\gamma_2 : [m_2, n_2] \to S$ from $b$ to $c$. Then the composite is

a path $\gamma_2 \circ \gamma_1 : [m_1, n_1 + n_2 - m_2] \to S$, given by:

$$\gamma_2 \circ \gamma_1(x) = \begin{cases} \gamma_1(x) & \text{if } x \in [m_1, n_1) \\ \gamma_2(x - n_1 + m_2) & \text{if } x \in [n_1, n_1 + n_2 - m_2] \end{cases} \tag{3.18}$$

This gives a well-defined category $Path(S)$, but has the awkward feature that our morphisms are not paths, but paths *equipped with parametrization*. So another standard possibility is to take morphisms from $a$ to $b$ to be paths $\gamma : [0,1] \to S.\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The obvious composition rule for $\gamma_1 \in \hom(a,b)$ and $\gamma_2 \in \hom(b,c)$ is that

$$\gamma_2 \circ \gamma_1(x) = \begin{cases} \gamma_1(2x) & \text{if } x \in [0, \frac{1}{2}) \\ \gamma_2(2x - 1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases} \tag{3.19}$$

However, this composition rule is not associative, and resolving this involves the use of a bicategory, either implicitly or explicitly. We get this bicategory $\mathrm{Path}_2(S)$, by first defining, for $a, b \in S$, a category $\hom(a,b)$ with:

- objects: paths from $a$ to $b$
- morphisms: homotopies between paths, namely a homotopy from $\gamma_1$ to $\gamma_2$ is $H : [0,1] \times [0,1] \to S$ such that $H(x,0) = \gamma_1(x)$, $H(x,1) = \gamma(x)$, $H(0,y) = a$, $H(1,y) = b$ for all $(x,y) \in [0,1] \times [0,1]$.

Then we have a unit law for the identity morphism (the constant path) at each point, and an associator for composition. Both of these are homotopies which reparametrize composite paths.

Finally, we note that, if we define horizontal and vertical composition of homotopies in the same way as above (in each component), then this composition is again not associative. So to get around this, we say that the bicategory we want has its hom-categories $\underline{\hom}(a, b)$, where the morphisms are *isomorphism classes* of homotopies. The isomorphisms in question will not be homotopies themselves (to avoid extra complications), but rather smooth maps which fix the boundary of the homotopy square.

We call the resulting bicategory $\mathrm{Path}_2(S)$.

A similar construction is possible for a double category.

**Example 2** We have seen that a double category it is rather analogous to a bicategory, so we would like to construct one analogous to the bicategory in Example 1. To do this, we construct a model having the following:

- A category Obj of objects is the path *category* $Path(S)$:
- A category Mor of morphisms: this has the following data:
  - objects: paths $\gamma : [m, n]$ in $S$

- morphisms: homotopies $H : [p, q] \times [m, n]$ between paths (these have source and target maps which are just $s : H(-,-) \to H(-, m)$ and $t : H(-,-) \to H(-, n)$.

These categories have source and target maps $s$ and $t$ which are functors from Mor to Obj. The object map for $s$ is just evaluation at 0, and for $t$ it is evaluation at 1. The morphism maps for these functors are $s : H(-,-) \to H(p,-)$ and $t : H(-,-) \to H(q,-)$.

We call the result the double category of homotopies, $\mathbf{H}(S)$.

We observe here that the double category $\mathbf{H}(S)$ is similar to the bicategory $\mathrm{Path}_2(S)$ in one sense. Both give a picture in which objects are points in a topological space, morphisms are 1-dimensional objects (paths), and higher morphisms involve 2-dimensional objects (homotopies). There are differences, however: the most obvious is that $\mathrm{Path}_2(S)$ involves only homotopies with fixed endpoints: its 2D objects are *bigons*, whereas in $\mathbf{H}(S)$ the 2D objects are "squares" (or images of rectangles under smooth maps).

A more subtle difference, however, is that, in order to make composition strictly associative in $\mathbf{H}(S)$, it was necessary to change how we parametrize the homotopies. There are no associators here, and so we make sure composition is strict

by not rescaling our source object (the product of two intervals) as we did in $\mathrm{Path}_2(S)$.

This is rather unsatisfactory, and in fact improving it leads to a general definition of a *double bicategory*, which has a large class of examples, namely *double cospans*. A special, restricted case of these is the double bicategory of cobordisms with corners we want.

# Part II

# Higher Categories for Extended TQFT's

# Chapter 4

# Verity Double Bicategories

The term double bicategory seems to have been originally introduced by Dominic Verity [86], and the structure it refers to is the one we want to use. There is some ambiguity here since the term double bicategory appears to describe is an internal bicategory in **Bicat** (the category of all bicategories). This is analogous to the definition of double category. Indeed, it is what we will mean by a double bicategory, and we discuss these in Section A. Since the two are closely related, and both will be important for us, we will refer to double bicategories in the sense of Verity by the term Verity double bicategories, while reserving *double bicategory* for the former. For more discussion of the relation between these, see Section A.

We wish to describe a structure which is sufficient to capture the possible compositions of cobordisms with corners just as **2Cob** does for cobordisms. These have "horizontal" composition along the manifolds with boundary which form their source and target. They also have "vertical" composition along the bound-

aries of those manifolds and of the cobordisms joining them (which, together, again form cobordisms) . However, to allow the boundaries to vary, we do not want to consider them as diffeomorphism classes of cobordisms, but simply as cobordisms. However, composition is then not strictly associative, but only up to diffeomorphism.

Thus, we want something like a double category, but with weakened axioms, just as bicategories were defined by weakening those for a category. The concept of a "weak double category" has been defined (for instance, see Marco Grandis and Robert Paré [45], and Martins-Ferreira's [71] discussion of them as "pseudo-categories"). Thomas Fiore [35] describes these as "Pseudo Double Categories", arising by "categorification" of the theory of categories, and describes examples motivated by conformal field theory. A detailed discussion is found in Richard Garner's Ph.D. thesis [44]. In these cases, the weakening only occurs in only one direction—either horizontal or vertical. That is, the associativity of composition, and unit laws, in that direction apply only up to certain higher morphisms, called *associators* and *unitors*. In the other direction, the category axioms hold strictly. In a sense, this is because the weakening uses the squares of the double category as 2-morphisms: specifically, squares with two sides equal to the identity. Trying to do this in both directions leads to some difficulties.

In particular, if we have associators for horizontal morphisms given by squares of the form:

$$\begin{array}{ccccc} a & \xrightarrow{f;g} & c & \xrightarrow{h} & d \\ {\scriptstyle 1_a}\downarrow & & \not\Downarrow_{f,g,h} & & \downarrow{\scriptstyle 1_d} \\ a & \xrightarrow{f} & b & \xrightarrow{g;h} & d \end{array} \tag{4.1}$$

then unless composition of vertical morphisms is strict, then to make a equation (for instance, the pentagon equation) involving this square, we would need to use unit laws (or associators) in the vertical direction to perform this composition. This would again be a square with identities on two sides, and the problem arises again. In fact, there is no consistent way to do this. Instead, we need to introduce a new kind of 2-morphism separate from the squares, as we shall see in Section 4.1. The result is what Dominic Verity has termed a double bicategory [86].

The problem of weakening the concept of a double category so that the unit and associativity properties hold up to higher-dimensional morphisms can be contrasted with a different approach. One might instead try to combine the notions of *bicategory* and *double category* in a different way. This is by "doubling" the notion of bicategory, in the same way that double categories did with the notion of category. Just as a double category is an internal category in **Cat**, the result would be an internal bicategory in **Bicat**.

We would like to call this a *double bicategory*: however, this term has already been used by Dominic Verity to describe the structure we will mainly be interested

in. Since the former concept is also important for us in certain lemmas, and is most naturally called a double bicategory, we will refer to the latter as a *Verity double bicategory*. For more discussion of the relation between these, see Section A.

## 4.1 Definition of a Verity Double Bicategory

The following definition of a Verity double bicategory is due to Dominic Verity [86], and is readily seen as a natural weakening of the definition of a double category. Just as the concept of *bicategory* weakens that of *2-category* by weakening the associative and unit laws, Verity double bicategories will do the same for double categories. The following definition can be contrasted with that for a double category in Section 3.4.

**Definition 5** *(**Verity**) A **Verity double bicategory** **C** is a structure consisting of the following data:*

- *a class of **objects*** Obj,
- ***horizontal** and **vertical bicategories** **Hor** and **Ver** having* Obj *as their objects*

- *for every square of horizontal and vertical morphisms of the form*

$$\begin{array}{ccc} a & \xrightarrow{h} & b \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ c & \xrightarrow{h'} & d \end{array} \tag{4.2}$$

*a class of* ***squares*** $\mathbf{Squ}$*, with maps* $s_h, t_h : \mathbf{Squ} \to \mathrm{Mor}(\mathbf{Hor})$ *and* $s_v, t_v : \mathbf{Squ} \to \mathrm{Mor}(\mathbf{Ver})$*, satisfying an equation for each corner, namely:*

$$\begin{aligned} s(s_h) &= s(s_v) \\ t(s_h) &= s(t_v) \\ s(t_h) &= t(s_v) \\ t(t_h) &= t(t_v) \end{aligned} \tag{4.3}$$

*The squares should have horizontal and vertical composition operations, defining the vertical composite* $F \otimes_V G$

$$\begin{array}{ccc} x & \longrightarrow & x' \\ \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle F} & \downarrow \\ y & \longrightarrow & y' \\ \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle G} & \downarrow \\ z & \longrightarrow & z' \end{array} \quad = \quad \begin{array}{ccc} x & \longrightarrow & x' \\ \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle F\otimes_V G} & \downarrow \\ z & \longrightarrow & z' \end{array} \tag{4.4}$$

*and horizontal composite* $F \otimes_H G$*:*

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle F} & \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle G} & \downarrow \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array} \quad = \quad \begin{array}{ccc} x & \longrightarrow & z \\ \downarrow & \swarrow\!\!\!\swarrow{\scriptstyle F\otimes_H G} & \downarrow \\ x' & \longrightarrow & z' \end{array} \tag{4.5}$$

*The composites have the usual relation to source and target maps, satisfy the interchange law*

$$(F \otimes_V F') \otimes_H (G \otimes_V G') = (F \otimes_H G) \otimes_V (F' \otimes_H G') \tag{4.6}$$

*and there is a unit for composition of squares:*

$$\begin{array}{ccc} x & \xrightarrow{1_x} & x \\ {\scriptstyle f}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle 1_f} & \downarrow{\scriptstyle f} \\ y & \xrightarrow[1_y]{} & y \end{array} \tag{4.7}$$

*(and similarly for vertical composition).*

*There is a left and right action by the horizontal and vertical 2-morphisms on* **Squ**, *giving* $F \star_V \alpha$,

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \swarrow\!\!\!\swarrow {\scriptstyle F} & \downarrow \\ x' & \longrightarrow & y' \end{array} \quad \overset{\alpha}{\Rightarrow} \quad = \quad \begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \swarrow\!\!\!\swarrow {\scriptstyle F \star_V \alpha} & \downarrow \\ x' & \longrightarrow & y' \end{array} \tag{4.8}$$

*(and similarly on the left) and* $F \star_H \alpha$,

$$\begin{array}{ccc} & \overset{\alpha}{\Downarrow} & \\ x & \longrightarrow & y \\ \downarrow & \swarrow\!\!\!\swarrow {\scriptstyle F} & \downarrow \\ x' & \longrightarrow & y' \end{array} \quad = \quad \begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \alpha \star_H F} & \downarrow \\ x' & \longrightarrow & y' \end{array} \tag{4.9}$$

*The actions also satisfy interchange laws:*

$$(F \otimes_H F') \star_H (\alpha \otimes_V \alpha') = (F \star_H \alpha) \otimes_h (F' \star_H \alpha') \tag{4.10}$$

*(and similarly for the vertical case) and are compatible with composition:*

$$(F \otimes_H G) \star_V \alpha = F \otimes_H (G \star_V \alpha) \tag{4.11}$$

*(and analogously for vertical composition). They also satisfy additional compatibility conditions: the left and right actions of both vertical and horizontal 2-morphisms satisfy the "associativity" properties,*

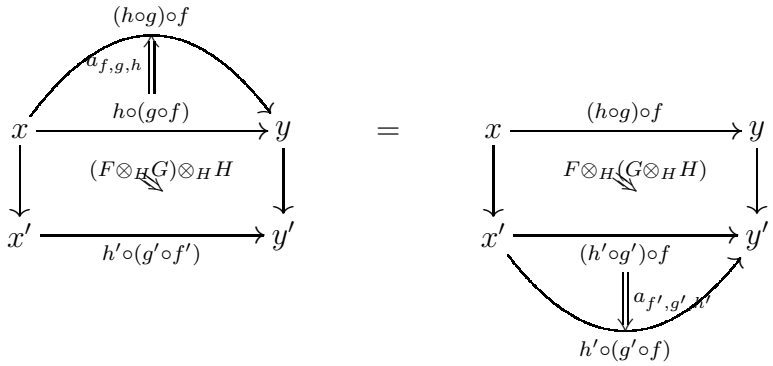
$$\alpha \star (S \star \beta) = (\alpha \star S) \star \beta \tag{4.12}$$

*for both $\star_H$ and $\star_V$. Moreover, horizontal and vertical actions are independent:*

$$\alpha \star_H (\beta \star_V S) = \beta \star_V (\alpha \star_H S) \tag{4.13}$$

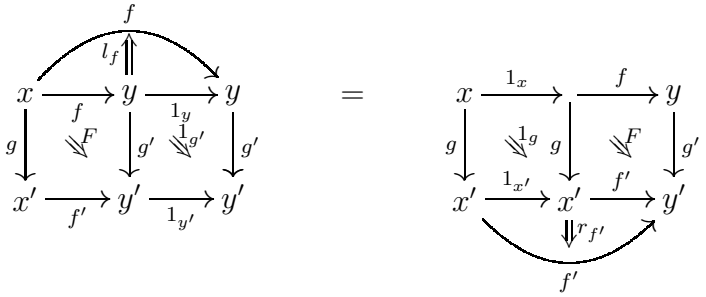
*and similarly for the right action.*

*Finally, the composition of squares agrees with the associators for composition by the action in the sense that given three composable squares $F$, $G$, and $H$:*




(4.14)

*and similarly for vertical composition. Likewise, unitors in the horizontal and vertical bicategories agree with the identity for composition of squares:*




(4.15)

*and similarly for vertical units.*

We will see in Chapter 5 that the higher categories defined this way are well suited to dealing with cobordisms with corners. In Section A we will consider how this definition arises as a special case of a broader concept of double bicategory which we define there. For now, in Section 4.2, we will consider how Verity double bicategories can give rise to ordinary bicategories.

## 4.2 Bicategories from Double Bicategories

There are numerous connections between double categories and bicategories (or their strict form, 2-categories). One is Ehresmann's double category of quintets, relating double categories to 2-categories: a double category by taking the squares to be 2-morphisms between composite pairs of morphisms, such as $\alpha : g' \circ f \to f' \circ g$.

Furthermore, it is well known that double categories can yield 2-categories in three different ways. Two obvious cases are when there are only identity horizontal morphisms, or only vertical morphisms, respectively, so that squares simply collapse into bigons with the two nontrivial sides. Notice that it is also true that a Verity double bicategory in which **Hor** is trivial (equivalently, if **Ver** is trivial) is again a bicategory. The squares become 2-morphisms in the obvious way, the action of 2-morphisms on squares then is just composition, and the composition rules for squares and bigons are the same. The result is clearly a bicategory.

The other, less obvious, case, is when the horizontal and vertical categories on the objects are the same: this is the case of *path-symmetric* double categories, and the recovery of a bicategory was shown by Brown and Spencer [19]. Fiore [35] shows how their demonstration of this fact is equivalent to one involving *folding structures*.

In this case we again can interpret squares as bigons by composing the top and right edges, and the left and bottom edges. Introducing identity bigons completes the structure. These new bigons have a natural composition inherited from that for squares. It turns out that this yields a structure satisfying the definition of a 2-category. Here, our goal will be to show half of an analogous result, that a Verity double bicategory similarly gives rise to a bicategory when the horizontal and vertical bicategories are equal. We will also show that a double bicategory for

which the horizontal (or vertical) bicategory is trivial can be seen as a bicategory. The condition that **Hor** = **Ver** holds in our general example of double cospans: both horizontal and vertical bicategories in any $2\mathrm{Cosp}(\mathbf{C})_0$ are just $\mathrm{Cosp}(\mathbf{C})$.

**Theorem 1** *Any Verity double bicategory* $(\mathrm{Obj}, \mathbf{Hor}, \mathbf{Ver}, \mathbf{Squ}, \otimes_H, \otimes_V, \star_H, \star_V)$ *for which* **Hor** = **Ver** *produces a bicategory by taking squares to be 2-cells.*

**Proof**: We begin by defining the data of this bicategory, which we call **B**. Its objects and morphisms are the same as those of **Hor** (equivalently, **Ver**). We describe the 2-morphisms by observing that **B** must contain all those in **Hor** (equivalently, **Ver**), but also some others, which correspond to the squares in **Squ**.

In particular, given a square

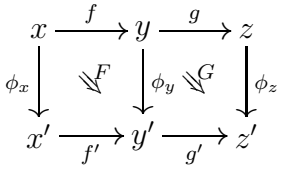
$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & \swarrow S & \downarrow{\scriptstyle g'} \\ c & \xrightarrow[f']{} & d \end{array} \tag{4.16}$$

there should be a 2-morphism

$$a \;\overset{g'\circ f}{\underset{f'\circ g}{\Downarrow S}}\; d \tag{4.17}$$

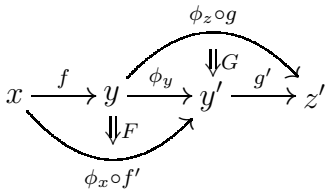
The composition of squares corresponds to either horizontal or vertical composition of 2-morphisms in **B**, and the relation between these two is given in terms of the interchange law in a bicategory:

Given a composite of squares,



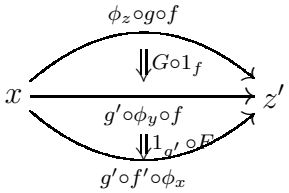

(4.18)

there will be a corresponding diagram in **B**:




(4.19)

Using horizontal composition with identity 2-morphisms ("whiskering"), we can write this as a vertical composition:




(4.20)

So the square $F \otimes_H G$ corresponds to $(1 \circ G) \cdot (F \circ 1)$ for appropriate identities 1. Similarly, the vertical composite of $F' \otimes_V G'$ must be the same as $(1 \circ F) \cdot (G \circ 1)$. Thus, every composite of squares, which can all be built from horizontal and vertical composition, gives a corresponding composite of 2-morphisms in **B**, which are generated by those corresponding to squares in **Squ**, subject to the relations imposed by the composition rules in a bicategory.

To show the Verity double bicategory gives a bicategory, it now suffices to show that all such 2-morphisms not already in **Hor** arise as squares (that is, the

structure is closed under composition). So suppose we have any composable pair of 2-morphisms which arise from squares. If the squares have an edge in common, then we have the situation depicted above (or possibly the corresponding form in the vertical direction). In this case, the composite 2-morphism corresponds exactly to the composite of squares, and the axioms for composition of squares ensure that all 2-morphisms generated this way are already in our bicategory. In particular, the unit squares become unit 2-morphisms when composed with left and right unitors.

Now, if there is no edge in common to two squares, the 2-morphisms in **B** must be made composable by composition with identities. In this case, all the identities can be derived from 2-morphisms in **Hor**, or from identity squares in **Squ** (inside commuting diagrams). Clearly, any identity 2-morphism can be factored this way. Then, again, the composite 2-morphisms in **B** will correspond to the composite of all such squares and 2-morphisms in **Squ** and **Hor**.

Finally, the associativity condition (4.14) for the action of 2-morphisms on squares ensures that composition of squares agrees with that for 2-morphisms, so there are no extra squares from composites of more than two squares. □

This allows us to think of $\mathbf{nCob}_2$ not only as a Verity double bicategory, but in the more familiar form of a bicategory.

It is also worth considering here the situation of a double bicategory with horizontal bicategory trivial (i.e. in which all horizontal morphisms and 2-morphisms are identities). In this case, one can define a 2-morphism from a square with and bottom edges being identities, whose source is the object whose identity is the corresponding edge, and similarly for the target. The composition rules for squares in the vertical direction, then, are just the same as those for a bicategory. Likewise, the axioms for action of a 2-morphism on a square reduce to the composition laws for a bicategory if one replaces the square by a 2-cell.

Next we describe the class of examples we will use to develop a double bicategory of cobordisms with corners.

## 4.3 Double Cospans

Now we construct a class of examples of double bicategories. These examples are analogous to the example of bicategories of spans, discussed in Section 3.3. These span-ish examples of Verity double bicategories are will give the Verity double bicategory of cobordisms with corners as a special case, which is similar in flavour to the topological examples of bicategories and double categories in Section 3.5. However, these will be based on *cospans*. Cospans in $\mathbf{C}$ are the same as spans in the opposite category, $\mathbf{C}^{\mathrm{op}}$. In Remark 1 we described Bénabou's

demonstration that Span($\mathbf{C}$) is a bicategory for any category $\mathbf{C}$ with pullbacks. Similarly, there is a bicategory of cospans in a category $\mathbf{C}$, with pushouts.

There will be an analogous fact about our example of a double bicategory: double cospans, described explicitly in Section A. Here, we are interested in a more restricted structure:

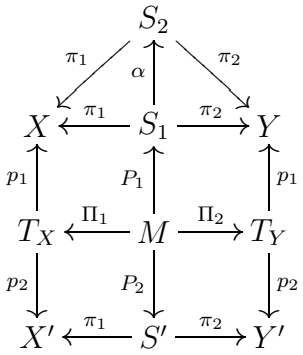
**Definition 6** *For a category* $\mathbf{C}$ *with finite limits, the Verity double bicategory* $2\mathrm{Cosp}(\mathbf{C})_0$*, has:*

- *the objects are objects of* $\mathbf{C}$
- *the horizontal and vertical bicategories* $\mathbf{Hor} = \mathbf{Ver}$ *are both equal to a sub-bicategory of* $\mathrm{Cosp}(\mathbf{C})$*, which includes only invertible cospan maps*
- *the squares are isomorphism classes of commuting diagrams of the form:*

$$\begin{array}{ccccc} X & \xrightarrow[\iota_X]{} & S & \xleftarrow[\iota_Y]{} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle i_Y} \\ T_X & \xrightarrow[J_X]{} & M & \xleftarrow[J_Y]{} & T_Y \\ \uparrow{\scriptstyle i'_Y} & & \uparrow{\scriptstyle I'} & & \uparrow{\scriptstyle i_{Y'}} \\ X' & \xrightarrow[\iota_{X'}]{} & S' & \xleftarrow[\iota_{Y'}]{} & Y' \end{array} \tag{4.21}$$

*where two diagrams of the form (4.21) are isomorphic if they differ only in the middle objects, say* $M$ *and* $M'$*, and their maps to the edges, and if there is an isomorphism* $f : M \to M'$ *making the combined diagram commute.*

*The action of 2-morphisms $\alpha$ in* **Hor** *and* **Ver** *on squares is by composition in diagrams of the form:*




(4.22)

*(where the resulting square is as in 4.21, with $S_2$ in place of $S$ and $\alpha \circ P_1$ in place of $P_1$).*

*Composition (horizontal or vertical) of squares of cospans is, as in* $2\mathrm{Cosp}(\mathbf{C})$*, given by composition (by pushout) of the three spans of which the square is composed. The composition operators for diagrams of cospan maps are by the usual ones in* $\mathrm{Cosp}(\mathbf{C})$.

**Remark 2** Notice that **Hor** and **Ver** as defined are indeed bicategories: eliminating all but the invertible 2-morphisms leaves a collection which is closed under composition by pushouts.

We show more fully that this is a Verity double bicategory in Theorem 2, but for now we note that the definition of horizontal and vertical composition of squares is defined on equivalence classes. One must show that this is well defined.

We will get this result indirectly as a result of Lemmas 6 and 7, but it is instructive to see directly how this works in Cosp($\mathbf{C}$).

**Lemma 1** *The composition of squares in Definition 6 is well-defined.*

**Proof**: Suppose we have two representatives of a square, bounded by horizontal cospans $(S, \pi_1, \pi_2)$ from $X$ to $Y$ and $(S', \pi_1, \pi_2)$ from $X'$ to $Y'$, and vertical cospans $(T_X, p_1, p_2)$ from $X$ to $X'$ and $(T_Y, p_1, p_2)$ from $Y$ to $Y'$. The middle objects $M_1$ and $M_2$ as in the diagram (4.21). If we also have a composable diagram—one which coincides along an edge (morphism in **Hor** or **Ver**) with the first—then we need to know that the pushouts are also isomorphic (that is, represent the same composite square).

In the horizontal and vertical composition of these squares, the maps from the middle object $M$ of the new square to the middle objects of the new sides (given by composition of cospans) arise from the universal property of the pushouts on the sides being composed (and the induced maps from $M$ to the corners, via the maps in the cospans on the other sides). Since the middle objects are defined only up to isomorphism class, so is the pushout: so the composition is well defined, since the result is again a square of the form (4.21). □

We use this, together with Lemmas 6 and 7, (proved in Section A) to show the following:

**Theorem 2** *If* $\mathbf{C}$ *is a category with finite colimits, then* $2\mathrm{Cosp}(\mathbf{C})_0$ *is a Verity double bicategory.*

**Proof**: In the construction of $2\mathrm{Cosp}(\mathbf{C})_0$, we take isomorphism classes of double cospans as the squares. We also restrict to invertible cospan maps in the horizontal and vertical bicategories.

That is, take 2-isomorphism classes of morphisms in **Mor** in the double bicategory, where the 2-isomorphisms are invertible cospan maps, in both horizontal and vertical directions. We are then effectively discarding all morphisms and 2-morphisms in **2Mor**, and the 2-morphisms in **Mor** except for the invertible ones. In particular, there may be "squares" of the form (4.21) in $2\mathrm{Cosp}(\mathbf{C})$ with non-invertible maps joining their middle objects $M$, but we have ignored these, and also ignore non-invertible cospan maps in the bicategories on the edges. Thus, we consider no diagrams of the form (A.2) except for invertible ones, in which case the middle objects (say, $M$ and $M'$) are representatives of the same isomorphism class. Similar reasoning applies to the 2-morphisms in **2Mor**.

The resulting structure we get from discarding these will again be a double bicategory. In particular, the new **Mor** and **2Mor** will be bicategories, since they are, respectively, just a category and a set made into a discrete bicategory by adding identity morphisms or 2-morphisms as needed. On the other hand, for the composition, source and target maps to be weak 2-functors amounts to saying

that the structures built from the objects, morphisms, and 2-cells respectively are again bicategories, since the composition, source, and target maps satisfy the usual axioms. But the same argument applies to those built from the morphisms and 2-cells as within **Mor** and **2Mor**. So we have a double bicategory.

Next we show that the horizontal and vertical action conditions (Definition 27 of section A.3) hold in 2Cosp(**C**). A square in 2Cosp(**C**) is a diagram of the form (4.21), and a 2-cell is a map of cospans. Given a square $M_1$ and 2-cell $\alpha$ with compatible source and targets as in the action conditions, we have a diagram of the form shown in (4.22). Here, $M_1$ is the square diagram at the bottom, whose top row is the cospan containing $S_1$. The 2-cell $\alpha$ is the cospan map including the arrow $\alpha : S_1 \to S_2$. There is a unique square built using the same objects as $M_1$ except using the cospan containing $S_2$ as the top row. The map to $S_2$ from $M$ is then $\alpha \circ P_1$.

To satisfy the action condition, we want this square $M_2$, which is the candidate for $M_1 \star_V \alpha$, to be unique. But suppose there were another $M_2'$ with a map to $S_2$. Since we are in 2Cosp(**C**)$_0$, $\alpha$ must be invertible, which would give a map from $M_2'$ to $S_1$. We then find that $M_2'$ and $M_2$ are representatives of the same isomorphism class, so in fact this is the same square. That is, there is a unique morphism in **2Mor** taking $M_1$ to $M_2$ (a diagram of the form A.3, oriented vertically) with

invertible cospan maps in the middle and bottom rows. This is the unique filler for the pillow diagram required by definition 27.

The argument that $2\mathrm{Cosp}(\mathbf{C})_0$ satisfies the action compatibility condition is similar.

So $2\mathrm{Cosp}(\mathbf{C})_0$ is a double bicategory in which, there there is at most one unique morphism in **Mor**, and at most unique morphisms and 2-morphisms in **2Mor**, for any specified source and target, and the horizontal and vertical action conditions hold. So $2\mathrm{Cosp}(\mathbf{C})_0$ can be interpreted as a Verity double bicategory (Lemma 7). □

**Remark 3** We observe here that the compatibility condition (4.14) relating the associator in the horizontal and vertical bicategories to composition for squares can be seen from the fact that the associators are maps which come from the universal property of pushouts. This is by the parallel argument to that we gave for spans in Section 3.3. The same argument applies to the middle objects of the squares, and gives associator isomorphisms for that composition. Since these become the identity when we reduce to isomorphism classes, we get a commuting pillow as in (4.14). A similar argument shows the compatibility condition for the unitor, (4.15).

Note that the analogous theorem beginning with a category $\mathbf{C}$ with finite limits and using spans is equivalent to this case, by taking $\mathbf{C}^{\mathrm{op}}$.

In Section 5.2 we use a similar argument to obtain a Verity double bicategory of cobordisms with corners. First, however, we must see how these are defined. This is the task of Chapter 5. In Appendix A.2 we show that $\mathrm{Cosp}(\mathbf{C})$ is a Verity double bicategory. For now, we will examine how cobordisms form a special topological example of this sort of Verity double bicategory.

# Chapter 5

# Cobordisms With Corners

Our motivation here for studying Verity double bicategories is to provide the right formal structure for our special example of higher categories of cobordisms. The objects in these categories are manifolds of some dimension, say $k$. In this case, the morphisms are $(k+1)$-dimensional cobordisms between these manifolds: that is, manifolds with boundary, such that the boundary decomposes into two components, with one component as the source, and one as the target. The 2-cells are equivalence classes $(k+2)$-dimensional cobordisms between $(k+1)$-dimensional cobordisms: these can be seen as manifolds with corners, where the corners are the $k$-dimensional objects. Specifically, with these as with the cobordisms in our definition of **nCob**, only the highest-dimensional level consists of isomorphism classes. This means that composition of the horizontal and vertical cobordisms will need to be weak, which is why we use Verity double bicategories as defined in Definition 5.

Observe that we could continue building a ladder in which, at each level, the $j+1$-cells are cobordisms between the $j$-cells, which are cobordisms between the $(j-1)$-cells. The two levels we consider here are enough to give a Verity double bicategory of $n$-dimensional cobordisms with corners, where we think of the top dimension ($k+2$ in the above) as $n$. We will see that these can be construed using the double cospan construction of Section 4.3.

## 5.1 Collars on Manifolds with Corners

Here we will use a modified form of our construction from Section 4.3 of a Verity double bicategory $2\mathrm{Cosp}(\mathbf{C})$ in order to show an example of a Verity double bicategory of cobordisms with corners, starting with $\mathbf{C}$ the category of smooth manifolds. To begin with, we recall that a smooth manifold with corners is a topological manifold with boundary, together with a certain kind of $C^\infty$ structure. In particular, we need a maximal compatible set of coordinate charts $\phi : \Omega \to [0,\infty)^n$ (where $\phi_1$, $\phi_2$ are compatible if $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism). The fact that the maps are into the positive sector of $\mathbb{R}^n$ distinguishes a manifold with corners from a manifold.

Jänich [48] introduces the notion of $\langle n \rangle$-manifold, reviewed by Laures [60]. This is build on a manifold with faces:

**Definition 7** *A face of a manifold with corners is the closure of some connected component of the set of points with just one zero component in any coordinate chart. An $\langle n \rangle$-manifold is a manifold with faces together with an $n$-tuple $(\partial_0 M, \ldots, \partial_{n-1} M)$ of faces of $M$, such that*

- $\partial_0 M \cup \ldots \partial_{n-1} M = \partial M$
- $\partial_i M \cap \partial_j M$ *is a face of* $\partial_i M$ *and* $\partial_j M$

The case we will be interested in here is the case of $\langle 2 \rangle$-manifolds. In this notation, a $\langle 0 \rangle$-manifold is just a manifold without boundary, a $\langle 1 \rangle$-manifold is a manifold with boundary, and a $\langle 2 \rangle$-manifold is a manifold with corners whose boundary decomposes into two components (of codimension 1), whose intersections form the corners (of codimension 2). We can think of $\partial_0 M$ and $\partial_1 M$ as the "horizontal" and "vertical" part of the boundary of $M$.

**Example 3** Let $M$ be the solid 3-dimensional illustrated in Figure 5.1. The boundary decomposes into 2-dimensional manifolds with boundary. Denote by $\partial_0 M$ the boundary component consisting of the top and bottom surfaces, and $\partial_1 M$ be the remaining boundary component (a topological annulus).

In this case, $\partial_0 M$ is the disjoint union of the manifolds with corners $S$ (two annuli) and $S'$ (topologically a three punctured sphere); $\partial_1 M$ is the disjoint union

of two components, $T_X$ (which is topologically a three-punctured sphere) and $T_Y$ (topologically a four-punctured torus).

Then we have $\partial_0 M \cup \partial_1 M = \partial M$. Also, $\partial_0 M \cap \partial_1 M$ is a 1-dimensional manifold without boundary, which is a face of both $\partial_0 M$ and $\partial_1 M$ (in fact, the shared boundary). In particular, it is the disjoint union $X \cup Y \cup X' \cup Y'$.

We have described a Verity double bicategory of double cospans in a category with all pushouts. We could then form such a system of cobordisms with corners in a category obtained by co-completing **Man**, so that all pushouts exist. The problem with this is that the pushout of two cobordisms $M_1$ and $M_2$ over a submanifold $S$ included in both by maps $S \xrightarrow{i_1} M_1$ and $S \xrightarrow{i_2} M_2$ may not be a cobordism. If the submanifolds are not on the boundaries, certainly the result may not even be a manifold: for instance, two line segments with a common point in the interior. So to get a Verity double bicategory in which the morphisms are smooth manifolds with boundary, certainly we can only consider the case where we compose two cobordisms by a pushout along shared submanifolds $S$ which are components of the boundary of both $M_1$ and $M_2$.

However, even if the common submanifold is at the boundary, there is no guarantee that the result of the pushout will be a smooth manifold. In particular, for a point $x \in S$, there will be a neighborhood $U$ of $x$ which restricts to $U_1 \subset M_1$ and $U_2 \subset M_2$ with smooth maps $\phi_i : U_i \to [0, \infty)^n$ with $\phi_i(x)$ on the

boundary of $[0,\infty)^n$ with exactly one coordinate equal to 0. One can easily combine these to give a homeomorphism $\phi : U \to \mathbb{R}^n$, but this will not necessarily be a diffeomorphism along the boundary $S$.

To solve this problem, we use the *collaring theorem*: For any smooth manifold with boundary $M$, $\partial M$ has a *collar*: an embedding $f : \partial M \times [0,\infty) \to M$, with $(x, 0) \mapsto x$ for $x \in \partial M$. This is a well-known result (for a proof, see e.g. [47], sec. 4.6). It is an easy corollary of this usual form that we can choose to use the interval $[0, 1]$ in place of $[0,\infty)$ here.

Gerd Laures ([60], Lemma 2.1.6) describes a generalization of this theorem to $\langle n\rangle$-manifolds, so that for any $\langle n\rangle$-manifold $M$, there is an $n$-dimensional cubical diagram ($\langle n\rangle$-*diagram*) of embeddings of cornered neighborhoods of the faces. It is then standard that one can compose two smooth cobordisms with corners, equipped with such smooth collars, by gluing along $S$. The composite is then the topological pushout of the two inclusions. Along the collars of $S$ in $M_1$ and $M_2$, charts $\phi_i : U_i \to [0,\infty)^n$ are equivalent to charts into $\mathbb{R}^{n-1} \times [0,\infty)$, and since the

the composite has a smooth structure defined up to a diffeomorphism[1] which is the identity along $S$.

## 5.2 Cobordisms with Corners

Suppose we take the category **Man** whose objects are smooth manifolds with corners and whose morphisms are smooth maps. Naively, would would like to use the cospan construction from Section 4.3, we obtain a Verity double bicategory 2Cosp(**Man**). While this approach will work with the category **Top**, however, it will not work with **Man** since this does not have all colimits. In particular, given two smooth manifolds with boundary, we can glue them along their boundaries in non-smooth ways, so to ensure that the pushout exists in **Man** we need to specify a smoothness condition. This requires using collars on the boundaries and corners.

For each $n$, we define a Verity double bicategory within **Man**, which we will call $\mathbf{nCob}_2$:

**Definition 8** *The Verity double bicategory* $\mathbf{nCob}_2$ *is given by the following data:*

[1]Note that the precise smooth structure on this cobordism depends on the collar which is chosen, but that there is always such a choice, and the resulting composites are all equivalent up to diffeomorphism. That is, they are equivalent up to a 2-morphism in the bicategory. So strictly speaking, the composition map is not a functor but an anafunctor. It is common to disregard this issue, since one can always define a functor from an anafunctor by using the axiom of choice. This is somewhat unsatisfactory, since it does not generalize to the case where our categories are over a base in which the axiom of choice does not hold, but this is not a problem in our example. This issue is discussed further by Makkai [69].

- *The objects of* $\mathbf{nCob}_2$ *are of the form* $P = \hat{P} \times I^2$ *where* $\hat{P}$ *may be any* $(n-2)$ *manifolds without boundary and* $I = [0,1]$.

- *The horizontal and vertical bicategories of* $\mathbf{nCob}_2$ *have*

  - *objects: as above*
  - *morphisms: cospans* $P_1 \xrightarrow{i_1} S \xleftarrow{i_2} P_2$ *where* $S = \hat{S} \times I$ *and* $\hat{S}$ *may be any of those cospans of* $(n-1)$*-dimensional manifolds-with-boundary which are cobordisms with collars such that the* $\hat{P}_i \times I$ *are objects, the maps are injections into* $S$*, a manifold with boundary, such that* $i_1(P_1) \cup i_2(P_2) = \partial S \times I$*,* $i_1(P_1) \cap i_2(P_2) = \emptyset$*,*
  - *2-morphisms: cospan maps which are diffeomorphisms of the form* $f \times \mathrm{id} : T \times [0,1] \to T' \times [0,1]$ *where* $T$ *and* $T'$ *have a common boundary, and* $f$ *is a diffeomorphism*$T \to T'$ *compatible with the source and target maps—i.e. fixing the collar.*

  *where the source of a cobordism* $S$ *consists of the collection of components of* $\partial S \times I$ *for which the image of* $(x, 0)$ *lies on the boundary for* $x \in \partial S$*, and the target has the image of* $(x, 1)$ *on the boundary*

- *squares: diffeomorphism classes of* $n$*-dimensional manifolds* $M$ *with corners satisfying the properties of* $M$ *in the diagram of equation (4.21), where isomorphisms are diffeomorphisms preserving the boundary*

- *the action of the diffeomorphisms on the "squares" (classes of manifolds $M$) is given by composition of diffeomorphisms of the boundary cobordisms with the injection maps of the boundary $M$*

*The source and target objects of any cobordism are the collars, embedded in the cobordism in such a way that the source object $P = \hat{P} \times I^2$ is embedded in the cobordism $S = \hat{S} \times I$ by a map which is the identity on $I$ taking the first interval in the object to the interval for a horizontal morphism, and the second to the interval for a vertical morphism. The same condition distinguishing source and target applies as above.*

*Composition of squares works as in* $2\mathrm{Cosp}(\mathbf{C})_0$.

We will see that $\mathbf{nCob}_2$ is a Verity double bicategory in Section 5.3, but for now it suffices to note that since it is composed of double cospans, we can hope to define composition to be just that in the Verity double bicategory$2\mathrm{Cosp}(\mathbf{C})_0$ where $\mathbf{C}$ is the category of manifolds with corners. The proof that this is a Verity double bicategory will entail showing that $\mathbf{nCob}_2$ is closed under this composition.

**Lemma 2** *Composing horizontal morphisms in* $\mathbf{nCob}_2$ *this way produces another horizontal morphism in* $\mathbf{nCob}_2$*. Similarly, composition of vertical morphisms produces a vertical morphism, and composition of squares produces another square.*

**Proof**: The horizontal and vertical morphisms are products of the interval $I$ with $\langle 1 \rangle$-manifolds, whose boundary is $\partial_0 S$), equipped with collars. Suppose we are given two such cobordisms $S_1$ and $S_2$, and an identification of the source of $S_2$ with the target of $S_1$ (say this is $P = \hat{P} \times I$). Then the composite $S_2 \circ S_1$ is topologically the pushout of $S_1$ and $S_2$ over $P$. Now, $P$ is smoothly embedded in $S_1$ and $S_2$, and any point in the pushout will be in the interior of either $S_1$ or $S_2$ since for any point on $\hat{P}$ each end of the interval $I$ occurs as the boundary of only one of the two cobordisms. So the result is smooth. Thus, **2Cob** is closed under such composition of morphisms.

The same argument holds for squares, since it holds for any representative of the equivalence class of some manifold with corners, $M$, and the differentiable structure will be the same, since we consider equivalence up to diffeomorphisms which preserve the collar exactly. □

This establishes that composition in $\mathbf{nCob}_2$ is well defined, and composites are again cobordisms in $\mathbf{nCob}_2$. We show that it is a Verity double bicategory in Section 5.3.

**Example 4** We can represent a typical manifestation of the diagram (4.21) as in Figure 5.1.

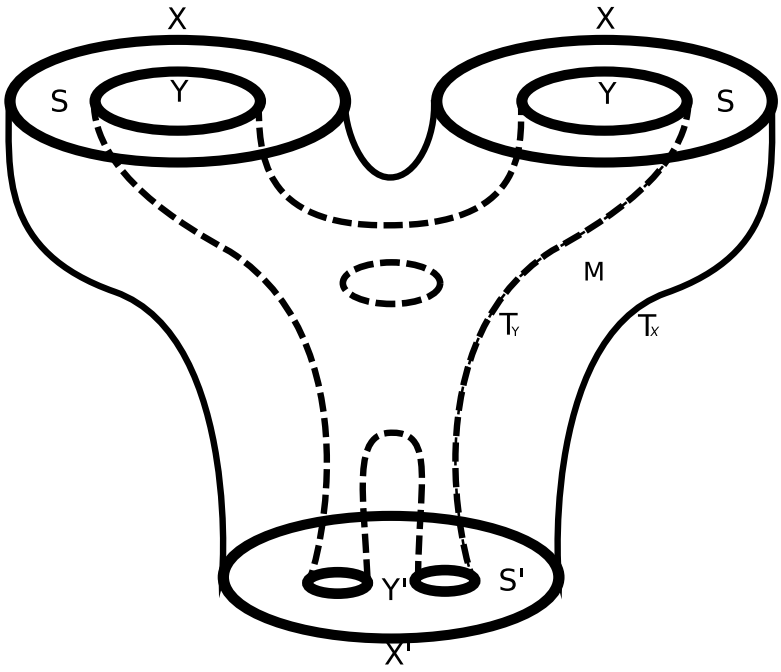



**Figure 5.1:** A Square in $\mathbf{nCob}_2$ (Thickened Lines Denote Collars)

Consider how this picture is related to (4.21). In the figure, we have $n = 3$, so the objects are (compact, oriented) 1-dimensional manifolds, thickened by taking a product with $I^2$. $X$ (top, solid lines) and $Y$ (top, dotted lines) are both isomorphic to $(S^1 \cup S^1) \times I^2$, while $X'$ and $Y'$ (bottom, solid and dotted respectively) are both isomorphic to $S^1 \times I^2$.

The horizontal morphisms are (thickened) cobordisms $S$, and $S'$, which are a pair of thickened annuli and a two-holed disk, respectively, with the evident injection maps from the objects $X, Y, X', Y'$. The vertical morphisms are the thickened cobordisms $T_X$ and $T_Y$. In this example, $T_X$ happens to be of the same

form as $S'$ (a two-holed disk), and has inclusion maps from $X$ and $X'$, the two components of its boundary, as the "source" and "target" maps. $T_Y$ is homotopy equivalent to a four-punctured torus, where the four punctures are the components of its boundary: two circles in $Y$ and two in $Y'$, which again have the obvious inclusion maps. Reading from top to bottom, we can describe $T_Y$ as the story of two (thick) circles which join into one circle, then split apart, then rejoin, and finally split apart again.

Finally, the "square" in this picture is the manifold with corners, $M$, whose boundary has four components, $S, S', T_X, \text{and} T_Y$, and which has corners precisely along the boundaries of these manifolds. These boundaries' components are divided between the objects $X, Y, X', Y'$. The embeddings of these thickened manifolds and cobordisms gives a specific way to equip $M$ with collars.

Given any of the horizontal or vertical morphisms (thickened cobordisms $S$, $S'$, $T_X$ and $T_Y$), a 2-morphism would be a diffeomorphism to some other cobordism equipped with maps from the same boundary components (objects), which fixes the collar on that cobordism (the embedded object). Such a diffeomorphism is necessarily a homeomorphism, so topologically the picture will be similar after the action of such a 2-morphism, but we would consider two such cobordisms as separate morphisms in **Hor** or **Ver**.

**Remark 4** We note the resemblance between this example and $\mathrm{Path}(S)_2$ and $\mathbf{H}(S)$ defined previously. In those cases, we are considering manifolds embedded in a topological space $S$, and only a low-dimensional special case (the square $[0,1] \times [0,1]$ is a manifold with corners). Instead of homotopies, which make sense only for embedded spaces, $\mathbf{nCob}_2$ has diffeomorphisms. However, in both cases, we consider the squares to be *isomorphism classes* of a certain kind of top-dimensional object (homotopies or cobordisms). This eliminates the need to define morphisms or cells in our category of dimension higher than 2. We may omit this restriction if we move to the more general setting of a double bicategory, as described in Section A.

We conclude this section by illustrating composition in both directions in $\mathbf{nCob}_2$, and in particular illustrating the interchange law (4.6) for cobordisms with corners. Figure 5.2 shows four cobordisms with corners, arranged to show three examples of horizontal composition and three of vertical composition. The vertical composites, denoted by $\otimes_V$, can be seen as "gluing" the vertically stacked cobordisms along the boundary between them, which is the bottom face of the cobordisms on top, and the top face of those on bottom. The horizontal composites, denoted by $\otimes_H$, are somewhat less obvious. In the figure, they can be seen as "gluing" the right-hand cobordism along a common face. In each case, the

common face is the "inside" face of the left-hand cobordism, and the "outside" face of the right-hand one.

## 5.3 A Bicategory Of Cobordisms With Corners

Now we want to show that cobordisms of cobordisms form a Verity double bicategory under the composition operations we have described. We have shown in Theorem 2 that there is a Verity double bicategory denoted $2\mathrm{Cosp}(\mathbf{C})_0$ for any category $\mathbf{C}$ with finite colimits. We want to show that the reduction from the full $2\mathrm{Cosp}(\mathbf{C})_0$ to just the particular cospans in $\mathbf{nCob}_2$ leaves this fact intact.

The argument that double cospans form a Verity double bicategory can be slightly modified to show the same about cobordisms with corners, which are closely related. We note that there are two differences. First, the category of manifolds with corners does not have all finite colimits, or indeed all pushouts. Second, we are not dealing with all double cospans of manifolds with corners, so $\mathbf{nCob}_2$ is not $2\mathrm{Cosp}(\mathbf{C})_0$ for any $\mathbf{C}$. In fact, the second difference is what allows us to deal with the first.

**Theorem 3** $\mathbf{nCob}_2$ *is a Verity double bicategory.*

**Proof**: First, recall that objects in $\mathbf{nCob}_2$ are manifolds with corners of the form $P = \hat{P} \times I^2$ for some manifold $\hat{P}$, and notice that both horizontal and vertical

morphisms are cospans. In general, if we have two cospans in the category of manifolds with corners sharing a common object, we cannot take a pushout and get a manifold with corners. However, we are only considering a subcollection of all possible cospans of smooth manifolds with corners, all all those we consider have pushouts which are again smooth manifolds with corners (Lemma 2).

In particular, since composition of squares is as in $2\mathrm{Cosp}(\mathbf{C})_0$, before taking diffeomorphism classes of manifolds $M$ in $\mathbf{nCob}_2$, we would again get a double bicategory made from cobordisms with corners, together with the embeddings used in its cospans, and collar-fixing diffeomorphisms. This is shown by arguments identical to those used in the proof of Lemma 6.

When we reduce to diffeomorphism classes of these manifolds, then just as in the proof of nTheorem 2, we can cut down this double bicategory to a structure, and the result will satisfy the horizontal and vertical action conditions, giving a Verity double bicategory, since it satisfies the conditions of Lemma 7.

So in fact, by the same arguments as in these other cases, $\mathbf{nCob}_2$ is a Verity double bicategory. □

By the argument of Section 4.2, this means that we can also think of $\mathbf{nCob}_2$ as a bicategory, which we will do for the purposes of constructing an Extended TQFT as a weak 2-functor. To do this, we next describe, in Chapter 6 some versions of a bicategory of 2-vector spaces, and in particular 2-Hilbert spaces.

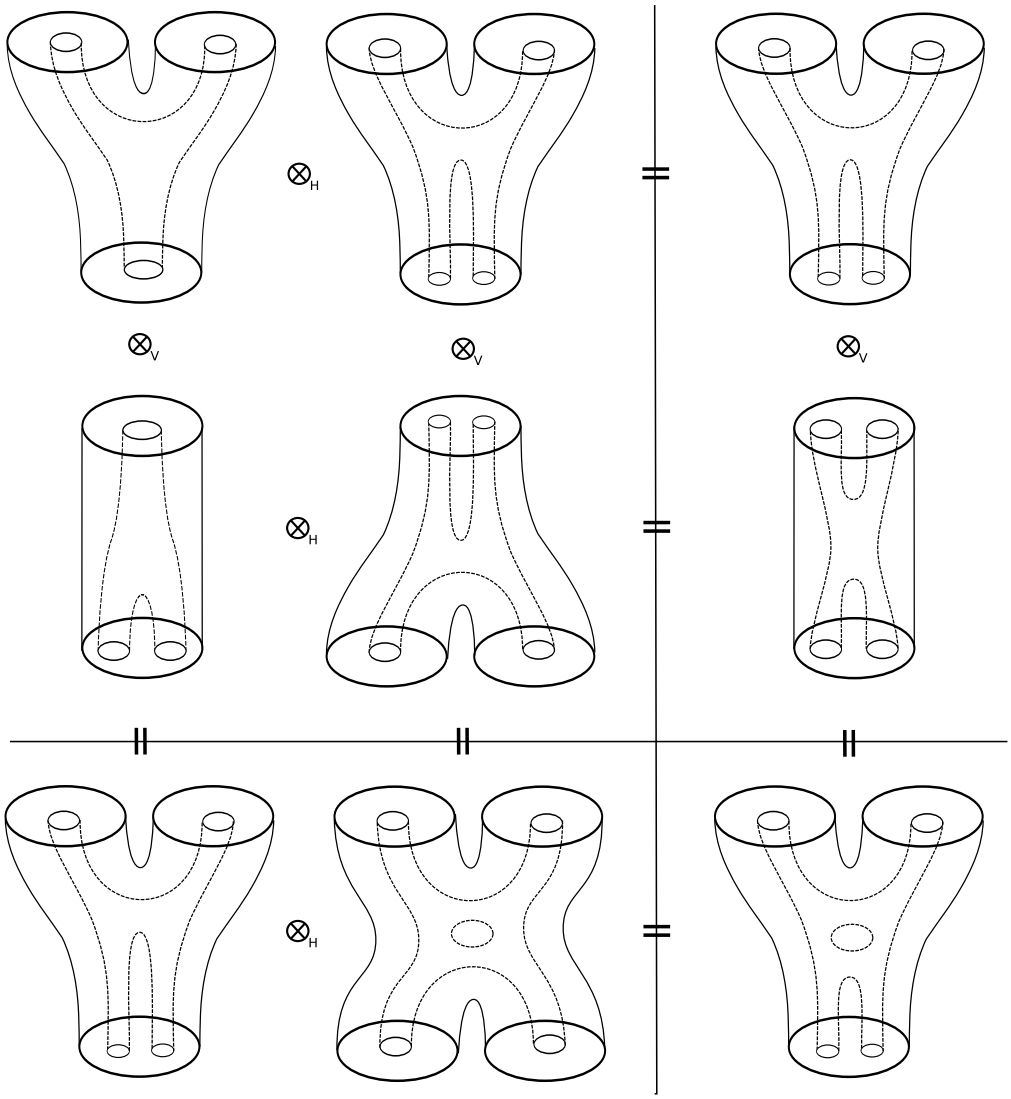



**Figure 5.2:** Compositions in $\mathbf{nCob}_2$ Satisfy the Interchange Law

# Chapter 6

# 2-Vector Spaces

## 6.1 Kapranov-Voevodsky 2-Vector Spaces

We want to find a way of describing an extended TQFT—one acting on manifolds with corners. We would like to find something analogous to Atiyah's characterization of a TQFT as a functor between a category of cobordisms and a category of vector spaces. We have now established that there is a bicategory $\mathbf{nCob}_2$ of in which we can interpret objects as manifolds, morphisms as cobordisms, and 2-morphisms as cobordisms between cobordisms (which are diffeomorphism classes of manifolds with corners). The next constituent we need is a bicategory to take the place of the category of vector spaces. There are several notions of a bicategory of "2-vector spaces" available, and each gives rise to a notion of an extended TQFT as a 2-functor into this bicategory.

There are two major philosophies of how to categorify the concept "vector space". A Baez-Crans (BC) 2-vector space is a category object in **Vect**—that is, a category having a vector space of objects and of morphisms, where source, target, composition, etc. are linear maps. This is a useful concept for some purposes—it was developed to give a categorification of Lie algebras. The reader may refer to the paper of Baez and Crans [6] for more details. However, a BC 2-vector space turns out to be equivalent to a 2-term chain complex and, this is not the concept of 2-vector space which concerns us here.

The other, and prior, approach is to define a 2-vector space as a category having operations such as a monoidal structure analogous to the addition on a vector space. In particular, We will restrict our attention to *complex* 2-vector spaces, though the generalization to an arbitrary base field $K$ is straghtforward.

This ambiguity about the correct notion of "2-vector space" is typical of the problem of categorificiation. Since the categorified setting has more layers of structure, there is a choice of level to which the structure in the concept of a vector space should be lifted. Thus in the BC 2-vector spaces, we have literal vector addition and scalar multiplication within the objects and morphisms. In KV 2-vector spaces and their cousins, we only have this for morphisms, and for objects there is a categorified analog of these operations, as wel shall see.

Indeed, there are different sensible generalizations of vector space even within this second philosophy, however. Josep Elgueta [34] shows several different types of "generalized" 2-vector spaces, and relationships among them. In particular, while KV 2-vector spaces can be thought of as having a *set* of basis elements, a generalized 2-vector space may have a general *category* of basis elements. The free generalized 2-vector space on a category is denoted $\mathbf{Vect}[\mathcal{C}]$. Then KV 2-vector spaces arise when $\mathcal{C}$ is a discrete category with only identity morphisms. This is essentially a set $S$ of objects. Thus it should not be surprising that KV 2-vector spaces have a structure analogous to free vector spaces generated by some finite set - which are isomorphic to $\mathbb{C}^k$.

All such concepts of 2-vector space are $\mathbb{C}$-linear additive categories with some properties, so we begin by explaining this. To begin with, we have *additivity* for categories, the equivalent of linear structure in a vector space. This is related to biproducts, which are both categorical products and coproducts, in compatible ways. The motivating example for us is the *direct sum* operation in **Vect**. Such an operation plays the role in a 2-vector space which vector addition plays in a vector space. To be precise:

**Definition 9** *A **biproduct** for a category* $\mathbf{C}$ *is an operation giving, for any objects* $x$ *and* $y$ *in* $\mathbf{V}$ *an object* $x \oplus y$ *equipped with morphisms* $\iota_x, \iota_y$ *from* $x$ *and* $y$ *respectively into* $x \oplus y$*; and morphisms* $\pi_x, \pi_y$ *from* $x \oplus y$ *into* $x$ *and* $y$ *respectively,*

*which satisfy the biproduct relations:*

$$\pi_x \circ \iota_x = \mathrm{id}_x \;\; \textit{and} \; \pi_y \circ \iota_y = \mathrm{id}_y \tag{6.1}$$

*and similarly for $y$, and*

$$\iota_x \circ \pi_x + \iota_y \circ \pi_y = \mathrm{id}_{x \oplus y} \tag{6.2}$$

Whenever biproducts exist, they are always both products and coproducts.

**Definition 10** *A $\mathbb{C}$-**linear additive category** is a category $\mathbf{V}$ with biproduct $\oplus$, and such that that for any $x, y \in \mathbf{V}$, $\hom(x, y)$ is a vector space over $\mathbb{C}$, and composition is a bilinear map. A $\mathbb{C}$-linear functor between $\mathbb{C}$-linear categories is one where morphism maps are $\mathbb{C}$-linear.*

The standard example of this approach is the Kapranov-Voevodsky (KV) definition of a 2-vector space [51], which is the form we shall use (at least when the situation is finite-dimensional). To motivate the KV definition, consider the idea that, in categorifying, one should replace the base field $\mathbb{C}$ with a monoidal category. Specifically, it turns out, with **Vect**, the category of finite dimensional complex vector spaces. This leads to the following replacements for concepts in elementary linear algebra:

- Vectors = $k$-tuples of scalars $\mapsto$ 2-vectors = $k$-tuples of vector spaces
- Addition $\mapsto$ Direct Sum

- Multiplication $\mapsto$ Tensor Product

So just as $\mathbb{C}^k$ is the standard example of a complex vector space, $\mathbf{Vect}^k$ will be the standard example of a 2-vector space. However, the axiomatic definition allows for other possibilities:

**Definition 11** *A **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear additive category in which every object can be written as a finite biproduct of simple objects (i.e. objects $x$ where $\hom(x,x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear functor which preserves biproducts.*

**Remark 5** It is a standard fact that preserving biproducts and preserving exact sequences are equivalent in this setting: in a KV 2-vector space, every object is equivalent to a direct sum of simple objects, so every exact sequence splits. The above definition of a 2-linear map is sometimes given in the equivalent form requiring that the functor preserve exact sequences. Indeed, since every object is a finite biproduct of simple objects, a 2-vector space is an *abelian* category. For more on these, see Freyd [42].

Now, it is worth mentioning that Yetter shows [88] (in his Proposition 13), that the original definition of Kapranov and Voevodsky gives an equivalent result to a definition of a 2-vector space $\mathbf{V}$ as a finitely semi-simple **Vect**-module. A **Vect**-module $\mathbf{V}$ is finitely semi-simple if there is a finite set $S \subset Ob(\mathbf{V})$ of simple

objects, such that every objects of $\mathbf{V}$ is a finite product of objects in $S$. The advantage of this definition is simply that it is a straightforward categorification of the usual definition of a vector space as a $\mathbb{C}$-module.

The reader is referred to Yetter's paper (Definition 2) for a precise version of the definition of a **Vect**-module, but remark that to be a **Vect**-module requires that $\mathbf{V}$ has an "action" of **Vect** on it. That is, there is a functor

$$\odot : \mathbf{Vect} \times \mathbf{V} \to \mathbf{V} \tag{6.3}$$

which satisfies the usual module axioms only up to two isomorphisms, similar to the associator and unitor, which satisfy some further coherence conditions. We will see the meaning this action when we consider a standard example, where this is literally a tensor product.

**Example 5** The standard example [51] of a KV 2-vector space highlights the analogy with the familar vector space $\mathbb{C}^k$. The 2-vector space $\mathbf{Vect}^k$ is a category whose objects are $k$-tuples of vector spaces, maps are $k$-tuples of linear maps. The *additive* structure of the 2-vector space $\mathbf{Vect}^k$ comes from applying the direct sum in **Vect** componentwise.

Note that there is an equivalent of *scalar multiplication*, using the tensor product:

$$V \otimes \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} = \begin{pmatrix} V \otimes V_1 \\ \vdots \\ V \otimes V_k \end{pmatrix} \tag{6.4}$$

and

$$\begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \oplus \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} V_1 \oplus W_1 \\ \vdots \\ V_k \oplus W_k \end{pmatrix} \tag{6.5}$$

As the correspondence with linear algebra would suggest, 2-linear maps $T : \mathbf{Vect^k} \to \mathbf{Vect^l}$ amount to $k \times l$ matrices of vector spaces, acting by matrix multiplication using the direct sum and tensor product instead of operations in $\mathbb{C}$:

$$\begin{pmatrix} T_{1,1} & \dots & T_{1,k} \\ \vdots & & \vdots \\ T_{l,1} & \dots & T_{l,k} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^{k} T_{1,i} \otimes V_i \\ \vdots \\ \bigoplus_{i=1}^{k} T_{l,i} \otimes V_i \end{pmatrix} \tag{6.6}$$

The natural transformations between these are matrices of linear transformations:

$$\alpha = \begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,k} \\ \vdots & & \vdots \\ \alpha_{l,1} & \dots & \alpha_{l,k} \end{pmatrix} : \begin{pmatrix} T_{1,1} & \dots & T_{1,k} \\ \vdots & & \vdots \\ T_{l,1} & \dots & T_{l,k} \end{pmatrix} \longrightarrow \begin{pmatrix} T'_{1,1} & \dots & T'_{1,k} \\ \vdots & & \vdots \\ T'_{l,1} & \dots & T'_{l,k} \end{pmatrix} \tag{6.7}$$

where each $\alpha_{i,j} : T_{i,j} \to T'_{i,j}$ is a linear map in the usual sense.

These natural transformations give 2-morphisms between 2-linear maps, so that $\mathbf{Vect}^k$ is a bicategory with these as 2-cells:

$$\mathbf{Vect^k} \;\overset{F}{\underset{G}{\Downarrow \alpha}}\; \mathbf{Vect^l} \tag{6.8}$$

In our example above, the finite set of simple objects of which every object is a sum is the set of 2-vectors of the form

$$\begin{pmatrix} 0 \\ \vdots \\ \mathbb{C} \\ \vdots \\ 0 \end{pmatrix} \tag{6.9}$$

which have the zero vector space in all components except one (which can be arbitrary). These are like categorified "standard basis vectors", so we call them *standard basis 2-vectors* . Clearly every object of $\mathbf{Vect}^k$ is a finite biproduct of these objects, and each is simple (its vector space of endomorphisms is 1-dimensional).

The most immediately useful fact about KV 2-vector spaces is the following well known characterization:

**Theorem 4** *Every KV 2-vector space is equivalent as a category to* $\mathbf{Vect}^k$ *for some* $k \in \mathbb{N}$.

**Proof**: Suppose $\mathbf{K}$ is a KV 2-vector space with a basis of simple objects $X_1 \ldots X_k$. Then we construct an equivalence $E : \mathbf{K} \to \mathbf{Vect}^k$ as follows:

$E$ should be an additive functor with $E(X_i) = V_i$, where $V_i$ is the $k$-tuple of vector spaces having the zero vector space in every position except the $i^{th}$, which has a copy of $\mathbb{C}$. But any object $X$, is a sum $\bigoplus_i X_i^{n_i}$, so by linearity (i.e. the fact that $E$ preserves biproducts) $X$ will be sent to the sum of the same number of copies of the $V_i$, which is just a $k$-tuple of vector spaces whose $i^{th}$ component is $\mathbb{C}^{n_i}$. So every object in $K$ is sent to an $k$-tuple of vector spaces. By $\mathbb{C}$-linearity, and the fact that hom-vector spaces of simple objects are one-dimensional, this determines the images of all morphisms.

But then the weak inverse of $E$ is easy to construct, since sending $V_i$ to $X_i$ gives an inverse at the level of objects, by the same linearity argument as above. At the level of morphisms, the same argument holds again. □

This is a higher analog of the fact that every finite dimensional complex vector space is isomorphic to $\mathbb{C}^k$ for some $k \in \mathbb{N}$. So, indeed, the characterization of 2-vector spaces in our example above is generic: every KV 2-vector space is equivalent to one of the form given. Moreover, our picture of 2-linear maps is also generic, as shown by this argument, analogous to the linear algebra argument for representation of linear maps by matrices:

**Lemma 3** *Any 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ is naturally isomorphic to a map of the form (6.6).*

**Proof**: Any 2-linear map $T$ is a $\mathbb{C}$-linear additive functor between 2-vector spaces. Since any object in a 2-vector space can be represented as a biproduct of simple objects—and morphisms likewise—such a functor is completely determined by its effect on the basis of simple objects and morphisms between them.

But then note that since the automorphism group of a simple object is by definition just all (complex) multiples of the identity morphism, there is no choice about where to send any such morphism. So a functor is complely determined by the images of the basis objects, namely the 2-vectors $V_i = (0, \ldots, \mathbb{C}, \ldots, 0) \in \mathbf{Vect}^n$, where $V_i$ has only the $i^{th}$ entry non-zero.

On the other hand, for any $i$, $T(V_i)$ is a direct sum of some simple objects in $\mathbf{Vect}^m$, which is just some 2-vector, namely a $k$-tuple of vector spaces. Then the fact that the functor is additive means that it has exactly the form given. □

And finally, the analogous fact holds for natural transformations between 2-linear maps:

**Lemma 4** *Any natural transformation $\alpha : T \to T'$ from a 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ to a 2-linear map $T' : \mathbf{Vect}^n \to \mathbf{Vect}^m$, both in the form (6.6) is of the form (6.7).*

**Proof**: By Lemma 3, the 2-linear maps $T$ and $T'$ can be represented as matrices of vector spaces, which act on an object in $\mathbf{Vect}^n$ as in (6.6). A natural transformation $\alpha$ between these should assign, to every object $X \in \mathbf{Vect}^n$, a morphism $\alpha_X : T(X) \to T'(X)$ in $\mathbf{Vect}^m$, such that the usual naturality square commutes for every morphism $f : X \to Y$ in $\mathbf{Vect}^n$.

Suppose $X$ is the $n$-tuple $(X_1, \ldots, X_n)$, where the $X_i$ are finite dimensional vector spaces. Then

$$T(X) = (\oplus_{k=1}^n V_{1,k} \otimes X_k, \ldots, \oplus_{k=1}^n V_{m,k} \otimes X_k) \tag{6.10}$$

where the $V_{i,j}$ are the components of $T$, and similarly

$$T'(X) = (\oplus_{k=1}^n V'_{1,k} \otimes X_k, \ldots, \oplus_{k=1}^n V'_{m,k} \otimes X_k) \tag{6.11}$$

where the $V'_{i,j}$ are the components of $T'$.

Then a morphism $\alpha_X : T(X) \to T'(X)$ consists of an $m$-tuple of linear maps:

$$\alpha_j : \oplus_{k=1}^n V_{j,k} \otimes X_k \to \oplus_{k=1}^n V'_{j,k} \otimes X_k \tag{6.12}$$

but by the universal property of the biproduct, this is the same as having an $(n \times m)$-indexed set of maps

$$\alpha_{jk} : V_{j,k} \otimes X_k \to \oplus_{r=1}^n V'_{j,r} \otimes X_r \tag{6.13}$$

and by the dual universal property, this is the same as having $(n \times n \times m)$-indexed maps

$$\alpha_{jkr} : V_{j,k} \otimes X_k \to V'_{j,r} \otimes X_r \tag{6.14}$$

However, we must have the naturality condition for every morphism $f : X \to X'$:

$$\begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(X') \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_{X'}} \\ T'(X) & \xrightarrow[T'(f)]{} & T'(X') \end{array} \tag{6.15}$$

Note that each of the arrows in this diagram is a morphism in $\mathbf{Vect}^m$, which are linear maps in each component—so in fact we have a separate naturality square for each component.

Also, since $T$ and $T'$ act on $X$ and $X'$ by tensoring with fixed vector spaces as in (6.10), one has $T(f)_i = \oplus_i f_i \otimes 1_{V_{ij}}$, having no effect on the $V_{ij}$. We want to show that the components of $\alpha$ affect *only* the $V_{ij}$.

Additivity of all the functors involved implies that the assignment $\alpha$ of maps to objects in $\mathbf{Vect}^n$ is additive. So consider the case when $X$ is one of the standard basis 2-vectors, having $\mathbb{C}$ in one position (say, the $k^{th}$), and the zero vector space in every other position. Then, restricting to the naturality square in the $k^{th}$ position, the above condition amounts to having $m$ maps (indexed by $j$):

$$\alpha_{j,k} : V_{j,k} \to V'_{j,k} \tag{6.16}$$

So by linearity, a natural transformation is determined by an $n \times m$ matrix of maps as in (6.7). $\square$

The fact that 2-linear maps between 2-vector spaces are functors between categories recalls the analogy between linear algebra and category theory in the

concept of an *adjoint*. If $V$ and $W$ are inner product spaces, the adjoint of a linear map $F : V \to V$ is a map $F^\dagger$ for which $\langle Fx, y\rangle = \left\langle x, F^\dagger y\right\rangle$ for all $x \in V_1$ and $y \in V_2$. A (right) adjoint of a functor $F : \mathbf{C} \to \mathbf{D}$ is a functor $G : \mathbf{D} \to \mathbf{C}$ for which $\hom_D(Fx, y) \cong \hom_C(x, Gy)$ (and then $F$ is a left adjoint of $G$).

In the situation of a KV 2-vector space, the categorified analog of the adjoint of a linear map is indeed an adjoint functor. (Note that since a KV 2-vector space has a specified basis of simple objects, it makes sense to compare it to an inner product space.) Moreover, the adjoint of a functor has a matrix representation which is much like the matrix representation of the adjoint of a linear map. We summarize this as follows:

**Theorem 5** *Given any 2-linear map $F : V \to W$, there is a 2-linear map $G : W \to V$ which is both a left and right adjoint to $F$, and $G$ is unique up to natural isomorphism.*

**Proof**: By Theorem 4, we have $V \simeq \mathbf{Vect}^n$ and $W \simeq \mathbf{Vect}^m$ for some $n$ and $m$. By composition with these equivalences, we can restrict to this case. But then we have by Lemma 3 that $F$ is naturally isomorphic to some 2-linear map given by

matrix multiplication by some matrix of vector spaces $[F_{i,j}]$:

$$\begin{pmatrix} F_{1,1} & \dots & F_{1,n} \\ \vdots & & \vdots \\ F_{m,1} & \dots & F_{m,n} \end{pmatrix} \tag{6.17}$$

We claim that a (two-sided) adjoint functor $F^\dagger$ is given by the "dual transpose matrix" of vector spaces $[F_{i,j}]^\dagger$:

$$\begin{pmatrix} F_{1,1}^\dagger & \dots & F_{1,m}^\dagger \\ \vdots & & \vdots \\ F_{n,1}^\dagger & \dots & F_{n,m}^\dagger \end{pmatrix} \tag{6.18}$$

where $F_{i,j}^\dagger$ is the vector space dual $(F_{j,i})^*$ (note the transposition of the matrix).

We note that this prescription is symmetric, since $[T]^{\dagger\dagger} = [T]$, so if $G$ is always left adjoint of $F$, then $F$ is also a left-adjoint of $G$, hence $G$ a right adjoint of $F$. So if this prescription gives a left adjoint, it gives a two-sided adjoint. Next we check that it does.

Suppose $x = (X_i) \in \mathbf{Vect}^n$ is the 2-vector with vector space $X_i$ in the $i^{th}$ component, and $y = (Y_j) \in \mathbf{Vect}^m$ has vector space $Y_j$ in the $j^{th}$ component. Then $Fx \in \mathbf{Vect}^m$ has $j^{th}$ component $\oplus_{k=1}^n V_{k,j} \otimes X_k$. Now, a map in $\mathbf{Vect}^m$ from $Fx$ to $y$ consists of a linear map in each component, so it is an $m$-tuple of maps:

$$f_j : \oplus_{k=1}^n V_{k,j} \otimes X_k \to Y_j \tag{6.19}$$

for $j = 1 \ldots m$. But since the direct sum (biproduct) is a categorical product, this is the same as an $m \times n$ matrix of maps:

$$f_{kj} : V_{k,j} \otimes X_k \to Y_j \tag{6.20}$$

for $k = 1 \ldots n$ and $j = 1 \ldots m$, and $\hom(Fx, y)$ is the vector space of all such maps.

By the same argument, a map in $\mathbf{Vect}^n$ from $x$ to $Gy$ consists of an $n \times m$ matrix of maps:

$$g_{jk} :: X_k \to V_{j,k}^* \otimes Y_j \cong \hom(V_{j,k}, Y_j) \tag{6.21}$$

for $k = 1 \ldots n$ and $j = 1 \ldots m$, and $\hom(x, Gy)$ is the vector space of all such maps.

But then we have a natural isomorphism $\hom(Fx, y) \cong \hom(x, Gy)$ by the duality of hom and $\otimes$, so in fact $G$ is a right adjoint for $F$, and by the above argument, also a left adjoint.

Moreover, no other nonisomorphic matrix defines a 2-linear map with these properties, and since any functor is naturally isomorphic to some matrix, this is the sole $G$ which works. $\square$

We conclude this section by giving an example of a 2-vector space which we shall return to again later. It is motivated by the attempt to generalize the FHK construction of a TQFT from a group, as described in Section 2.3. During the

construction of the vector space assigned to a circle, one makes use of the group algebra of a finite group $G$—the set of complex linear combinations of group elements. There is a categorified analog:

**Example 6** As an example of a KV 2-vector space, consider the *group 2-algebra* on a finite group $G$, defined by analogy with the group algebra:

The group algebra $\mathbb{C}[G]$ consists of the set of elements formed as formal linear combinations elements of $G$:

$$b = \sum_{g \in G} b_g \cdot g \tag{6.22}$$

where all but finitely many $b_g$ are zero. We can think of these as complex functions on $G$. The algebra multiplication on $\mathbb{C}[G]$ is given by the multiplication in $G$:

$$b \star b' = \sum_{g,g' \in G} (b_g b'_{g'}) \cdot gg' \tag{6.23}$$

This does not correspond to the multiplication of functions on $G$, but to *convolution*:

$$(b \star b')_g = \sum_{h \cdot h' = g} b_h b'_{h'} \tag{6.24}$$

Similarly, the *group 2-algebra* $A = \mathbf{Vect}[G]$ is the *category* of $G$-graded vector spaces. That is, direct sums of vector spaces associated to elements of $G$:

$$V = \bigoplus_{g \in G} V_g \tag{6.25}$$

where $V_g \in \mathbf{Vect}$ is a vector space. This is a $G$-graded vector space. We can take direct sums of these pointwise, so that $(V \oplus V')_g = V_g \oplus V'_g)$, and there is a "scalar" product with elements of **Vect** given by $(W \otimes V)_g = W \otimes V_g$. There is also a *group 2-algebra* product of $G$-graded vector spaces, involving a convolution on $G$:

$$(V \star V')_h = \bigoplus_{g \cdot g' = h} V_g \otimes V'_{g'} \tag{6.26}$$

The category of $G$-graded vector spaces is clearly a KV 2-vector space, since it is equivalent to $\mathbf{Vect}^k$ where $k = |G|$. However, it has the additional structure of a 2-algebra because of the group operation on the finite set $G$.

The analogy between group algebras and group 2-algebras highlights one motivation for thinking of 2-vector spaces. This is the fact that, in quantum mechanics, one often "quantizes" a classical system by taking the Hilbert space of $\mathbb{C}$-valued functions on its phase space. Similarly, one approach to finding a higher-categorical version of a quantum field theory is to take **Vect**-valued functions. We have noted in Section 2.3 that, given a finite group, the Fukuma-Hosono-Kawai construction gives 2D TQFT, whose Hilbert space of states on a circle is just $\mathbb{C}[G]$. For this reason, we expect that Example 6 should be relevant to categorifying this theory. However, it is not quite sufficient, as we discuss in Section 6.2.

## 6.2 KV 2-Vector Spaces and Finite Groupoids

The group 2-algebra of Example 6 shows that we can get a 2-vector space as a category of functions from some finite set $S$ into **Vect**, and this may have extra structure if $S$ does. However, this is somewhat unnatural, since **Vect** is a category and $S$ a mere set. It seems more natural to consider functor categories into **Vect** from some category **C**. These are the generalized 2-vector spaces described by Elgueta [34]. Then the above way of looking at a KV 2-vector space can be reduced to the situation when **C** is a discrete category with a finite set of elements.

However, there are interesting cases where **C** is not of this form, and the result is still a KV vector space. A relevant class of examples, as we shall show, come from special kinds of groupoids.

**Definition 12** *An* ***essentially finite*** *groupoid is one which is equivalent to a finite groupoid. A* ***finitely generated*** *groupoid is one with a finite set of objects, and all of whose morphisms are generated under composition by a finite set of morphisms. An* ***essentially finitely generated*** *groupoid is one which is equivalent to a finitely generated one.*

We first show that finite groupoids are among the special categories **C** we want to consider:

**Lemma 5** *If* $\mathbf{X}$ *is an essentially finite groupoid, the functor category* $[\mathbf{X}, \mathbf{Vect}]$ *is a KV 2-vector space.*

**Proof**: To begin with, we note that **Vect** is trivially a KV 2-vector space. In particular, it is a $\mathbb{C}$-linear additive category, which we use to give $[\mathbf{X}, \mathbf{Vect}]$ the same structure.

Define a biproduct $\oplus$ on $[\mathbf{X}, \mathbf{Vect}]$ as follows. Given two functors $F_1, F_2 : \mathbf{X} \to \mathbf{Vect}$, define for both objects and morphisms,

$$(F_1 \oplus F_2)(x) = F_1(x) \oplus F_2(x) \tag{6.27}$$

where we are using both the direct sum of vector spaces, and the fact that linear maps between vector spaces inherit a direct sum. The projections and injections are defined pointwise. Since the biproduct axioms (6.1) and (6.2) hold pointwise, this is indeed a biproduct.

Now $\mathbf{X}$ is equivalent to a skeleton of itself, $\underline{\mathbf{X}}$, which contains a single object in each isomorphism class. Since $\mathbf{X}$ is essentially finite, this is also a finite set of objects, and each object has a finite set of endomorphisms. Since these are all invertible, $\mathbf{X}$ is in fact equivalent to a finite coproduct of finite groups, thought of as single-object categories.

But then a functor $F : \underline{\mathbf{X}} \to \mathbf{Vect}$ is just a direct sum of functors from these groups. A functor from a group $G$ (as a one-object category) to **Vect** is just a

finite dimensional representation of $G$. Now, Schur's Lemma states that the only intertwining operators from an irreducible representation to itself are multiples of the identity. That is, it ensures that all such representations are simple objects. On the other hand, every representation is a finite direct sum of irreducible ones.

So in particular, the finite dimensional representations of a finite group form a KV 2-vector space. A direct sum of such categories is again a KV 2-vector space, and so $[\underline{\mathbf{X}}, \mathbf{Vect}]$ is one.

But $[\mathbf{X}, \mathbf{Vect}]$ is equivalent to this, so it is a KV 2-vector space. □

We notice that we are speaking here of groupoids, and any groupoid $\mathbf{X}$ is equivalent to its opposite category $\mathbf{X}^{op}$, by an equivalence that leaves objects intact and replaces each morphism by its inverse. So there is no real difference between $[\mathbf{X}, \mathbf{Vect}]$, the category of **Vect**-valued functors from $\mathbf{X}$, and $[\mathbf{X}^{op}, \mathbf{Vect}]$, the category of **Vect**-*valued presheaves* (henceforth simply "**Vect**-presheaves") on $\mathbf{X}$, where we emphasize that unlike ordinary presheaves, these are functors into **Vect**, rather than **Set**. So we have shown that **Vect**-presheaves on a groupoid $\mathbf{X}$ form a KV 2-vector space. We will work with these examples from now on.

Since many results about presheaves are well known, we will find it convenient to use this terminology for objects of $[\mathbf{X}, \mathbf{Vect}]$ for the sake of compatibility, and to highlight the connection to these results. We will ignore the distinction in the sequel, but remind readers here that our uses of the term "presheaf" are valid only

because we are working with groupoids. The importance of **Set**-valued presheaves to topos theory, and the richness of existing results for these, is one reason to keep this relationship in mind.

Now we want to show a result analogous to a standard result for presheaves (see, e.g. MacLane and Moerdijk [67], Theorem 1.9.2). This is that functors between underlying groupoids induce 2-linear maps between the 2-vector spaces of **Vect**-presheaves on them.

**Theorem 6** *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \to \mathbf{Y}$ *gives two 2-linear maps between KV 2-vector spaces:*

$$f^* : [\mathbf{Y}, \mathbf{Vect}] \to [\mathbf{X}, \mathbf{Vect}] \tag{6.28}$$

*called "pullback along* $f$*" and*

$$f_* : [\mathbf{X}, \mathbf{Vect}] \to [\mathbf{Y}, \mathbf{Vect}] \tag{6.29}$$

*called "pushforward along* $f$*". Furthermore,* $f_*$ *is the (two-sided) adjoint to* $f^*$*.*

**Proof**: First we define, for any functor $F : \mathbf{Y} \to \mathbf{Vect}$,

$$f^*(F) = F \circ f \tag{6.30}$$

which is a functor from $\mathbf{X}$ to **Vect**. This is just the pullback of $F$ along $f$.

To show that this is a 2-linear map (that is, a biproduct-preserving, $\mathbb{C}$-linear functor), we first note that it is trivially $\mathbb{C}$-linear since a linear combination of

maps in some hom-category in $[\mathbf{Y}, \mathbf{Vect}]$ is taken by $f^*$ to the corresponding linear combination in the hom-category in $[\mathbf{X}, \mathbf{Vect}]$, where maps are now between vector spaces thought of over $x \in \mathbf{X}$.

To check that the functor $f^* : [\mathbf{Y}, \mathbf{Vect}] \to [\mathbf{X}, \mathbf{Vect}]$ preserves biproducts, note that for any $x \in \mathbf{X}$ we have that $f^*(F_1 \oplus F_2)(x) = (F_1 \oplus F_2)(f(x)) = F_1(f(x)) \oplus F_2(f(x)) = (f^*F_1 \oplus f^*F_2)(x)$.

So indeed there is a 2-linear map $f^*$. But then by Theorem 5, there is a two-sided adjoint of $f^*$, denoted $f_*$. $\square$

**Remark 6** The argument in this proof for the existence of the adjoint to $f^*$ uses Theorem 5. While no such theorem exists for **Set**-valued presheaves, there is a corresponding theorem defining a "pushforward" of presheaves of sets. In fact, the only major difference between what we have shown for **Vect**-presheaves and the standard results for **Set**-presheaves is that the left and right adjoint are the same. This means that the "pushforward" map is an *ambidextrous adjunction* for the pullback (for much more on the relation between ambidextrous adjunctions and TQFTs, see Lauda [57]).

It seems useful, then, to have another approach to the "pushforward" map than the matrix-dependent view of Theorem 5. Fortunately, there is a more instrinsic way to describe the 2-linear map $f_*$, the adjoint of $f^*$, and we know this must be the same as the one given in matrix form.

**Definition 13** *For a given $y \in \mathbf{Y}$, define the diagram $D_y$ whose objects are objects $x \in \mathbf{X}$ equipped with maps $f(x) \to y$ in $\mathbf{Y}$, and whose morphisms are morphisms $a : x \to x'$ whose images make the triangles*

$$\begin{array}{ccc} f(x) & \xrightarrow{f(a)} & f(x') \\ & \searrow \quad \swarrow & \\ & y & \end{array} \tag{6.31}$$

*in $\mathbf{Y}$ commute. Given a $\mathbf{Vect}$-presheaf $G$ on $\mathbf{X}$, define $f_*(G)(y) = \operatorname{colim} G(D_y)$—a colimit in $\mathbf{Vect}$.*

*The pushforward of a morphism $b : y \to y'$ in $\mathbf{Y}$, $f_*(G)(b) : f_*(G)(y) \to f_*(G)(y')$ is left to the reader.*

This definition of the pushforward involved the diagram $D$, which is the *comma category* of objects $x \in \mathbf{X}$ equipped with maps from $f(x)$ to $y$. This is the appropriate categorical equivalent of a *preimage*—rather than requiring $f(x) = y$, one accepts that they may be isomorphic, in different ways. So this is a categorified equivalent of taking a sum over a preimage. It needs to be confirmed directly that it really is the adjoint.

**Theorem 7** *This $f_*$ is a 2-linear map, and a two-sided adjoint for $f^*$.*

**Proof**: The given $f_*$ certainly defines a $\mathbf{Vect}$-presheaf $f_*G$ on $\mathbf{Y}$, and the operation of taking colimits is functorial and preserves biproducts, so $f_*$ is a 2-linear map.

Consider the effect of $f_*$ on a 2-vector $G : \mathbf{X} \to \mathbf{Vect}$ by describing $f_*G : \mathbf{Y} \to \mathbf{Vect}$. If $F : \mathbf{Y} \to \mathbf{Vect}$ is as above, there should be a canonical isomorphism between $[G, f^*(F)]$ (a hom-set in $[\mathbf{X}, \mathbf{Vect}]$) and $[p_*(G), F]$ (a hom-set in $[\mathbf{Y}, \mathbf{Vect}]$).

The hom-set $[G, f^*(F)]$ is found by first taking the pullback of $F$ along $f$. This gives a presheaf on $\mathbf{X}$, namely $F(f(-))$. The hom-set is then the set of natural transformations $\alpha : G \to f^*F$. Each such $\alpha$, given an object $x$ in $\mathbf{X}$, picks a linear map $\alpha_x : F(f(x)) \to G(x)$ (subject to the naturality condition).

For an object $y$ in $\mathbf{Y}$, pulling back $F$ onto $\mathbf{X}$ gives the vector space $F(y)$ at each object $x$ with $f(x) = y$. This is the presheaf $f^*F$. So an element of $[f^*F, G]$ is an assignment, to every $x \in \mathbf{X}$, a linear map $f^*F = F(y) \to G(x)$.

To get the equivalence required for adjointness, given a linear map $h : f_*G(y) \to F(y)$, one should get a collection of maps $h_x : G(x) \to F(y)$ for each object $x$ in $D$ (which commute with all arrows in $D$). But $f_*(G)(x)$ was defined to be a colimit, hence there is a unique compatible map $i_x$ from each $G(x)$ into it, so take $h_x = h \circ i_x : G(x) \to F(y)$. This gives a map from $[p_*(G), F]$ to $[G, f^*(F)]$. To see that this is an equivalence, note that the colimit is a *universal* object with the specified maps. So given the collection of $h_x$, one gets the map $h$ from the universal property.

So $f_*$ is a left adjoint to $f^*$. By Theorem 5, it is therefore also a right-adjoint.

□

**Remark 7** For future reference, we will describe the pair of adjoint functors, $f^*$ and $f_*$ in even more detail, since this is used in the construction of our extended TQFT in Chapter 7. Since we will want to make use of the simplifying fact that any groupoid is equivalent to a skeletal groupoid, it is particularly helpful to consider this case.

A skeletal groupoid has exactly one object in each isomorphism class, so it is equivalent to a disjoint union of one-object groupoids - which can be interpreted as groups. Since $\mathbf{X}$ and $\mathbf{Y}$ are essentially finite, these are finite groups. So a **Vect**-presheaf on $\mathbf{X}$ is a functors which assigns a vector space $V_x$ to each object $x \in \mathbf{X}$, and a linear map $V \to V$ for each morphism (i.e. group element). This is just a representation of the finite group $\mathrm{Aut}(x)$ on $\mathbf{Vect}_x$.

If $\mathbf{X}$ and $\mathbf{Y}$ are skeletal, then $f : \mathbf{X} \to \mathbf{Y}$ on objects is just a set map, taking objects in $\mathbf{X}$ to objects in $\mathbf{Y}$. For morphisms, $f$ gives, for each object $x \in \mathbf{X}$, a homomorphism from the group $\hom(x, x)$ to the group $\hom(f(x), f(x))$.

So the pullback $f^*$ is fairly straightforward: given $F : \mathbf{Y} \to \mathbf{Vect}$, the pullback $f^*F = F \circ f : \mathbf{X} \to \mathbf{Vect}$ assigns to each $x \in \mathbf{X}$ the vector space $F(f(x))$, and gives a representation of $\mathrm{Aut}(x)$ on this vector space where $g : x \to x$ acts by $f(g)$.

This is the *pullback representation*. In the special case where $f$ gives an inclusion of groups, this is usually called the "restricted representation".

The adjoint process to the restriction of representations is generally called finding the *induced representation* (see, e.g. Burrows [20] for a classical discussion of this when $f$ is an inclusion). We will use the same term for the general case when $f$ is just a homomorphism, and slightly generalize the usual description.

The pushforward $f_*$, recall, assigns each object the vector spaces which is the colimit of its essential preimage. For any presheaf $V$, this is determined by the colimit for each component of that essential preimage. In particular, in the simple case where $\mathbf{X}$ and $\mathbf{Y}$ are discrete (i.e. have only identity isomorphisms, so they can be thought of as sets, and the essential preimage is just the usual preimage for sets), for each $y \in \mathbf{Y}$,

$$f_*F(y) \in \mathbf{Y} = \bigoplus_{g: f(x) \to y} F(x) \tag{6.32}$$

So we just get the biproduct of all vector spaces over the preimage.

In any component, which can be seen as a group $H$, the colimit is again a direct sum over the components of the essential preimage, but each component of the essential preimage amounts to the induced representiation of $F(x)$ under the homomorphism given by $f$. So the colimit is a direct sum of such representations.

To see what this does, consider the case where $\mathbf{X}$ and $\mathbf{Y}$ are just single groups, so we have a group homomorphism $f : G \to H$, and we have a representation of

$G$ on $V$. Now such a representation is the same as a representation of the group algebra $\mathbb{C}[G]$ on $V$ - i.e. it makes $V$ into a $\mathbb{C}[G]$-module. Furthermore, $f$ induces an algebra homomorphism $f : \mathbb{C}[G] \to \mathbb{C}[H]$.

To get a $\mathbb{C}[H]$-module from $V$ (i.e. in order to produce a representation of $H$, the pushforward of $V$), we first allow $\mathbb{C}[H]$ to act freely on $V$. Then, to be the pushforward - that is, the colimit of the diagram $D_y$ described above - we must take the quotient under the relation that all morphisms coming from $G$ act on $V$ by letting $f(g)$ have the same action as $g$. Taking the quotient, we get $f_*V = \mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$.

Then for general groupoids, we have a direct sum of such components:

$$f_*F(y) \in \mathbf{Y} = \bigoplus_{g : f(x) \to y} \mathbb{C}[H_y] \otimes_{\mathbb{C}[G_x]} V \tag{6.33}$$

where $H_y = \mathrm{Aut}(y)$ and $G_x = \mathrm{Aut}(x)$.

**Remark 8** To describe an adjunctions, we should describe its unit and counit. To begin with, we give a description of the "pull-push":

$$f_* \circ f^* : [\mathbf{Y}, \mathbf{Vect}] \to [\mathbf{Y}, \mathbf{Vect}] \tag{6.34}$$

The unit

$$\eta : \mathbb{1}_{[\mathbf{Y}, \mathbf{Vect}]} \Longrightarrow f_* \circ f^* \tag{6.35}$$

is a natural transformation which, for each $V \in [\mathbf{Y}, \mathbf{Vect}]$ gives a morphism. This is itself a natural transformation between functors:

$$\Delta_y : V(y) \to f_* \circ f^* V(y) \tag{6.36}$$

This takes $V(y)$ into the colimit described above by a diagonal map. If there is no special symmetry (the discrete case) and the colimit is just the direct sum $V \oplus \cdots \oplus V$, this map is obvious. If not, there is a canonical map into the colimit (a quotient space) which factors through the direct sum with the diagonal map. This is because the map from $V(y)$ to the pullback on any bject in its essential preimage in $\mathbf{X}$ is evidently the identity, and then one uses the injection into the colimit.

Now consider the other, "push-pull" side of the (two-sided) adjunction, $f^* \circ f_*$. Here, we first push a presheaf $V'$ from $\mathbf{X}$ to $\mathbf{Y}$, then pull back up to $\mathbf{X}$, has a similar effect on the vector spaces.

Here we start with a presheaf $V'$ on $\mathbf{X}$. The "push-pull" along $f$ just takes every vector space on an object and replaces it by a colimit over the diagram consisting of all objects with the same image in $\mathbf{Y}$, and morphisms agreeing with these maps: $\operatorname{colim} D_{f(x)}$. This is because this is the result of pushing $V'$ along $f$ at $f(x)$, which is then pulled back to $x$.

Then the unit

$$\eta : \mathbb{1}_{[\mathbf{X},\mathbf{Vect}]} \Longrightarrow f^* \circ f_* \tag{6.37}$$

is a natural transformation giving, for any presheaf $V'$, a morphism (i.e. natural transformation of functors):

$$\iota_x : V'(x) \to \operatorname{colim} D_{f(x)} \tag{6.38}$$

This is just the canonical map into the colimit.

Now, in Section 6.3 we discuss a generalization of 2-vector spaces based on the fields of measurable Hilbert spaces discussed by Crane and Yetter [28]. This generalization has much in common with KV 2-vector spaces, but corresponds to infinite dimensional Hilbert spaces in the way that they correspond to the finite dimensional case.

## 6.3 2-Hilbert Spaces

The KV 2-vector spaces we have discussed so far are a categorified analog of finite dimensional vector spaces. However, there are situations in which this is insufficient, and analogs of infinite dimensional vector spaces are needed. Moreover, and perhaps more important, we have not yet discussed the equivalent of an inner product on 2-vector spaces.

In fact, both of these issues are closely related to applications to quantum mechanics. A standard way to describe a quantum mechanical system, starting with the corresponding classical system, involves $L^2$ spaces, which in general will

be infinite dimensional, and possess an inner product. The relationship is that the Hilbert space of states of the quantum system is $L^2(X)$, where $X$ is the phase space of a classical system. A possible motivation for trying to find a higher analog for Hilbert spaces is to reproduce this framework for quantizing a classical theory in the categorified setting.

The form of an inner product on a KV 2-vector space is not difficult to infer from the intuition that categorification corresponds to replacing sums and products in vector spaces by $\oplus$ and $\otimes$ in 2-vector spaces, together with the fact that a KV 2-vector space has a specified basis. However, we will put off describing it until we have discussed infinite dimensional 2-vector spaces, since we can put the expression in a more general form.

One approach to infinite-dimensional 2-vector spaces is developed by Crane and Yetter [28], who develop a 2-category called **Meas**. This is a 2-category of categories, functors, and natural transformations, but in particular, the objects are all of the form **Meas(X)** for some measurable space $X$. This object can be interpreted as infinite-dimensional 2-vector spaces associated to $X$, analagous to the Hilbert space $L^2(X)$. A simplified form of the definition is as follows:

**Definition 14** *Suppose $(X, \mathcal{M})$ is a measurable space, so that $X$ is a set and $\mathcal{M}$ is a sigma-algebra of measurable subsets of $X$. Then* **Meas(X)** *is a category with:*

- *Objects: measurable fields of Hilbert spaces on $(X, \mathcal{M})$: i.e. $X$-indexed families of Hilbert spaces $\mathcal{H}_x$ such that the preimage of any $H \in$ **Hilb** is measurable.*

- *Morphisms: measurable fields of bounded linear maps between Hilbert spaces. That is, an $X$-indexed family*

$$f_x : \mathcal{H}_x \to \mathcal{K}_x$$

*so that $||f||$, the operator norm of $f$, is measurable. The field $f$ is bounded if $||f_x||$ is bounded.*

**Remark 9** The original definition given by Crane and Yetter is somewhat different, in the way it specifies how to identify when a function selecting $v_x \in \mathcal{H}_x \forall x \in X$ is measurable. This somewhat simplified definition should suffice for our later discussion, since we return to these ideas only briefly in Chapter 8.

The construction of fields of Hilbert spaces is due to Jacques Dixmier [31], although he described them, not as categories, but merely as Hilbert spaces with a particular decomposition in terms of the measurable space $X$. As with $L^2$ spaces, to get what we will call a 2-Hilbert space, we need to have a standard measure on $X$. This is used to define a *direct integral* of Hilbert spaces:

$$H = \int_X^{\oplus} \mathcal{H}_x \mathrm{d}\mu(x) \tag{6.39}$$

As a vector space, this is the direct sum of all $\mathcal{H}_x$. The measure enters when we define its inner product:

$$\langle \phi | \psi \rangle = \int_X \langle \phi_x | \psi_x \rangle \mathrm{d}\mu(x) \tag{6.40}$$

We will use this notation to define 2-linear maps of 2-Hilbert spaces.

The 2-vector space $\mathbf{Meas(X)}$ is the category of all measurable fields of Hilbert spaces on $X$. Then we have the 2-category of all such categories:

**Definition 15** *The 2-category* **Meas** *is the collection of all categories* $\mathbf{Meas(X)}$*, with functors between them, and natural transformations between functors.*

Crane and Yetter describe how functors between such categories arise from:

- a measurable field of Hilbert spaces $\mathcal{K}_{(x,y)}$ on $X \times Y$
- a $Y$-family $\mu_y$ of measures on $X$

Given these things, there is a functor $\Phi_{\mathcal{K},\mu_y} : \mathbf{Meas(X)} \to \mathbf{Meas(Y)}$ any field $\mathcal{H}_x$ on $X$:

$$\Phi_{\mathcal{K},\mu_y}(\mathcal{H})_y = \int_X^{\oplus} \mathcal{H}_x \otimes \mathcal{K}_{(x,y)} \mathrm{d}\mu_y(x) \tag{6.41}$$

This is a generalization of the 2-linear maps between Kapranov-Voevodsky 2-vector spaces: summing over an index set in matrix multiplication is a special case of integrating over $X$, when $X$ is a finite set with counting measure (and all

the vector spaces $\mathbb{C}^n$ which appear as components in a 2-vector or 2-linear map are equipped with the standard inner product). Indeed, these functors generalize the matrices (6.6). Yetter conjectures that all functors between categories like **Meas(X)** are of this form.

The 2-maps are ways to get from one functor to another. In this case, given $\Phi_{\mathcal{K},\mu_y}$ and $\Phi_{\mathcal{K}',\nu_y}$, if there is such a 2-map, it will be given by:

- A measurable field of bounded linear operators

$$\alpha_{(x,y)} : \mathcal{K}_{(x,y)} \to \mathcal{K}'_{(x,y)} \tag{6.42}$$

- A $Y$-indexed family $\left\{\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)_y\right\}$s.t.$y \in Y$, the *Radon-Nikodym* derivatives of $\nu_y$ w.r.t. $\mu_y$ (or, equivalently, )

Once again, the KV 2-vector space situation is a special case as above.

Now, just as integration is used to define the inner product on $L^2(X)$, the direct integral gives a categorified equivalent of an *inner product* of fields of Hilbert spaces:

$$\langle \mathcal{H} | \mathcal{H}' \rangle = \int_X^{\oplus} \mathcal{H}_x^* \otimes {\mathcal{H}'}_x \mathrm{d}\mu(x) \tag{6.43}$$

So in particular, the inner product is given by linearity, and the fact that, for $\phi_i \in \mathcal{H}$ and ${\phi'}_i \in \mathcal{H}'$:

$$\langle \phi_1^* \otimes {\phi'}_1 | \phi_2^* \otimes {\phi'}_2 \rangle = \int_X \langle \phi_1^* | \phi_2^* \rangle \cdot \langle \phi_1' | \phi_2' \rangle \mathrm{d}\mu(x) \tag{6.44}$$

where $\phi^*$ is the dual of $\phi$, namely $\langle\phi|-\rangle$.

We will mostly consider the finite-dimensional (Kapranov-Voevodsky) 2-vector spaces, which remain better understood than these infinite dimensional 2-Hilbert spaces in the style of Crane and Yetter. However, we return to these ideas to justify some of the physical motivation for this thesis in Chapter 8.

# Part III

# Extended TQFT's and Quantum Gravity

# Chapter 7

# Extended TQFTs as 2-Functors

We began in our preliminary section by discussing Atiyah's description of an $n$-dimensional TQFT as a functor

$$Z : \mathbf{nCob} \to \mathbf{Hilb} \tag{7.1}$$

The development since that point has been aimed at setting up what we need to give a parallel description of an *extended TQFT* in terms of 2-functors. This concept extends the definition of a TQFT to more general manifolds with corners, and is due to Ruth Lawrence.

One of the values of TQFT's has been as a method for finding invariants of manifolds, and in particular, for 3-manifolds (potentially with boundary). This is closely connected to the subject of knot theory, since knots are studied by their complement in some 3-manifold. One way to think of the invariants which appear this way is as ways of cutting up the manifold into pieces, assigning algebraic data to the pieces, and then recombining it. The possibility of recombining the pieces

unambiguously to form the invariant for the whole manifold is precisely what we want to express as some form of *functoriality*.

By now we have considered two examples of the process of categorification. The first involved passing from **nCob**, a category of manifolds and cobordisms between them, to $\mathbf{nCob}_2$, a (double) bicategory in which we allowed cobordisms between cobordisms. The second case was the passage from **Vect**, the category of vector spaces and linear maps, to **2Vect** with 2-vector spaces, 2-linear maps, and natural transformations.

In the first case we saw that in both **nCob** and $\mathbf{nCob}_2$, each level of structure involves entities of one higher dimension than the previous level. So in $\mathbf{nCob}_2$, the objects (manifolds) have codimension one higher in the total spaces represented by the (isomorphism class of) cobordisms, than is the case in **nCob**. One sometimes says that categorification allows us to "go up a dimension", or rather codimension. This theme appears in what is probably the prototypical example of higher categories (and indeed categories of any kind), namely homotopy theory. where we consider homotopies between spaces, homotopies between homotopies, and so forth.

We want to use this to develop the following definition:

**Definition 16** *An **extended TQFT** is a weak 2-functor*

$$Z : \mathbf{nCob}_2 \to \mathbf{2Hilb} \tag{7.2}$$

So in particular, such a $Z$ assigns:

- To an $(n-2)$-manifold, a 2-Hilbert space (i.e. a $\mathbb{C}$-linear additive category)
- To an $(n-1)$-manifold, a 2-linear map between 2-Hilbert spaces (an exact $\mathbb{C}$-linear functor)
- To an $n$-manifold, a 2-natural transformation between 2-linear maps

Where all this data satisfies the conditions for a weak 2-functor (e.g. it preserves composition and units up to coherent isomorphism, and so forth). To take this as a definition seems reasonable enough, but we then need to show how particular examples of extended TQFT's satisfy this definition.

## 7.1 $Z_G$ on Manifolds: The Dijkgraaf-Witten Model

Here we want to consider explicit construction of some extended TQFT's based on a finite group $G$. We saw in Section 2.3 that the Fukuma-Hosono-Kawai construction gave a way to define a regular 2D TQFT for any finite group. In that case, space of states for a circle which is just the centre of the group algebra $\mathbb{C}[G]$. In particular, this means that the space of states has a basis consisting of elements of the group $G$. Each state therefore consists of some linear combination of group elements. Extending this to higher dimensions is somewhat nonobvious,

but turns out to be related to the Dijkgraaf-Witten (DW) model [30]. This can also be described as a *topological gauge theory*.

The DW model describes a flat connection on a manifold $B$ (we use $B$ rather than $M$ here for consistency with our previous notation). Being flat, the nontrivial information about a connection is that which depends only on the topology of $B$. In particular, all the information available about the connection comes in the form of holonomies of the connection around loops. The holonomy is an element of the gauge group $G$, which is the "symmetry group" of some field. The element assigned to a loop gives the element of $G$ by which the field would be transformed if it is "parallel transported" around that loop. We then define:

**Definition 17** *A flat $G$-bundle on a connected, pointed manifold $B$ is a homomorphism*

$$A : \pi_1 B \to G \tag{7.3}$$

*We denote the set of all such functions as $\mathcal{A}_0(B)$.*

This definition is different from the usual concept of a "$G$-bundle" equipped with a flat connection in terms of fibre bundles, but the two concepts are equivalent, as established by Thurston [83].

Generally, a flat $G$-bundle on $B$ takes loops in $B$ into elements $G$. For any loop $\gamma$ in $B$, it assigns an element $A(\gamma) \in G$. This is the *holonomy* around the loop

$\gamma$. The $G$-connection is flat if the holonomy assigned to a loop is invariant under homotopy. In particular, any contractible loop must have trivial holonomy. On the other hand, nontrivial elements of the fundamental group of $M$ may correspond to nontrivial elements of $G$. These are thought of as describing the "parallel transport" of some object, on which $G$ acts as a symmetry group, around the loop. The usual picture in gauge theory has this object being the fibre of some bundle, such as a vector space, so that $G$ is a Lie group such as $SO(3)$. However, the same picture applies when $G$ is finite.

However, instead of the *set* of flat bundles here, we want to categorify this usual picture, to extend TQFT's to give a functor into **2Vect**. So there must be a category to take the place of $\mathcal{A}_0$, which has morphisms as well as objects. Fortunately, the structure of gauge theory which we have not captured in the definition of $\mathcal{A}_0$ does precisely this.

The principle here is that the fundamental group is too restrictive, and we should instead use the *fundamental groupoid* of $B$, and describe connections as functors.

**Definition 18** *The* ***fundamental groupoid*** $\Pi_1(B)$ *of a space $B$ is a groupoid with points of $B$ as its objects, and whose morphisms from $x$ to $y$ are just all homotopy classes paths in $B$ starting at $x$ and ending at $y$.*

The operation of taking $\Pi_1$ of a space can be thought of as a form of categorifying: instead of spaces considered as sets of points (with some topology), we now think of them as categories, whose set of objects is just the original space. In fact, these categories are groupoids, since we consider paths only up to homotopy, so every morphism is invertible. Moreover, a loop can be thought of as an automorphism of the chosen base point in $B$, so the fundamental group $\pi_1(B)$ is just the group of automorphisms of a single object in $\Pi_1(B)$.

Then, following the principle that a connection gives a group element in $G$ for each such loop, we can generalize this to the whole of $\Pi_1(B)$:

**Definition 19** *A **flat connection** is a functor*

$$A : \Pi_1(B) \to G \tag{7.4}$$

*where $G$ is thought of as a one-object groupoid (hence every $b \in B$ is sent to the unique object). A **gauge transformation** $\alpha : A \to A'$ from one connection to another is a natural transformation of functors: it assigns to each point $x \in B$ a group element in such a way that for each path $\gamma : x \to y$ the naturality square*

$$\begin{array}{ccc} \star & \xrightarrow{A(\gamma)} & \star \\ {\scriptstyle \alpha(x)}\downarrow & & \downarrow{\scriptstyle \alpha(y)} \\ \star & \xrightarrow[A'(\gamma)]{} & \star \end{array} \tag{7.5}$$

*commutes.*

**Remark 10** Using the notation that $[C_1, C_2]$ is the category whose objects are functors from $C_1$ to $C_2$ and whose morphisms are natural transformations, then we can say that flat connections and natural transformations form the objects and morphisms of the category

$$[\Pi_1(B), G] \tag{7.6}$$

Physicaly, a gauge transformation can be thought of as a change, at each point in $B$, of the way of measuring the internal degrees of freedom of the object which is transformed by $G$. In gauge theory, two connections which are related by a gauge transformation are usually considered to describe physically indistinguishable states - the differences they detect are due only to the system of measurement used.

We stop here to note that this definition is somewhat different from the usual notion of a smooth connection on a bundle—indeed, we have not used any concept of smoothness. To make all these connections into smooth connections on a definite bundle would be impossible. What we have described would have to be a sum over all possible bundles. However, for discrete $G$, we can ignore this issue.

So then the "configuration space" for an $(n-2)$-dimensional manifold $B$ in our extended TQFT will be a category whose objects are flat $G$-connections on $B$ and whose morphisms (all invertible) are gauge transformations between connections.

**Remark 11** If $\gamma : x \to x$ in $\Pi_1(B)$ is a loop, and $A$ and $A'$ are two connections related by a gauge transformation $\alpha$, we have $A'(\gamma) = \alpha(x)^{-1}A(\gamma)\alpha(x)$ - that is, the holonomies assigned by the two connections around the loop are conjugate. So physically distinct holonomies correspond to conjugacy classes in $G$.

In particular, in the case of 1-dimensional manifolds, if $B$ is just a circle, then the space of states of the field in the DW model has a basis consisting just of elements in the centre of $G$. (We remark here that this is the same as the TQFT for the FHK construction, which we have obtained now in a different way.)

But indeed, any category, and in particular the groupoid $\Pi_1(B)$, is equivalent to its skeleton. If $B$ is connected, all points are related by paths, so $\Pi_1(B) \cong \pi_1(B)$: the fundamental group, as a single-object category, is equivalent to the path category. However, the gauge transformations for connections measured from a fixed base point are determined by a single group element at the base point, which acts on holonomies around any loop by conjugation.

The groupoid $[\Pi_1(B), G]$, the configuration space for our theory, is the "moduli stack" of connections *weakly* modulo gauge transformations. This is a categorified equivalent of the usual physical configuration space, which consists of the set of equivalence classes of flat connetions modulo gauge transformations. Instead of imposing equations between connections related by gauge transformations, how-

ever, we simply add isomorphisms connecting these objects. This is the "weak quotient" of the space of connections by the action of a group.

Finally, using this, we can define a 2-vector space associated to any manifold:

**Definition 20** *For any compact manifold $B$, and finite group $G$, define $Z_G(B)$ to be the functor category $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$.*

as we verify in the following theorem.

**Theorem 8** *For any compact manifold $B$, and finite group $G$, the functor category $Z_G(B) = \big[[\Pi_1(B), G], \mathbf{Vect}\big]$ is a Kapranov-Voevodsky 2-vector space.*

**Proof**: First, note that for any space $B$,

$$\Pi_1(B) \equiv \coprod_{i=1}^{n} (\pi_1(B_i)) \tag{7.7}$$

where the sum is taken over all path components of $B$. That is, objects in $\Pi_1(B)$ are by definition isomorphic if and only if they are in the same path component. But this groupoid is equivalent to a skeletal version which has just one object for each isomorphism class—that is, one object for each path component. The automorphisms for the object corresponding to path component $B_i$ are then just the equivalence classes of paths from any chosen point to itself—namely, $\pi_1(B_i)$.

Moreover, if $B$ is a compact manifold, so is each component $B_i$, which is also connected. But the fundamental group for a compact, connected manifold is

finitely generated. So in particular, each $\pi_1(B_i)$ is finitely generated, and there are a finite number of components. So $\Pi_1(B)$ is an essentially finitely generated groupoid.

But if $\Pi_1(B)$ is essentially finitely generated, then since $G$ is a finite group, $[\Pi_1(B), G]$ is an essentially finite groupoid. This is because each functor's object map is determined by the images of the generators, and there are finitely many such assignments. Similarly, $\Pi_1(B)$ is equivalent to a skeleton of itself, and a natural transformation in this case is just given by a group element in $G$ for each component of $B$, so there are finitely many of these. By Lemma 5 this means that $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$ is a KV 2-vector space. $\square$

So we have a KV 2-vector space for each manifold, which is defined as **Vect**-valued functors, on the groupoid $[\Pi_1(B), G]$. As remarked earlier, we will describe these as **Vect**-presheaves, since $[\Pi_1(B), G]$ is isomorphic to $[\Pi_1(B), G]^{\mathrm{op}}$.

**Example 7** Consider the circle $S^1$.

The 2-vector space assigned to the cricle by our TQFT $Z_G$ is the Hilbert space of of flat connections modulo gauge transformations, on the circle:

$$\big[[\Pi_1(S^1), G], \mathbf{Vect}\big] \tag{7.8}$$

Now, $[\Pi_1(S^1), G]$ looks like the group $G$ equipped with the adjoint action on itself, in the following sense. The fundamental group of the circle is $\mathbb{Z}$, and $\Pi_1(S^1)$ is

thus equivalent to $\mathbb{Z}$ as a one-object category. Then taking maps into $G$, we note that each functor takes the unique object of $\mathbb{Z}$ to the unique object of $G$, and thus is determined entirely by the image of $1 \in \mathbb{Z}$. This will be some morphism $g \in G$ (i.e. an element of the group $G$), so we simply denote the corresponding functor by $g$.

A natural transformation between two functors $g$ and $g'$ assigns to the single object in $\mathbb{Z}$ a morphism $h \in G$—that is, it is again a group element. This must satisfy the naturality condition that $g'h = hg$, or simply $g' = hgh^{-1}$. So there is a natural transformation between functors for each conjugacy relation of this kind.

So $[\Pi_1(S^1), G]$ is equivalent to a groupoid whose objects correspond to elements of the group $G$, and whose morphisms are conjugacy relations between elements (which are clearly all invertible). This is also known as $G$ weakly modulo $G$, or $G/\!\!/G$. Another equivalent category is the skeleton of this, whose set of objects is the set of conjugacy classes of $G$. Each such object has a group of automorphisms $\mathrm{Stab}(g)$, the stabilizer of any element in it.

Finally, the 2-vector space corresponding to the circle is the category of functors from $G/\!\!/G$ into **Vect**. This gives a vector space for each object (element of $G$). It also assigns an isomorphism in **Vect** for each isomorphism in $G/\!\!/G$: the functors must be equivariant under conjugation by any $h \in G$. So the adjoint action of G on itself is already built into this 2-vector space, and an object of $Z(\mathbf{Vect}[G])$ is

functor $F : G \to \mathbf{Vect}$, which comes equipped with natural isomorphisms

$$R_g : F \to gFg^{-1} \tag{7.9}$$

such that

$$R_g R_h = R_{gh} \tag{7.10}$$

Where $gFg^{-1}$ is a functor whose image vector space at a point $h$ under $F$ becomes the image of the point $ghg^{-1}$.

So we have *$G$-equivariant functors* as the objects of the 2-vector space, and all $G$-equivariant natural transformations between them as the morphisms.

As a 2-vector space, this category of $G$-equivariant functors can be described in terms of its irreducible objects—since every other functor is isomorphic to a direct sum of these. Any equivariant functor will have the same value on every element of each conjugacy class in $G$, but an irreducible one will only assign nonzero to elements of ONE conjugacy class.

Moreover, since the action of $G$ by conjugation gives linear isomorphisms between the vector spaces over elements of $[\Pi_1(S^1), G]$, and since $[\Pi_1(S^1), G]$ is equivalent to its skeleton, we can think of this functor as specifying as a *conjugacy classes*, and single vector space $V$ assigned to it, together with a *linear representation* of $G$ on $V$.

So the objects of $ZS^1$ can be seen as consisting of pairs: a conjugacy class in $G$, and a representation of $G$.

This example of the circle returns to a previous remark about Example 6, the "group 2-algebra" $\mathbf{Vect}[G]$, the generalization of the group algebra $\mathbb{C}[G]$. As seen in Section 2.3, a TQFT based on the finite group $G$ assigns $Z(\mathbb{C}[G])$ to the circle. So one expects a categorified version to assign something like the centre of $\mathbf{Vect}[G]$ to a circle. What was not obvious in Example 6 was exactly what this is.

Irreducible elements of $Z(\mathbb{C}[G])$ are indeed specified by conjugacy classes of $G$, but as we see here, a difference appears because we think of functions on $G$ not precisely as a group, but as a groupoid of connections. Since the objects are the elements of $G$, and the morphisms are conjugations (as distinct from the view of a group as a one-object category), we get something new. The new ingredient is the representation of $G$. We return to this fact for infinite $G$ in Chapter 8.

**Example 8** Consider the torus $T^2 = S^1 \times S^1$. We want to find

$$Z_G(T^2) = \big[[\Pi_1(T^2), G], \mathbf{Vect}\big] \tag{7.11}$$

This will be equivalent to the category we get if we replace the fundamental groupoid $\Pi_1(T^2)$ by the equivalent skeletal groupoid. This is just the fundamental group of $T^2$, which is isomorphic to $\mathbb{Z}^2$. So we simplify here by using this version.

The category $[\Pi_1(T^2), G]$ has, as objects, functors from $\Pi_1(T^2)$ to $G$ (both seen as a categories with one object), and morphisms which consist of natural transformations. A functor $F \in [\mathbb{Z}^2, G]$ is then equivalent to a group homomorphism from $\mathbb{Z}^2$ to $G$. Since $\mathbb{Z}^2$ is the free abelian group on the two generators $(1, 0)$ and $(0, 1)$, the functor $F$ is determined by the images of these two generators. The only restriction on $F$ is that since $\mathbb{Z}^2$ is abelian, the images $g_1 = F(1, 0)$ and $g_2 = F(0, 1)$ must commute.

So the objects of $[\Pi_1(T^2), G]$ are indexed by commuting pairs of elements $(g_1, g_2) \in G^2$.

A natural transformation $g : F \to F'$ assigns to the single object $\star$ of $\mathbb{Z}^2$ a morphism in $G$—that is, a group element $h$. This must satisfy the naturality condition that this commutes for every $a \in \mathbb{Z}^2$:

$$\begin{array}{ccc} \star & \xrightarrow{F(a)} & \star \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \star & \xrightarrow[F'(a)]{} & \star \end{array} \tag{7.12}$$

Equivalently, since $h$ is invertible, we can write this in the form $hF(a)h^{-1} = F'(a)$ for all $a$. This will be true for all $a$ in $\mathbb{Z}^2$ as long as it is true for $(1, 0)$ and $(0, 1)$.

In other words, functors $F$ and $F'$ represented by $(g_1, g_2) \in G^2$ and $(g_1', g_2') \in G^2$, the natural transformations $\alpha : F \to F'$ correspond to group elements $h \in G$ which act in both components at once, so $(h^{-1}g_1h, h^{-1}g_2h) = (g_1', g_2')$.

So we have that the groupoid $[\Pi_1(T^2), G]$ is equivalent to $A/\!\!/G$, where $A = \{(g_1, g_2) \in G^2 : g_1 g_2 = g_2 g_1\}$, and the action of $G$ on $A$ comes from the action on $G^2$ as above.

So the 2-vector space $Z_G(T^2)$ is just the category of **Vect**-presheaves on $A$, equivariant under the given action of $G$. This assigns a vector space to each connection $(g_1, g_2)$ on $T^2$, and an isomorphism of these vector spaces for each gauge transformation $h : (g_1, g_2) \mapsto (h^{-1}g_1 h, h^{-1}g_2 h)$. Equivalently (taking a skeleton of this), we could say it gives a vector space for each equivalence class $[(g_1, g_2)] \in G^2$ under simultaneous conjugation, and a representation of $G$ on this vector space.

Both of these examples conform to a general pattern, which should be clear by now:

**Theorem 9** *The KV 2-vector space $Z_G(B)$ for any connected manifold $B$ is equivalent to* $\mathbf{Vect}^n$*, where $n$ is*

$$\sum_{[A] \in \mathcal{A}/G} |\{\textit{irreps of} \mathrm{Aut}(A)\}| \tag{7.13}$$

*where the sum is over equivalence classes of connections on $B$, and* $\mathrm{Aut}(A) \subset G$ *is the subgroup of $G$ which leaves $A$ fixed.*

**Proof**: The groupoid $[\Pi_1(B), G]$ is equivalent to its skeleton $S$. This has as objects the gauge equivalence classes of connections on $B$, and on each object, a

group of morphisms isomorphic to the group of gauge transformations fixing a representative (i.e. the automorphism group of any object in the original $[\Pi_1(B), G]$). Now we want to consider $[S, \mathbf{Vect}]$, which is equivalent to $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$. We know $[S, \mathbf{Vect}]$ is a KV vector space, hence equivalent to some $\mathbf{Vect}^n$, where $n$ is the number of nonisomorphic simple objects. So consider what these are.

A functor $F : S \to \mathbf{Vect}$ assigns a vector space to each equivalence class of connections (i.e. each object), but also a representation of the group of automorphisms of that object. This is $\mathrm{Aut}(A)$. Note that two functors giving inequivalent representations cannot have a natural isomorphism between them. On the other hand, any representation of $\mathrm{Aut}(A)$ is a direct sum of irreducible representations. So a simple objects in $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$ amount to a choice of $[A]$, and an irreducible representation of $\mathrm{Aut}(A)$. The theorem follows immediately. □

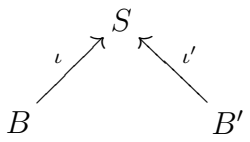
The next thing to consider is how $Z_G$ will act on cobordisms.

## 7.2 $Z_G$ on Cobordisms: 2-Linear Maps

We have described a construction which builds an extended TQFT from a finite group $G$. This takes a manifold $M$—possibly with boundary or corners—and produces a 2-vector space of states on it. This involved a 2-step construction:

first one finds $[\Pi_1(M), G]$, the moduli stack of flat connections; then one takes $\big[[\Pi_1(M), G], \mathbf{Vect}\big]$, which is the 2-vector space having $[\Pi_1(M), G]$ as basis.

This begins to describe the extended TQFT $Z : \mathbf{nCob}_2 \to \mathbf{2Vect}$ that we are interested in. However, $Z$ is to be a 2-functor, and so far we have only described what it does to objects of **Top**. This tells us its effect on objects in $\mathbf{nCob}_2$, and goes some way to describing its effect on morphisms, but recall that a morphism in $\mathbf{nCob}_2$ can be seen as a cospan in **Top**. A cobordism ("space") from a boundary $B$ to a boundary $B'$ is the cospan given by inclusion maps:




(7.14)

Our construction amounts to a sequence of functorial operations, which therefore give a corresponding sequence of spans (or cospans) in three different categories. Next we will consider each of these steps in turn, remarking on the co- or contravariance of the operation at each step.

The first step is the operation of taking the fundamental groupoid. This is somewhat more elaborate than the fundamental group of a (pointed) space, but it is closely related. Since any inclusion of spaces gives an inclusion of points, and

also of paths, we again have a cospan:

$$\begin{array}{ccc} & \Pi_1(S) & \\ {\scriptstyle \iota}\nearrow & & \nwarrow{\scriptstyle \iota'} \\ \Pi_1(B) & & \Pi_1(B') \end{array} \tag{7.15}$$

(Where we are abusing notation somewhat by using the same notation for the inclusion maps of spaces and path groupoids.)

In the next step, we apply a contravariant functor, $[-, G]$. Recall that we are thinking of the group $G$ as the category with one object $\star$ and the elements of $G$ as morphisms. Taking functors into $G$ is contravariant, since if we have a functor $F : X \to Y$, then any from $Y$ into $G$ becomes a map from $X$ into $G$ by pullback along $F$ (i.e. $\psi \mapsto \psi \circ F = F^*\psi$). That is, we get a functor $F^* : [Y, G] \to [X, G]$. So at this stage of the construction we have a span:

$$\begin{array}{ccc} & [\Pi_1(S), G] & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \pi'} \\ [\Pi_1(B), G] & & [\Pi_1(B'), G] \end{array} \tag{7.16}$$

For convenience here we have made the convention that the pullback maps along the inclusions are denoted $\iota^* = \pi$ and $\iota'^* = \pi'$.

Finally, to this span, we apply another functor, namely $[-, \mathbf{Vect}]$. This is contravariant for the same reason as $[-, G]$, and thus we again have a cospan:

$$\begin{array}{ccccc} & & \big[[\Pi_1(S), G], \mathbf{Vect}\big] & & \\ & \overset{\pi^*}{\nearrow} & & \overset{\pi'^*}{\nwarrow} & \\ \big[[\Pi_1(B), G], \mathbf{Vect}\big] & & & & \big[[\Pi_1(B'), G], \mathbf{Vect}\big] \end{array} \tag{7.17}$$

We now recall that the pullbacks $\pi^*$ and $\pi'^*$ have adjoints: this is a direct consequence of Theorem 6. This reveals how to transport a **Vect**-presheaf on $[\Pi_1(B), G]$ along this cospan. In fact, it gives two 2-linear maps, which are adjoint. Having written the cobordism as a morphism from $B$ to $B'$, we find a corresponding 2-linear map, though we observe that the adjoint is equally well defined. We first do a pullback along $\pi$, giving a **Vect**-presheaf on $S$. Then we use the adjoint map $\pi'_*$. So we have the following:

**Definition 21** *For any cobordism $S : B \to B'$ between compact manifolds, and finite group $G$, define $Z_G(S)$ to be the 2-linear map:*

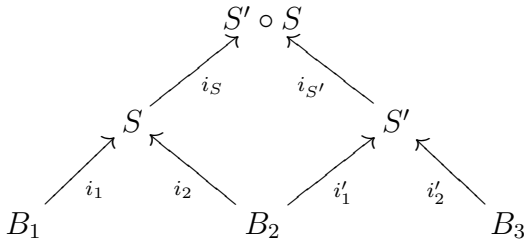
$$(\pi')_* \circ \pi^* : Z_G(B) \longrightarrow Z_G(B') \tag{7.18}$$

Here we have used the notation of Definition 20. Note that again by Theorem 6, both of these functors are 2-linear maps, so the composite $\pi'_* \circ \pi^*$ is also a 2-linear map. It remains to show that $Z_G$ preserves horizontal composition of functors *weakly*—that is, up to a natural isomorphism.

**Remark 12** We can think of the pullback-pushforward construction as giving—in the language of quantum field theory—a "sum over histories" for evolving a 2-vector built from the space of connections. Each 2-vector in $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$ picks out a vector space for each $G$-connection on $B$. The 2-linear map we have described tells us how to evolve this 2-vector along a cobordism (i.e. a change of spatial topology). First we consider the pullback to $\big[[\Pi_1(S), G], \mathbf{Vect}\big]$, which gives us a 2-vector consisting of all assignments of vector spaces to connections on $S$ which restrict to the chosen one on $B$. Each of these could be considered a "history" of the 2-vector along the cobordism. We then "push forward" this assignment to $B'$, which involves a colimit. This is more general than a sum, though so one could describe this as a "colimit of histories". It takes into account the symmetries between individual "histories" (i.e. connections on the cobordism, which are related by gauge transformations).

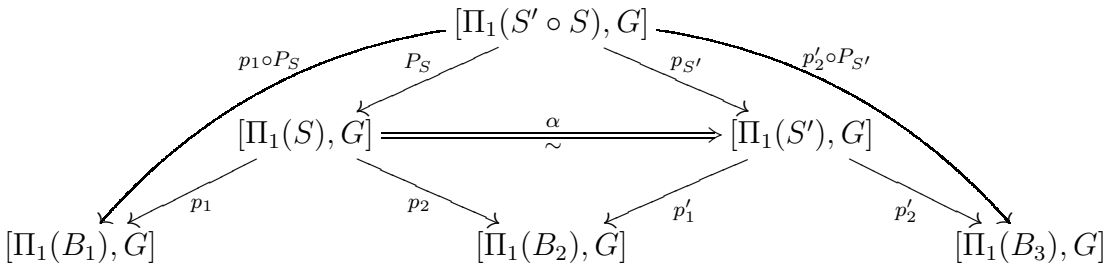
It still needs to be seen that this operation is compatible with composition of cobordisms. Now, a composite of two cobordisms is a special case of a composite of cospans. This is a composition in a bicategory cobordisms—either the horizontal or vertical bicategory in the Verity double bicategory defined in Chapter 5. It is

given by a pushout as described in Definition 4:




(7.19)

When we take the groupoid of connections, however, the corresponding diagram of spans between groupoids of connections weakly mod gauge transformations contains a weak pullback square. This is since the objects are now groupoids, it makes sense to speak of two connections being gauge equivalent, whereas the manifolds in cobordisms are sets, where elements can only be equal or unequal. So for connections on $S$ and $S'$, it is possible for the restrictions to the same set $B_2$ to be isomorphic, rather than merely equal. Thus, we should consider this larger groupoid, the weak pullback, whose objects come with a specified isomorphism between the two restrictions:




(7.20)

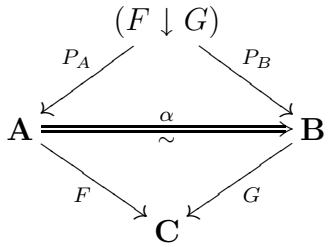
That this is a weak pullback square of functors between groupoids means that this diagram commutes up to the natural isomorphism $\alpha : p_2 \circ P_S \longrightarrow p_1' \circ P_{S'}$. In the case of groupoids, a weak pullback can be seen as an example of a *comma category* (the concept, though not the name, introduced by Lawvere in his doctoral thesis [62]). We briefly discuss this next before stating the theorem regarding composition.

**Remark 13** In general, suppose we have a diagram of categories $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$. Then an object in the comma category $(F \downarrow G)$ consists of a triple $(a, f, b)$, where $a \in \mathbf{A}$ and $b \in \mathbf{B}$ are objects, and $f : F(a) \to G(b)$ is a morphism in $\mathbf{C}$. A morphism in $(F \downarrow G)$ consists of a pair of morphisms $(h, k) \in \mathbf{A} \times \mathbf{B}$ making the square

$$\begin{array}{ccc} F(a_1) & \xrightarrow{f_1} & G(b_2) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(k)} \\ F(a_2) & \xrightarrow[f_2]{} & G(b_2) \end{array} \tag{7.21}$$

commute. Note that in a weak pullback, the morphisms $f$ would be required to be an *isomorphism*, but when we are talking about a weak pullback of groupoids, these conditions are the same.

The comma category has projection functors which complete the (weak) pullback square for the two projections:




(7.22)

such that $(F \downarrow G)$ is a universal object (in **Cat**) with maps into **A** and **B** making the resulting square commute up to a natural isomorphism $\alpha$. This satisfies the universal condition that, given any other category **D** with maps to **A** and **B**, there's an equivalence between $[\mathbf{D}, \mathbf{C}]$ and the comma category $(P_A^*, P_B^*)$ (where $P_S*$ and $P_T*$ are the functors from **D** to **B** which factor through $P_S$ and $P_T$ respectivery). This equivalence arises in a natural way. This is the weak form of the universal property of a pullback.

So suppose we restrict to the case of a weak pullback of groupoids. This is equivalent to the situation where **A**, **B** and **C** are skeletal - that is, each is just a disjoint union of groups. Then the set of objects of $(F \downarrow G)$ is a disjoint union over all the morphisms of **C** (which are all of the form $g : x \to x$ for some object $x$) of all the pairs of objects $a \in \mathbf{A}$ and $b \in \mathbf{B}$ with $g : F(a) \to G(b)$. In particular, since we assume **C** is skeletal, this means $F(a) = G(b)$, though there will be an

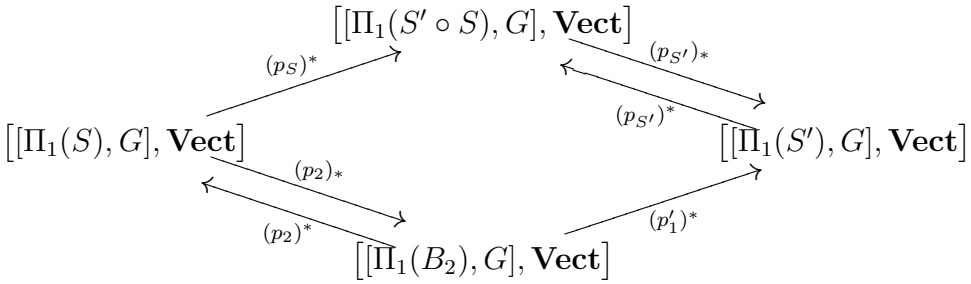
instance of this pair in $(F \downarrow G)$ for each $g$ in the group of morphisms on this object $F(a) = G(b)$.

So as the set of objects in $(F \downarrow G)$ we have a disjoint union of products of sets—for each $c \in \mathbf{C}$, we get $|\hom(c,c)|$ copies of $F^{-1}(c) \times G^{-1}(c)$. The set of morphisms is just the collection of commuting squares as in (7.21) above.

Note that if we choose a particular $c$ and $g : c \to c$, and choose objects $a$, $b$ with $F(a) = c$, $G(b) = c$, and if $H = \mathrm{Aut}(a)$, $K = \mathrm{Aut}(b)$ and $M = \mathrm{Aut}(c)$, then the group of automorphisms of $(a, g, b) \in (F \downarrow G)$ is isomorphic to the fibre product $H \times_M K$. In particular, it is a subgroup of the product group $H \times K$ consisting of only those pairs $(h, k)$ with $F(h)g = gG(k)$, or just $F(h) = gG(k)g^{-1}$. We can call it $H \times_M K$, keeping in mind that this fibre product depends on $g$. Clearly, the group of automorphisms of two isomorphic objects in $(F \downarrow G)$ are isomorphic groups.

In our example, the connections on $S$ and $S'$ need only restrict to gauge-equivalent connections on $B_2$—since two such connections can be "pasted" together using a gauge transformation. Moreover, we note that since all categories involved in our example are groupoids, we have the extra feature that every morphism mentioned must be invertible. This is what makes this a *weak* pullback rather than a *lax* pullback, where $\alpha$ is only a natural transformation.

We are interested in the weak pullback square in the middle of (7.20), since the two 2-linear maps being compared differ only by arrows in this square. The square as given is a weak pullback, with the natural isomorphism $\alpha$ "horizontally" across the square. When considering a corresponding square of categories of **Vect**-presheaves, the arrows are reversed. So, including the adjoints of $p_2^*$ and $p_{S'}^*$, namely $(p_2)_*$ and $(p_{S'})_*$, we have the square:




(7.23)

Note that there are two squares here—one by taking only the "pull" morphisms $(-)^*$ from the indicated adjunctions, and the other by taking only the "push" morphisms $(-)_*$. The first is just the square of pullbacks along morphisms from the weak pullback square of connection groupoids. Comparing these is the core of the following theorem, which gives one of the necessary properties for $Z_G$ to be a weak 2-functor.

**Theorem 10** *The process $Z_G$ weakly preserves composition. In particular, there is a natural isomorphism*

$$\beta_{S',S} : Z_G(S' \circ S) \to Z_G(S') \circ Z_G(S) \tag{7.24}$$

**Proof**: The process $Z_G$ acts by on $S' \circ S$ by taking the spans of groupoids in 7.20, and giving 2-linear maps:

$$(p'_2 \circ P_{S'})_* \circ (p_1 \circ P_S)^* \tag{7.25}$$

On the other hand, $Z_G(S') \circ Z_G(S)$ is found in the same diagram to be

$$(p'_2)_* \circ (p'_1)^* \circ (p_2)_* \circ (p_1)^* \tag{7.26}$$

So we want to show there is a natural isomorphism:

$$\beta_{S',S} : (p'_2 \circ P_{S'})_* \circ (p_1 \circ P_S)^* \to (p'_2)_* \circ (p'_1)^* \circ (p_2)_* \circ (p_1)^* \tag{7.27}$$

It suffices to show that there is an isomorphism between the upper and lower halves of the square in the middle:

$$\gamma : (P_{S'})_* \circ (P_S)^* \to (p'_1)^* \circ (p_2)_* \tag{7.28}$$

since then $\beta_{S',S}$ is obtained by tensoring with identities.

Now, as we saw when discussing comma squares, the objects of the weak pullback $[\Pi_1(S' \circ S), G]$ consist of pairs of connections, $A \in [\Pi_1(S), G]$, and $A' \in [\Pi_1(S'), G]$, together with a morphism in $B_2$, $g : p_2(A) \to {p'}_1(A')$. The morphisms from $(A_1, g_1, {A'}_1)$ to $(A_2, g_2, {A'}_2)$ in the weak pullback are pairs of morphisms, $(h, k) \in [\Pi_1(S), G] \times [\Pi_1(S'), G]$, making the square

$$\begin{array}{ccc} p_2(A_1) & \xrightarrow{g_1} & {p'}_1(A'_2) \\ {\scriptstyle p_2(h)}\downarrow & & \downarrow{\scriptstyle {p'}_1(k)} \\ p_2(A_2) & \xrightarrow[g_2]{} & {p'}_1(A'_2) \end{array} \tag{7.29}$$

commute.

We may assume that the groupoids we begin with are skeletal—so the objects consist of gauge equivalence classes of connections. Then recall from Remark 13 that in this weak pullback the set of objects in $[\Pi_1(S' \circ S), G]$ is a disjoint union of products of sets - for each $c \in [\Pi_1(B_2), G]$, we get $|\hom(c, c)|$ copies of ${p_2}^{-1}(c) \times {p'_1}^{-1}(c)$.

So first taking a **Vect**-presheaf $F$ on $[\Pi_1(S), G]$, we get that $(P_S)^*F$ is a **Vect**-presheaf on $[\Pi_1(S' \circ S), G]$. Now over any fixed object (connection) $A$, we have a set of objects in $[\Pi_1(S' \circ S), G]$ which restrict to it: there is one for each choice $(g, A')$ which is compatible with $A$ in the sense that $(A, g, A')$ is an object in the weak pullback - that is, $g : p_2(A) \to {p'}_1(A')$. Each object of this form is assigned the vector space $F(A)$ by $(P_S)^*F$.

Further, there are isomorphisms between such objects, namely pairs $(h, k)$ as above. There are thus no isomorphisms except between objects $(A, g_1, A')$ and $(A, g_2, A')$ for some fixed $A$ and $A'$. For any such fixed $A$ and $A'$, objects corresponding to $g_1$ and $g_2$ are isomorphic if

$$g_2 p_2(h) = {p'}_1(k) g_1 \tag{7.30}$$

. Denote the isomorphism class of any $g$ by $[g]$.

Then if $G_A$ is the group of automorphisms of any gauge equivalence class of connections $A$, and for notational convenience $M$ is here the group of automor-

phisms of $p_2(A)$ in $[\Pi_1(B_2), G]$ (note that this $M$ depends on $A$, which we are considering fixed for now), we get:

$$(P_{S'})_* \circ (P_S)^* F(A') = \bigoplus_A \Big( \bigoplus_{[g]:p_2(A)\to {p'}_1(A')} \mathbb{C}[G_{A'}] \otimes_{\mathbb{C}[G_A \times_M G_{A'}]} F(A) \Big) \quad (7.31)$$

since $G_A \times_M G_{A'}$ is the automorphism group of the object in $[\Pi_1(S' \circ S), G]$ which restricts to $A$ and $A'$ by gluing along $g$. The outside direct sum here is written over all connections $A$ on $S$, but note that the only ones which contribute any factor are those for which $g : p_2(A) \to {p'}_1(A')$ for some $g$. The inside direct sum is over all isomorphism classes of elements $g$ for which this occurs: in the colimit, vector spaces over objects with isomorphisms between them are identified.

Note that in the direct sum over $[g]$, there is a tensor product term for each class $[g] : p_2(A) \to {p'}_1(A')$. By the definition of the tensor product over an algebra, we can pass elements of $\mathbb{C}[G_A \times_M G_{A'}]$ through the tensor product. These are generated by pairs $(h, k) \in G_A \times G_{A'}$ where the images of $h$ and $k$ are conjugate by $g$ so that $p_2(h)g = g{p'}_1(k)$. These are just automorphisms of $g$: so this says we are considering objects only up to these isomorphisms.

This is the result of the "pull-push" side of the square applied to $F$. Now consider the "push-pull" side: $(p'_1)^* \circ (p_2)_*$.

First, pushing down to $B_2$, we get, on any connection $A''$ on $B_2$ (whose automorphism group is $M$):

$$(p_2)_* F(A'') = \bigoplus_{p_2(A)=A''} \mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A) \tag{7.32}$$

Then, pulling this back up to $S'$, we get (with $M$ again the symmetry group of $p_2(A)$) that:

$$(p_1')^* \circ (p_2)_* F(A') = \bigoplus_{g: p_2(A) \to {p'}_1(A')} \Big( \mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A) \Big) \tag{7.33}$$

Now we define a natural isomorphism

$$\gamma_{S,S'} : (P_{S'})_* \circ (P_S)^* \to (p_1')^* \circ (p_2)_* \tag{7.34}$$

as follows. For each $A'$, this must be an isomorphism between the above vector spaces. The first step is to observe that there is a 1-1 correspondence *between* the terms of the first direct sums, and then secondly to note that the corresponding terms are isomorphic.

Since the outside direct sums are over all connections $A$ on $S$ for which $p_2(A) = {p'}_1(A')$, it suffices to get an isomorphism between each term. That is, between

$$\bigoplus_{[g]: p_2(A) \to {p'}_1(A')} \mathbb{C}[G_{A'}] \otimes_{\mathbb{C}[G_A \times_M G_{A'}]} F(A) \tag{7.35}$$

and

$$\mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A) \tag{7.36}$$

In order to define this isomorphism, first note that both of these vector spaces are in fact $\mathbb{C}[G_{A'}]$-modules. An element of $G_{A'}$ acts on (7.35) in each component by the standard group algebra multiplication, giving an action of $\mathbb{C}[G_{A'}]$ by extending linearly. An element $g \in G_{A'}$ acts on (7.36) by the action of ${p'}_1(g)$ on $\mathbb{C}[M]$. Two $g \in [g]$ have the same action on this tensor product, since they differ precisely by $(h, k) \in G_A \times G_{A'}$, so that $g_2 p_2(h) = {p'}_1(k) g_1$.

Also, we notice that, in (7.35), for each $g \in M$, the corresponding term of the form $\mathbb{C}[G_{A'}] \otimes_{\mathbb{C}[G_A \times_M G_{A'}]} F(A)$ is generated by elements of the form $k \otimes v$, for $k \in \mathbb{C}[G_{A'}]$. and $v \in F(A)$. These are subject to the relations that, for any $(h, k_1) \in \mathbb{C}[G_A] \times \mathbb{C}[G_{A'}]$ such that $p_2(h) = g^{-1} {p'}_1(k_1) g$:

$$kk_1 \otimes v = k(h, k_1) \otimes v = k \otimes (h, k_1)v = k \otimes hv \tag{7.37}$$

since elements of $\mathbb{C}[G_A] \times \mathbb{C}[G_{A'}]$ act on $F(A)$ and $\mathbb{C}[G_{A'}]$ by their projections into the first and second components respectively.

Now, we define the map $\gamma_{A,A'}$. First, for any element of the form $k \otimes v \in \mathbb{C}[G_{A'}] \otimes_{\mathbb{C}[G_A \times_M G_{A'}]} F(A)$ in the $g$ component of the direct sum (7.35):

$$\gamma_{A,A'}(k \otimes v) = {p'}_1(k) g^{-1} \otimes v \tag{7.38}$$

which we claim is in $\mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A)$. This map extends linearly to the whole space.

To check this is well-defined, suppose we have two representatives $k_1 \otimes v_1$ and $k_2 \otimes v_2$ of the class $k \otimes v$. So these differ by an element of $\mathbb{C}[G_A \times_M G_{A'}]$, say $(h, k)$, so that

$$k_1 = k_2 k \tag{7.39}$$

, and

$$h v_1 = v_2 \tag{7.40}$$

where

$$p_2(h) = g {p'}_1(k) g^{-1} \tag{7.41}$$

But then

$$\begin{aligned} \gamma_{A,A'}(k_1 \otimes v_1) &= {p'}_1(k_1) g^{-1} \otimes v_1 \\ &= {p'}_1(k_2 k) g^{-1} \otimes v_1 \\ &= {p'}_1(k_2) g^{-1} g {p'}_1(k) g^{-1} \otimes v_1 \\ &= {p'}_1(k_2) g^{-1} p_2(h) \otimes v_1 \\ &= {p'}_1(k_2) g^{-1} \otimes h v_1 \end{aligned} \tag{7.42}$$

while on the other hand,

$$\begin{aligned} \gamma_{A,A'}(k_2 \otimes v_2) &= {p'}_1(k_2) g^{-1} \otimes v_2 \\ &= {p'}_1(k_2) g^{-1} \otimes h v_1 \end{aligned} \tag{7.43}$$

But these are representatives of the same class in $\mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A)$, so $\gamma$ is well defined on generators, and thus extends linearly to give a well-defined function on the whole space.

Now, to see that $\gamma$ is invertible, note that given an element $m \otimes v \in \mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A)$ (where we are fixing $A$, since both 2-vectors decompose into components corresponding to connections $A$), we can define

$$\gamma^{-1}(m \otimes v) = 1 \otimes v \in \bigoplus_{[g]:p_2(A) \to {p'}_1(A')} \mathbb{C}[G_{A'}] \otimes_{\mathbb{C}[G_A \times_M G_{A'}]} F(A) \tag{7.44}$$

in the component coming from the isomorphism class of $g = m^{-1}$ (we will denote this by $(1 \otimes v)_{m^{-1}}$ to make this explicit, and in general an element in the class of $g$ will be denoted with subscript $g$ whenever we need to refer to $g$).

Now we check that this is well-defined. Given $m_1 \otimes v_1$ and $m_2 \otimes v_2$ representing the same element $m \otimes v$ of $\mathbb{C}[M] \otimes_{\mathbb{C}[G_A]} F(A)$, we must have $h_1 \in G_A$ with

$$m_1 p_2(h_1) = m_2 \tag{7.45}$$

and

$$h_1 v_2 = v_1 \tag{7.46}$$

But then applying $\gamma^{-1}$, we get:

$$\gamma^{-1}(m_1 \otimes v_1) = (1 \otimes v_1)_{m_1^{-1}} = (1 \otimes h_1 v_2)_{m_1^{-1}} \tag{7.47}$$

and

$$\gamma^{-1}(m_2 \otimes v_2) = (1 \otimes v_2)_{m_2^{-1}} = (1 \otimes v_2)_{p_2(h_1)^{-1} m_1^{-1}} \tag{7.48}$$

but these are in the same component, since $g \sim g'$ when $g'p'_1(k) = p_2(h)g$ for some $h \in G_A$ and $k \in G_{A'}$. But then, taking $k = 1$ and $h = h_1^{-1}$, we get that $m_1^{-1} \sim m_2^{-1}$, and hence the component of $\gamma(m \otimes v)$ is well defined.

But then, consider $m \otimes v = \gamma((k \otimes v)_g) = p'_1(k)g^{-1} \otimes v$. Applying $\gamma^{-1}$ we get:

$$\gamma^{-1} \circ \gamma(k \otimes v)_g = (1 \otimes v)_{gp'_1(k)^{-1}} \tag{7.49}$$

so we hope that these determine the same element. But in fact, notice that the morphism in the weak pullback which gives that $g^{-1}$ and $p'_1(k)g^{-1}$ are isomorphic is just labelled by $(h, k) = (1, k)$, which indeed takes $k$ to 1 and leaves $v$ intact. So these are the corresponding elements under this isomorphism.

So $\gamma$ is invertible, hence an isomorphism. Thus we define

$$\beta_{S,S'} = 1 \otimes \gamma \otimes 1 \tag{7.50}$$

This is the isomorphism we wanted. □

**Remark 14** The weak pullback square gave a natural isomorphism:

$$\alpha^* : P_{S'}^* \circ (p'_1)^* \to P_S^* \circ p_2^* \tag{7.51}$$

Given a connection on a composite of cobordisms $S' \circ S$, $\alpha$ gives the gauge transformation of the restriction, on their common boundary $B_2$, needed so the gluing of connections on $S$ and $S'$ is compatible.

We proved that the other square—the "mate" under the adjunctions, also has a natural isomorphism ("vertically" across the square), namely that there exists:

$$\beta_{S,S'} : (P_{S'})_* \circ (P_S)^* \to (p'_1)^* \circ (p_2)_* \tag{7.52}$$

In fact, these are related by the units for both pairs of adjoint functors:

$$\eta_{S'} : 1_{Z_G(S' \circ S)} \to (P_{S'})_* \circ (P_{S'})^* \tag{7.53}$$

and

$$\eta_2 : 1_{Z_G(S)} \to (p_2)_* \circ (p_2)^* \tag{7.54}$$

So the desired "vertical" natural transformation across the square 7.23 is determined by the condition that it complete the following square of natural transformations to make it commute:

$$\begin{array}{ccc} (P_{S'})_* \circ (P_S)^* & \xRightarrow{1 \otimes \eta_2} & (P_{S'})_* \circ (P_S)^* \circ p_2^* \circ (p_2)_* \\ \| \\ \| \beta_{S,S'} \\ \Downarrow & & \Big\Downarrow 1 \otimes (\alpha^*)^{-1} \otimes 1 \\ (p'_1)^* \circ (p_2)_* & \xRightarrow{1 \otimes \eta_{S'}} & (P_{S'})_* \circ (P_{S'})^* \circ (p'_1)^* \circ (p_2)_* \end{array} \tag{7.55}$$

The crucial element of this is the fact that the (weak) pullback square for the groupoids of connections in the middle of the composition diagram gives rise to a square of **Vect**-presheaf categories. To get this we used that the adjunction between the pullback and pushforward along the $\pi$ maps had unit and counit 2-morphisms which turn a natural transformation vertically across the first square to one horizontally across the second. Note, however, that we do not expect

this to be invertible. When it is, the square is said to satisfy the *Beck-Chevalley* (BC) condition. This is discussed by Bénabou and Streicher [17], MacLane and Moerdijk [67], and by Dawson, Paré and Pronk [29].

**Remark 15** It is useful to consider a description of the two functors between which we have found this natural isomorphism $\beta_{S,S'}$—namely, the two 2-linear maps across the central square in (7.20). See Remark 7 for the general case. In this situation, these behave as follows:

First, the "push-pull": given a functor $f : [\Pi_1(S), G] \to \mathbf{Vect}$ (i.e. in $Z_G(S)$), in the first stage, push forward to a functor in $Z_G(B_2)$. This gives, for each connection $C$ on $B_2$, a vector space which is the colimit of a diagram of the vector spaces $f(C_i)$ for all connections $C_i$ on $S$ which restrict to $C$ on $B_2$. In the second stage, pull back to $[\Pi_1(S'), G]$: for each connection $C'$ on $S'$, find the connection $C$ it restricts to on $B_2$, and assign $C'$ the vector space obtained for $C$ above. Namely, the colimit of the diagram of vector spaces $f(C_i)$ for connections $C_i$ which also restrict to $C$.

Next, the "pull-push" given a functor $f[\Pi_1(S), G] \to \mathbf{Vect}$, in the first stage, pull back to a functor on $[\Pi_1(ST), G]$. This gives, at each connection $C$ on $S' \circ S$, a vector space which is just $f(C|_S)$, the one assigned to the connection given by $C$ restricted to $S$. At the second stage, push this forward to a functor in $[\Pi_1(S'), G]$. This gives, at each connection $C'$ on $S'$, the colimit of a diagram whose objects

are all the $f(C_i|_S)$ obtained in the first stage, for any $C_i$ which restricts to $C'$ on $T$.

In both cases there is a colimit over a diagram including all possible connections on $S$ which match some specified one on $S'$. This "matching" can occur either by inclusion in a bigger entity (the composite being the minimal cobordism $S' \circ S$ containing both $S$ and $S'$). Or it can occur just by matching along the shared boundary $B_2$. However, since the composition of $S$ and $S'$ is weak, the groupoid of connections on $S' \circ S$ only needs to have inclusions of the groupoids $[\Pi_1(S'), G]$ and $[\Pi_1(S), G]$ which agree on $B_2$ up to gauge equivalence. This gauge equivalence is part of the specification of an object in the weak pullback of the groupoids of connections.

**Remark 16** We can describe more explicitly the effect of $\beta$. Suppose we have a composite of cobordisms, $S' \circ S$. ,By Lemma 3, we have that the functors $(P_{S'})_* \circ (P_S)^*$ and $(p'_1)^* \circ (p_2)_*$ can be written in the form of a matrix of vector spaces as in (6.6). The matrix components for each 2-linear map are given by colimits of diagrams of vector spaces in groupoids of connections on $S'$ matching a specified one on $S$.

However, the criterion for "matching" is different: when we push first, then pull, the connections must match exactly on $B_2$; when pulling first, then pushing, the connections must both be restrictions of one on $S' \circ S$, but are only required

to match up to gauge equivalence $\alpha$ on $B_2$. The isomorphism $\beta_{A,A'}$ is just the isomorphism between the colimits which induced by the permutation of vector spaces associated to these gauge transformations.

Recall that the source, $Z_G(S' \circ S)$, is given by a matrix indexed by gauge equivalence classes of connections $[A_1]$ on $B_1$ and $[A_3]$ on $B_3$. The entries are isomorphic to $\mathbb{C}^n$ where $n$ is the number of classes of connections on $S_2 \circ S_1$ restricting to representatives of $[A_1]$ and $[A_3]$.

On the other hand, this can be seen (by the isomorphisms $\beta$) as a matrix product of $Z_G(S')$ with $Z_G(S)$, which has components given by a direct sum over equivalence classes $[A_2]$ of connections on $B_2$:

$$[Z_G(S_2) \circ Z_G(S_1)]_{[A_1],[A_3]} \overset{\beta_{S,S'}}{\to} \bigoplus_{[A_2]} [Z_G(S)]_{[A_1],[A_2]} \otimes [Z_G(S)]_{[A_2],[A_3]} \tag{7.56}$$

Recall that $[Z_G(S)]_{[A_1],[A_2]} \cong \mathbb{C}^m$, where $m$ is the number of gauge equivalence classes of connections on $S$ which restrict to $[A_1]$ and $[A_2]$. Similarly $[Z_G(S')]_{[A_2],[A_3]} \cong \mathbb{C}^{m'}$, where $m'$ is the number of classes of connections on $S'$ which restrict to $[A_2]$ and $A_3$. Indeed, the components are just the vector spaces whose bases are these equivalence classes.

The isomorphism $\beta$ identifies the composite, whose components count connections on $S' \circ S$, with this product. This consists of identification maps in each component. A component indexed by $[A_1]$ and $[A_3]$ comes from the groupoid of all connections on $S' \circ S$ which restrict to $[A_1]$ and $[A_3]$ on $B_1$ and $B_3$. Each such

connection restricts to connections on $S'$ and $S$ by the maps $\pi_{S'}$ and $\pi_S$. These in turn restrict by $\pi_2$ and $\pi_2'$ to $B_2$ to gauge-equivalent (by $\alpha$) connections - and those restricting to different $[A_2]$ are in different components of $[\Pi_1(S' \circ S), G]$. Over each $[A_2]$, the we have the product groupoid of the groupoids of all connections on $S$ and $S'$ restricting to this $[A_2]$ (and to $[A_1]$ and $[A_3]$ respectively). So the groupoid of such connections on $S' \circ S$ is isomorphic to a fibred product over $Z_G(B_2)$.

Then the $([A_1], [A_3])$ component of $Z_G(S' \circ S)$ is a vector space whose basis is the set of components of this groupoid, and $\beta$ is an isomorphism which takes which takes the vector spaces over this to those in (7.56).

**Example 9** Consider the "pair of pants" cobordism (the "multiplication" cobordism from the generators of **2Cob**): This can be seen as a morphism $S : B \to B'$

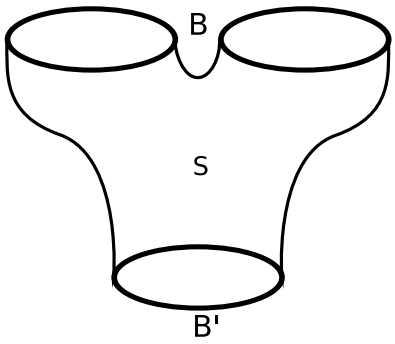



**Figure 7.1:** The "Pair of Pants"

in **2Cob**, where $B = S^1 \cup S^1$ and $B' = S^1$. The 2-linear map corresponding to

it can be found by the above procedure. To begin with, recall the 2-vector space on $S^1$ found in Example 7. It is equivalent to $[G/\!\!/G, \mathbf{Vect}]$, the 2-vector space of **Vect**-presheaves on $G$ which are equivariant under conjugation by elements of $G$.

The groupoid of connections on $S^1 \cup S^1$ can be found using the fact that the path groupoid is just $\Pi_1(S^1) \cup \Pi_1(S^1)$, a disjoint union of two copies of the groupoid $\Pi_1(S^1) \cong \mathbb{Z}$. Notice that this is different from the group $\mathbb{Z}^2$, since a group is a one-object groupoid, whereas here we have a two-object groupoid, each object having a group of morphisms isomorphic to $\mathbb{Z}$. A functor from this into $G$ amounts to two choices $g, g' \in G$, but a gauge transformation amounts to a conjugation by some $h \in G$ at *each* of the two objects (one chosen base points in each component), so:

$$\begin{aligned} [\Pi_1(S^1 \cup S^1), G] &\cong (G \times G)/\!\!/(G \times G) \\ &\cong (G/\!\!/G)^2 \end{aligned} \tag{7.57}$$

where $G \times G$ acts on itself by conjugation componentwise. This just says that a connection on the space consisting of two circles is the same as a choice of connection on each one separately. This is illustrated in Figure 7.2, where we show the pants as a disc with two holes, and label a connection on $S$ with its restrictions to the boundary. The connection on $S$ has holonomies $g$ and $g'$ around the two holes. On $S^1 \cup S^1$, this restricts to a connection with holonomies $g$ and

$g'$ respectively, and on $S^1$ to the product (since the circle around the outside $S^1$ is homotopic to the composite of the two loops).

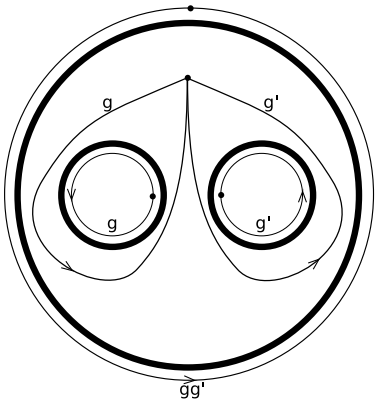



**Figure 7.2:** Connection for Pants

On the other hand, the manifold with boundary, $S$, is homeomorphic to a two-punctured disc, whose path groupoid has a skeleton with one-object, and group of morphisms $\pi_1(S) = F(\gamma_1, \gamma_2)$, the free group on two generators. Functors from this into $G$ amount to homomorphisms $(g, g') : F(\gamma_1, \gamma_2) \to G$. That is, a choice of two elements of $G$ (the images of the generators). A gauge transformation amounts to conjugation at the single object (a chosen base point in $S$—indicated in Figure 7.2 as a dot on the loop). So we have the span of connection groupoids:

$$[\Pi_1(S), G] \cong (G \times G)/\!\!/G \tag{7.58}$$

where $G$ acts on $G \times G$ by conjugation in both components at once. Then the span (7.16):

$$\begin{array}{ccc} & (G \times G)/\!\!/G & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ (G/\!\!/G)^2 & & G/\!\!/G \end{array} \tag{7.59}$$

Both projections are restrictions of a connection on $S$ to the corresponding connection on the components of the boundary. It is easily seen that $p_1$ leaves objects intact and takes the morphism corresponding to conjugation by $h$ to that corresponding to conjugation by $(h, h)$. The projection $p_2$ maps object $(g, g')$ to $gg'$, and the morphism for conjugation by $h$ to, again, conjugation by $h$.

The gauge-equivalent connections on $S$ have holonomies of the form $(hgh^{-1}, hg'h^{-1})$ for any $h \in G$, and those for $S^1$ are compatible, since they have holonomies of the form $hgg'h^{-1}$ for $h \in G$. Those for $S^1 \cup S^1$ can be any connection with holonomies $(hgh^{-1}, h'g'(h')^{-1})$ for any choices of $(h, h') \in G^2$, so that connections which are gauge equivalent on $S^1 \cup S^1$ may be restrictions of inequivalent connections on $S$.

Finally, suppose we have a functor $f : [\Pi_1(S^1 \cup S^1), G] \to \mathbf{Vect}$, and transport it to $(p_2)_* \circ p_1^*(f) : [\Pi_1(S^1), G] \to \mathbf{Vect}$. To see what this does, note that since $Z_G(S^1 \cup S^1$ that any such $f$ can be written as a sum of irreducible functors (since $Z_G(S^1 \cup S^1)$ is a KV 2-vector space). So we can consider one of these, say $f$, which assigns a copy of $\mathbb{C}$ to each connection in some gauge-equivalence class, say $([g], [g'])$, and 0 to all others. This $f$ assigns an isomorphism, compatibly, to each

gauge transformation (i.e. pair of elements $(h, h')$). Such an isomorphism amounts to multiplication by a complex number—so we get a representation $\rho : G \times G \to \mathbb{C}$.

Now pull $f$ back to $p_1^*(f) : [\Pi_1(S), G] \to \mathbf{Vect}$, a functor $f(p_1(A))$. This assigns a copy of $\mathbb{C}$ to any connection on $S$ which restricts to any representative of $([g], [g'])$—note that these are not all equivalent. To any gauge transformation given by conjugation by $h$, it assigns the isomorphism $\rho(h, h)$. So we get the representation $\rho \circ \Delta : G \to \mathbb{C}$ for each equivalence class (where $\Delta : G \to G \times G$ is the diagonal map).

Then push $p_1^*(f)$ forward to $(p_2)_* \circ p_1^*(f) : [\Pi_1(S^1), G] \to \mathbf{Vect}$. To each connection on $S^1$ (represented by $g_1 \in G$) the colimit over the diagram of all connections restricting to $g_1$. That is, over all $(g, g')$ such that $gg' = g_1$. So then we get a copy of $\mathbb{C}$ for each pairs of representatives of $[g]$ and $[g']$ which give $g_1$ as a product: note that there may be more than one such, which are not gauge equivalent in $[\Pi_1(S), G]$. The diagram of all these amounts (by taking its skeleton) to just a disjoint union of gauge-equivalence classes in $[\Pi_1(S), G]$.

For each class (since all copies of $\mathbb{C}$ over it are equipped with compatible isomorphisms) we just get one copy of $\mathbb{C}$. The group $G$ thought of as the group of gauge transformations acts on each copy of $\mathbb{C}$. If it acts nontrivially, then in the colimit, at least two points in that $\mathbb{C}$ will be identified (since the isomorphisms

given by the $G$-action must agree with the maps into the colimit). If this happens, that copy of $\mathbb{C}$ collapses to zero.

So finally we have that

$$(\pi_2)_* \circ \pi_1^*(g_1) \cong \bigoplus_{(g,g') \in ([g],[g']) | gg' = g_1} \mathbb{C}[\mathrm{Aut}(g_1)] \otimes_{\mathbb{C}[\mathrm{Aut}(g,g')]} \mathbb{C} \tag{7.60}$$

where the direct sum is over all non-equivalent $(g, g')$ representing $([g], [g'])$ and satisfying $gg' = g_1$, and the action of $G$ on each component is as we have described. On morphisms, we get the direct sum of the isomorphisms between these copies of $\mathbb{C}$.

We can describe this as a categorified "convolution of class functions" on $G$. This is related to Example 6, the group 2-algebra on a group. Note that this is almost the 2-vector space of **Vect**-presheaves on the groupoid of connections on $S^1$ - except that here only "equivariant" functors, where there are isomorphisms between spaces over conjugate elements of $G$, are considered. For such functors, the "pants" morphism amounts to multiplication in the group 2-algebra.

An important special case of a higher cobordism for our extended TQFT is the one where the objects in $\mathbf{nCob}_2$ are empty manifolds $\emptyset$. Then cobordisms between these are themselves manifolds *without* boundary, and cobordisms between these have boundary, but no nontrivial corners. So we have just a cobordism from one manifold to another. It is reasonable to expect that in this case, the extended

TQFT based on a group $G$ should give results equivalent to those obtained from a TQFT based on the same group, suitably reinterpreted.

**Example 10** Consider a manifold $S$, thought of as a cobordism $S : \emptyset \to \emptyset$. We expect that finding our $Z_G(S)$ for such a cobordism should be like finding the vector space assigned to the manifold $S$ by an ordinary TQFT.

To see this, first note that $\Pi_1(\emptyset) = \emptyset$, the empty category, and since this is the initial category, there is a single functor from it to $G$, hence $[\Pi_1(\emptyset), G] = \mathbf{1}$, the category with one object and one morphism. Thus, $Z(\emptyset) \cong \mathbf{Vect}$

Now, since every connection on $S$ "restricts" to the unique trivial connection on $\emptyset$, the 2-linear map takes **Vect** to **Vect**, and can be represented as a $1 \times 1$ matrix of vector spaces. In other words, the operators both involve tensoring with a single vector space.

Too see which vector spaces this is, begin with a 2-vector in $Z(\emptyset) \cong \mathbf{Vect}$. This amounts to a choice of a vector space, say $V \in \mathbf{Vect}$. Pulling back to $S$, we simply get the functor assigning a copy of $V$ to every object of the groupoid $[\Pi_1(S), G]$. Isomorphisms from $V$ to $V$ must be assigned to every arrow in this groupoid. But there is a unique isomorphism is $[\Pi_1(\emptyset), G]$, namely the identity—so the pullback to $S$ must assign the identity to every arrow.

So in fact, taking the pushforward gives a colimit of a diagram which has a single copy of $V$ for each isomorphism class in $[\Pi_1(S), G]$, which decomposes as a

direct sum of these classes. This is since the colimit for just one class is just $V$, and for the whole groupoid is the direct sum of one copy of $V$ from each isomorphism class.

So we have that:

$$(\pi_2)_* \circ \pi_1^*(-) \cong (- \otimes \mathbb{C}^k) \tag{7.61}$$

Where $k = |\underline{[\Pi_1(S), G]}|$ is the number of connected components of $[\Pi_1(S), G]$.

If we reinterpret this as assigning $\mathbb{C}^k$ to $S$, thought of as a manifold, this does indeed recover the usual formula obtained from a TQFT. The TQFT based on the finite group $G$ will assign to a manifold the Hilbert space of complex-valued functions on the space of connections (strictly) modulo gauge transformations. This is equivalent to what we have just found.

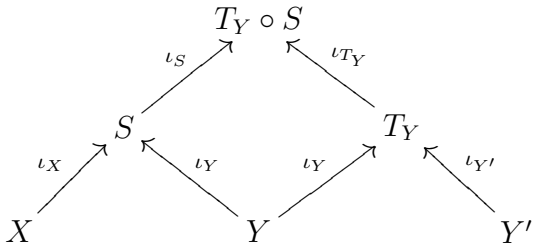
The final element of our weak 2-functor is its effect on 2-morphisms, so this is the subject of the next section.

## 7.3 $Z_G$ on Cobordisms of Cobordisms

Now we consider the situation of a cobordism between cobordisms. We want to describe our extended TQFT as a weak 2-functor, so we want a bicategory derived from our double bicategory $\mathbf{nCob}_2$. By Theorem 1, this is possible, but we need to see just what a 2-morphism in this corresponding bicategory looks like.

Recall that the source and target morphisms of the corresponding 2-morphism are those obtained by composing horizontal and vertical morphisms which form the edges of the square.

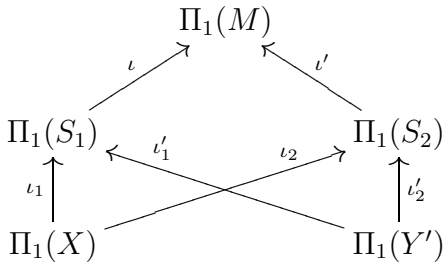
Given a square in $\mathbf{nCob}_2$, we have a diagram of the form (4.21). When we turn this into a 2-cell, the source morphism will be a cospan in the category of manifolds with corners. It is found by taking the following pushout:




(7.62)

The pushout square is the central square here, where we get the object $T_Y \circ S$ equipped with injections $\iota_S$ and $\iota_{T_Y}$ which make the square commute, and which is universal in the sense that any other object with injections from $S$ and $T_Y$ factors through $T_Y \circ S$. So in particular, the maps into $M$ can be factorized as the maps into $T_Y \circ S$ and the canonical injection $\iota : T_Y \circ S \to M$. A similar argument applies to the target morphism, so the situation we are interested in can be represented

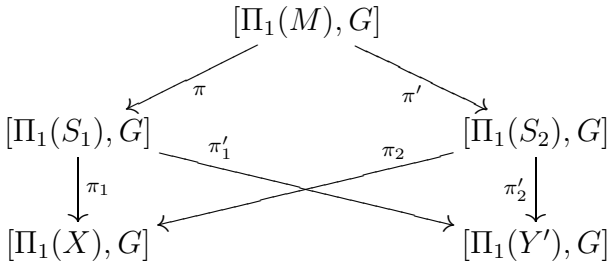
as a cospan of cospans in the following way:




(7.63)

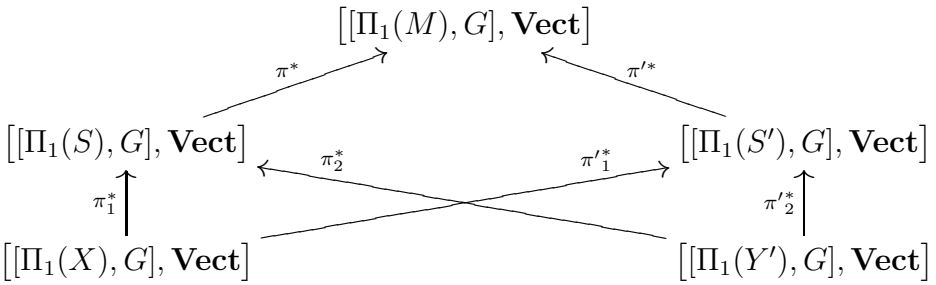
with $S_1 = T_Y \circ S$ and $S_2 = S' \circ T'_X$.

Given this situation, which is a 2-morphism for the bicategory of cobordisms, we want to get a 2-morphism in the bicategory **2Vect**. That is to say, a natural transformation $\alpha_M$ between a pair of 2-linear maps. The 2-linear maps in question are those we get by the construction (7.17). So in particular,




(7.64)

And finally, quantizing these configuration groupoids by taking functors into **Vect**:




(7.65)

Now, recall that each of the pullback maps appearing here has an adjoint, so we have functors $F_1 = (\pi'_1)_* \circ \pi_1^*$ and $F_2 = (\pi'_2)_* \circ \pi_2^*$ from $Z(X) = \big[[\Pi_1(X), G], \mathbf{Vect}\big]$ to $Z(Y') = \big[[\Pi_1(Y'), G], \mathbf{Vect}\big]$. A natural transformation will take an object $f \in Z_G(X)$ and give a morphism $Z_G(M)(f) : F_1(f) \to F_2(f)$ in $Z(Y')$ satisfying the usual naturality condition. Now, an object in $Z_G(X)$, namely a 2-vector, is a **Vect**-presheaf on the groupoid of $G$-connections on $X$ weakly mod gauge transformations.

The hoped-for morphism $Z_G(M)(f)$ in $Z(Y')$ is just a natural transformation between two such functors $g, g' : [\Pi_1(Y'), G] \to \mathbf{Vect}$. That is, it assigns, for each connection $A \in [\Pi_1(Y), G]$, a linear map between the two vector spaces: $(Z_G(M)(f))(A) : g(A) \to g'(A)$. We want to get $Z_G(M)$ from the cobordism with corners, $M$. This we define by means of a "pull-push" process, similar to the one used to define the 2-linear maps in the first place.

However, as remarked in Section 6.2, any natural transformation between a pair of 2-linear maps between KV 2-vector spaces can be represented as a matrix

of linear operators, as in (6.7). The matrix in question is indexed by gauge equivalence classes of connections on $X$ and on $Y$. Writing $Z(S)$ in the matrix form means that given a pair $([A], [A'])$ of such classes, there is a vector space $Z(S)_{[A],[A']}$. Recall that we found these vector spaces by the "pull-push" process for presheaves along inclusion maps.

A natural transformation between such functors is a matrix of linear maps, so we will have

$$Z_G(M)_{[A],[A']} : Z_G(S)_{[A],[A']} \to Z_G(S')_{[A],[A']} \tag{7.66}$$

But now we can use the fact that the top level of the tower of spans of groupoids in (7.65) is of the same form as that for cobordisms between manifolds given in (7.17). The component linear maps arise by applying a similar "pull-push" process to that used in Section 7.2 to define $Z_G$ on cobordisms.

Since there are canonical bases $[A] \in \mathcal{A}_0(S)$ and $[A'] \in \mathcal{A}_0(S')$ for the vector spaces $Z(S) \cong \mathbb{C}^k$ and $Z(S') \cong \mathbb{C}^{k'}$, so we can represent $T$ as a $k \times k'$ matrix. We then need to describe the effect of $T$ on a vector in $\mathbb{C}^k$. Such a vector amounts to an assignment of a scalar to each gauge equivalence class of connections in $[\Pi_1(S), G]$. In particular, to find the component $T_{[A'],[A]}$ indexed by the class $[A']$ of connections on $S'$, and the class $[A]$ on $S$, take the vector corresponding to the function equal to 1 on $[A]$ and 0 elsewhere.

The linear map $T$ acts by the "pull-push" operation. The first stage—pullback gives a function on $[\Pi_1(M), G]$ which is 1 on any gauge-equivalence class of connections $[B]$ on $M$ restricting to $[A]$ on $S$. Pushing this forward involves taking a sum over all classes of connection restricting to $[A']$ on $S'$. Clearly, the only nonzero contributions are from those connections which restrict to $[A]$ on $S$. The action of $T$ extends linearly to all of $V$, so it is represented by a $k \times k'$ matrix whose entries are indexed by classes of connections.

So indeed, all discussion of the construction of the natural transformation will parallel the construction of the 2-linear maps, but at a lower categorical level, since we get a matrix of scalars rather than vector spaces—this time in each component $([A], [A'])$. The resulting linear map (and its matrix representation) can then be "lifted" to a natural transformation between 2-linear maps.

A more tricky question is what contribution to expect from those which do restrict to $[A]$. Naively, one might expect to simply take a sum of the function values (all equal to 1 at the moment) over all such connections. Since this ignores the morphisms in $[\Pi_1(M), G]$, one might perhaps imagine the sum should be over only equivalence classes of connections. However, one should suspect that this is also incorrect, since when we found a pushforward for **Vect**-presheaves, we took not a direct sum over equivalence classes, but a colimit.

In fact, the correct prescription involves the *groupoid cardinality* of the groupoid of those connections which contribute to the sum. This concept is described by Baez and Dolan [10], and related to Leinster's [63] concept of the Euler characteristic of a category. For a more in-depth discussion of groupoid cardinality, and also of its role (closely related to the role it plays here) in a simple model in quantum mechanics, see the author's paper [72] on the categorified harmonic oscillator.

The cardinality of a groupoid $\mathbf{G}$ is:

$$|\mathbf{G}| = \sum_{[x] \in \underline{\mathbf{G}}} \frac{1}{|\operatorname{Aut}(x)|} \tag{7.67}$$

the sum is over isomorphism classes in $\mathbf{G}$. This quantity is invariant under equivalence of categories, and should be the pushforward of the constant function 1. So we define:

**Definition 22** *Given cobordism between cobordisms, $M : S \to S'$, for $S, S' : B \to B'$, then*

$$Z_G(M) : Z_G(S) \to Z_G(S') \tag{7.68}$$

*is a natural transformation given by a matrix of linear operators:*

$$Z_G(M)_{j,k} : Z_G(S)_{j,k} \to Z_G(S')_{j,k} \tag{7.69}$$

*where the vector space $Z_G(S)_{j,k}$ is the $(j,k)$ component of the matrix for the 2-linear map $Z(S)$. This is indexed by choices $(j,k)$, where $j$ identifies an equivalence class $[A]$ of connections on $B$.*

*The linear map $Z_G(M)_{j,k} = T$ is described by the matrix:*

$$T_{[A],[A']} = |(j \times j')^{-1}(A, A')| \tag{7.70}$$

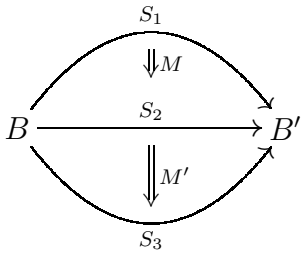
*the groupoid cardinality of the essential preimage of $(A, A')$, where $A$ is a connection on $S$ and $A'$ a connection on $S'$.*

(That is, of the groupoid of all connections on $M$ simultaneously restricting to a connection gauge equivalent to $A$ on $S$ and $A'$ on $S'$.)

Since this is a matrix of linear transformations between the correct vector spaces, it defines a natural transformation. This is the last element of the extended TQFT $Z_G$ which needs to be defined—Theorem 13 will show that its behaviour on manifolds, cobordisms, and cobordisms between cobordisms satisfy the axioms of a weak 2-functor. Two parts of this we prove here separately. The first is strict preservation of vertical composition; the second is preservation of horizontal composition as strictly as possible (i.e. up to the isomorphisms $\beta$ which make comparison possible - as we will see).

**Theorem 11** *The assignment $Z_G(M)$ to cobordisms with corners given by (7.70) preserves vertical composition strictly: $Z_G(M'M) = Z_G(M') \circ Z_G(M)$.*

**Proof**: Vertical composition is just component-wise composition of linear operators. So it suffices to show that given any component, composition is preserved. That is, given a vertical composite of two cobordisms between cobordisms:




(7.71)

we get matrices $Z(S_1)_{(j,k)}$, $Z(S_2)_{(j,k)}$, and $Z(S_3)_{j,k}$, of vector spaces indexed by connections-and-representations on $B$ and $B'$ as in Definition 22.

For the following, fix a component—i.e. a gauge equivalence class of connections $[A]$ on $B$ and representation of $\mathrm{Aut}([A])$, and similarly for $B'$.

Then we have two linear operators. The first is

$$Z_G(M)_{j,k} = T : Z(S_1)_{(j,k)} \to Z(S_2)_{(j,k)} \tag{7.72}$$

and is given as a matrix, indexed by equivalence classes of connections $[A_1]$ on $S_1$ and $[A_2]$ on $S_2$, as follows. The component $T_{[A_1],[A_2]}$ is the groupoid cardinality of the groupoid of all connections on $M$ which are gauge equivalent to ones restricting to both $A_1$ and $A_2$—that is, the *essential preimage* of $(A_1, A_2)$. Denote this by $|\widehat{(A_1, A_2)}|$.

The second operator

$$Z_G(M')_{j,k} = T' : Z(S_2)_{(j,k)} \to Z(S_3)_{(j,k)} \tag{7.73}$$

is likewise a matrix, indexed by equivalence classes of connections $[A_2]$ on $S_2$ and $[A_3]$ on $S_3$, where $T'_{[A_2],[A_3]} = |\widehat{(A_2, A_3)}|$, the groupoid cardinality of the essential preimage of $(A_2, A_3)$ (a groupoid of connections on $M'$).

The product of these is then just given by matrix multplication, so that

$$(T'T)_{[A_1],[A_3]} = \sum_{[A_2]} |\widehat{(A_1, A_2)}| \times |\widehat{(A_2, A_3)}| \tag{7.74}$$

That is, to get the component of the image of a delta functon on $[A_1]$ in the connection $[A_2]$, one takes a sum over equivalence classes of connections $[A_2]$ on $B_2$. The sum is of of the products of the groupoid cardinalities of connections on $M$ and $M'$ restricting to this $[A_2]$.

We need to show this is the same as the linear operator obtained from the same $(j, k)$ component for the 2-morphism $Z_G(M'M)$. But we know that

$$Z_G(M'M)_{(j,k)} = R : Z(S_1)_{(j,k)} \to Z(S_3)_{(j,k)} \tag{7.75}$$

has component

$$|\widehat{(A_1, A_3)}| \tag{7.76}$$

the groupoid cardinality of the essential preimage of $(A_1, A_3)$, which is a groupoid of connections on $M'M$. So we really just need the fact that groupoid cardinalities behaves well with respect to sum and product.

In particular, $\widehat{(A_1, A_3)}$ is a groupoid of connections on $M'M$, but each of these has a restriction to $S_2$, and if two connections on $M'M$ have gauge-inequivalent

restrictions to $S_2$, they must be gauge inequivalent. So $(\widehat{A_1, A_3})$ is a direct sum over the possible gauge-equivalence classes of restrictions $[A_2]$ to $S_2$. Since the groupoid cardinality of a direct sum of groupoids is the sum of their cardinalities, we thus have

$$|(\widehat{A_1, A_2})| = \sum_{[A_2]} |(\widehat{A_1, A_2, A_3})| \tag{7.77}$$

where $(\widehat{A_1, A_2, A_3})$ is the groupoid of connections on $M'M$ which restrict to all the $A_i$ simultaneously.

However, we claim this is just the cartesian product of groupoids. This is since $M'M$ is an equivalence class of manifolds with corners, where a standard representative for $M'M$ is a representative for $M'$ and for $M$, identified at the images of the common inclusions of $S_2$. By a generalization of the Meyer-Vietoris theorem (see, for instance, Brown [18]) we have $\Pi_1(M'M)$ likewise is a disjoint union of $\Pi_1(M')$ and $\Pi_1(M)$, modulo the equivalence of the images of $\Pi_1(S_2)$. But then, taking functors into $G$, we have $[\Pi_1(M'M), G]$ is a subgroupoid of the product $[\Pi_1(M'), G] \times [\Pi_1(M), G]$, containing only the objects (connections) such that the connections in the two components agree on $S_2$. Since we have fixed a particular connection $A_2$ on $S_2$, we just get the cartesian product of groupoids of connections on $M'$ and $M$ respectively which restrict to $A_2$.

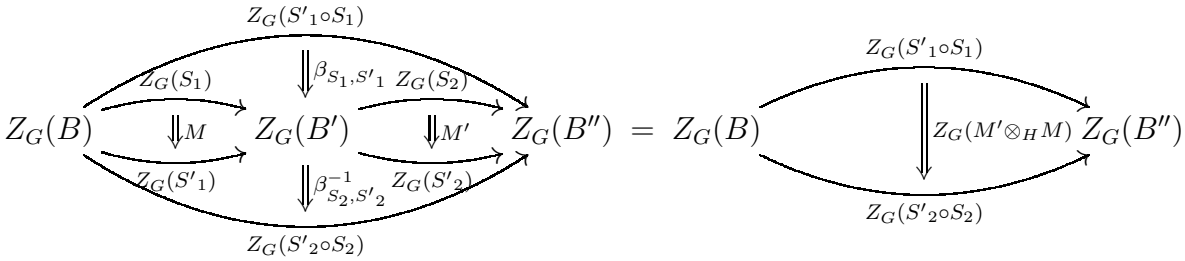
Now, since the groupoid cardinality of a cartesian product of groupoids is the product of their groupoid cardinalities, we have

$$R_{[A_1],[A_3]} = (T'T)_{[A_1],[A_3]} \tag{7.78}$$

so $Z_G$ preserves vertical composition of 2-morphisms strictly. □

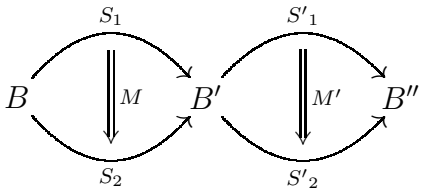
A similar result holds for vertical composition.

**Theorem 12** *The assignment $Z_G(M)$ to cobordisms with corners given by (7.70) preserves horizontal composition strictly, up to the isomorphism weakly preserving composition of the source and target morphisms:*




(7.79)

**Proof**: The horizontal composition involves "matrix multiplication" at the level of composition of 2-linear maps. Given a horizontal composite




(7.80)

the functor $Z_G$ assigns 2-linear maps to the cobordisms $S_1$, $S_2$, $S'_1$, and $S'_2$, and natural transformations to $M$ and $M'$. Then the horizontal composite is a natural

transformation

$$Z_G(M' \otimes_H M) : Z_G(S'_1) \circ Z_G(S_1) \to Z_G(S'_2) \circ Z_G(S_2) \tag{7.81}$$

As discussed in Remark 16, the isomorphisms $\beta$ allow comparison of the horizontal composite of natural transformations $Z_G(M') \otimes Z_G(M)$ with the natural transformation $Z_G(M' \otimes_H M)$. The presence of the $\beta$ isomorphisms only allows us to disregard the distinction between $Z_G(S_2 \circ S_1)$ and $Z_G(S_2) \circ Z_G(S_1)$ (and likewise for the $S'$).

So first consider $Z_G(M') \otimes Z_G(M)$, the horizontal composite of the natural transformations in **2Vect** corresponding to the cobordisms with corners. Each of these natural transformations can be represented as a matrix of linear maps:

$$Z_G(M)_{[A_1],[A_2]} : V_{[A_1],[A_2]} \to W_{[A_1],[A_2]} \tag{7.82}$$

where the $V$'s are the coefficients of $Z_G(S_1)$ and $W$'s are those of $Z_G(S_2)$. The coefficients of $Z_G(M')$ are similarly

$$Z_G(M')_{[A_2],[A_3]} : {V'}_{[A_2],[A_3]} \to {W'}_{[A_2],[A_3]} \tag{7.83}$$

Then the horizontal product $Z_G(M') \otimes Z_G(M)$ will be given by the matrix of linear maps:

$$\bigoplus_{[A_2]} \left( Z_G(M)_{[A_1],[A_2]} \otimes Z_G(M')_{[A_2],[A_3]} \right) : \bigoplus_{[A_2]} \left( V_{[A_1],[A_2]} \otimes {V'}_{[A_2],[A_3]} \right) \to \bigoplus_{[A_2]} \left( W_{[A_1],[A_2]} \otimes {W'}_{[A_2],[A_3]} \right) \tag{7.84}$$

The $([A_1], [A_3])$ component of this product is a linear map given as a block matrix, with one block for each gauge equivalence class of connections $[A_2]$ on $B_2$, and whose blocks consist of the tensor product of the matrices from the components of $Z_G(M)$ and $Z_G(M')$. So suppose these are, respectively, $n \times m$ and $n' \times m'$ dimensional matrices. Then the result is an $(n \times n') \times (m \times m')$ matrix, where the $((i, i'), (j, j'))$ component is the product of the $(i, j)$ component of $Z_G(M)$ and the $(i', j')$ component of $Z_G(M')$.

Recall that these indexes mark connections on the cobordisms: the $(i, j)$ component of $Z_G(M)$ is the groupoid cardinality of the groupoid of connections on $M$ which match the $i^{th}$ on $S_1$ and the $j^{th}$ on $S_2$; and the $(i', j')$ component of $Z_G(M')$ is the groupoid cardinality of the groupoid of connections on $M'$ which match the $i'^{th}$ on ${S'}_1$ and the $j'^{th}$ on ${S'}_2$. But this is the groupoid cardinality of the product groupoid whose objects are just these pairs, since groupoid cardinality respects products.

Next consider $Z_G(M' \otimes_H M)$, the natural transformation in **2Vect** corresponding to the horizontal composite of the cobordisms with corners. Again, this can be represented as a matrix of linear maps indexed by pairs $([A_1], [A_3])$ just as above:

$$Z_G(M' \otimes_H M)_{[A_1],[A_3]} : U_{[A_1],[A_3]} \to X_{[A_1],[A_3]} \tag{7.85}$$

where the $U$ have a basis of equivalence classes connections on ${S'}_1 \circ S_1$, and the $X$ on ${S'}_2 \circ S_2$, which restrict to $[A_1]$ and $[A_3]$.

But on the other hand, using the $\beta$ isomorphisms to identify the source and targets allows us to compare this directly to the other side.

But the groupoid of connections on ${S'}_1 \circ S_1$ has the restriction maps $p_S$ and $p_{S'}$ to give connections on $S$ and $S'$. Moreover, the connections obtained this way agree up to gauge equivalence on $B_2$ (since composition of cobordisms is given by a weak pushout). The gauge equivalence up to which these agree is given by the natural isomorphism $\alpha$ from the weak pullback of connection groupoids. So the components $U_{[A_1],[A_3]}$ decompose as a direct sum over $[A_2]$ on $B_2$ of pairs of connections, one on $S_1$, and one on ${S'}_1$, each of which restricts to $[A_2]$ and either $[A_1]$ or $[A_3]$. Similarly for the vector spaces $X_{[A_1],[A_3]}$.

Now, the groupoid of all connections on $M' \otimes_H M$ is a fibred product over $[\Pi_1(B_2), G]$, since each such connection restricts to just one gauge equivalence class on $B_2$. Then for each such $[A_2]$, the groupoid of connections decomposes as a product over choices of restrictions to the $S$ on each side. So it is just a product of the groupoids of connections on $M'$ and $M$, separately, which restrict $[A_2]$. Restrictions to ${S'}_1 \circ S_1$ and ${S'}_2 \circ S_2$ each give separate restrictions to the two halves. Then the cardinality of this groupoid in any component (i.e. with any particular restrictions to source and target) is just the product of the groupoid cardinalities for the corresponding restrictions on $M'$ and $M'$.

But this is exactly what we found for $Z_G(M') \otimes Z_G(M)$. So the two sides are equal as required. $\square$

Again, a special instance of an extended TQFT is when it "restricts" to a TQFT.

**Example 11** Returning to the example of cobordisms between empty manifolds, suppose we have two such cobordisms $S$ and $S'$, and a cobordism with (trivial!) corners $M : S \to S'$. In fact, the effect should be similar to that of evaluating a TQFT on $M$ thought of as a cobordism between manifolds, in a precisely analogous way that $Z_G(S)$ can be thought of as a TQFT giving a vector space for the manifold $S$.

In particular, we have, by the argument in Example 10, that:

$$Z(S) \cong (- \otimes \mathbb{C}^k) \tag{7.86}$$

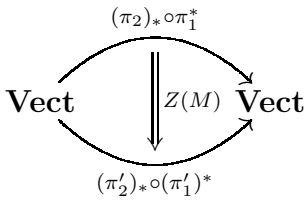
and

$$Z(S') \cong (- \otimes \mathbb{C}^{k'}) \tag{7.87}$$

where $k$ and $k'$ are the number of isomorphism classes of connections on $S$ and $S'$ respectively. If we think of these as being vector spaces $\mathbb{C}^k$ and $\mathbb{C}^{k'}$ assigned by a TQFT, then a cobordism should assign a linear map $T : \mathbb{C}^k \to \mathbb{C}^{k'}$. Indeed, such a linear map will give rise to a natural transformation from $Z(S)$ to $Z(S')$ by

giving, for any objects $V \in \mathbf{Vect}$ on the left side of the diagram, the map $1_V \otimes T$ on the right side. Moreover, all such natural transformations arise this way.

Now, the diagram from (7.65) gives rise to a 2-morphism in **Cat**:




(7.88)

Here, $Z(M)$ arises from the 2-linear map

$$\pi'_* \circ \pi^* : \big[[\Pi_1(S), G], \mathbf{Vect}\big] \to \big[[\Pi_1(S'), G], \mathbf{Vect}\big] \tag{7.89}$$

as described in Definition 22.

Having now described the effect of the extended TQFT at each level - manifolds, cobordisms, and cobordisms with corners—it remains to check that these really define a 2-functor of the right kind. This is the task of Section 7.4.

## 7.4 Main Theorem

Now let us recap the discussion so far. For each finite group $G$, we want to get a weak 2-functor from the bicategory associated to the double bicategory of cobordisms with corners into 2-vector spaces, $Z_G : \mathbf{nCob}_2 \to \mathbf{2Vect}$. This has three aspects, for which we then must verify some properties.

To a compact $(n-2)$-manifold, $Z_G$ assigns a 2-vector space. This consists of **Vect**-presheaves on the groupoid of $G$-connections on $B$ weakly modulo gauge transformations.

To a cobordism between $(n-2)$-manifolds, $S : B \to B'$ in $\mathbf{nCob}_2$, $Z_G$ assigns a span of the groupoids of $G$-connections, as in (7.16). Then a **Vect**-presheaf $F$ on $[\Pi_1(B), G]$ can be transported along the span by first pulling back onto $[\Pi_1(S), G]$ along the restriction $\pi$ of connections on $S$ to connections on $B$. We then push forward this pullback $\pi^* F$ along the restriction $\pi'$ of $[\Pi_1(S), G]$ to $[\Pi_1(B'), G]$ to give a **Vect**-presheaf $\pi'_* \circ \pi^* F$ on $[\Pi_1(B'), G]$.

To a cobordism between cobordisms, $Z_G$ assigns a natural transformation in a similar fashion. Given two functors corresponding to cobordisms, as above, if there is a cobordism between them, it defines a way to push forward a vector in any of the component vector spaces of the functor, written as a matrix. This is done by pulling back the function on the basis defined by the vector, and then pushing forward using a weight given by the groupoid cardinality.

This construction is to give a weak 2-functor. This must be equipped with natural isomorphisms $\beta_{S,S'} : Z_G(S' \circ S) \to Z_G(S') \circ Z_G(S)$ giving weak preservation of composition, as described in Theorem 10. It also must have a natural transformation $U_B : 1_{Z_G(B)} \tilde{\to} Z_G(1_B)$ giving weak preservation of units. Note that for any $(n-2)$-manifold $B$, the idenity $1_B$ is a cobordism $I \times B$, which has the manifold

$B$ embedded as $\{(0,b)|b \in B\}$ and $\{(0,b)|b \in B\}$ (and this cobordism is exactly the collar on both source and target). Then we note that there is an equivalence of categories between $[\Pi_1(B), G]$ and $[\Pi_1(1_B), G]$ since $B$ and $1_B$ have the same homotopy type. So $Z_G(1_B)$, which uses a "pull-push" through the groupoid of connections on $I \times B$, is equivalent to the identity on $1_{Z_G(B)}$.

**Definition 23** *Given a finite group $G$, the extended TQFT $Z_G$ is a 2-functor defined as follows:*

- *For a closed compact manifold $B$, the weak 2-functor assigns a 2-vector space:*

$$Z_G(B) = \big[[\Pi_1(B), G], \mathbf{Vect}\big] \tag{7.90}$$

- *For a cobordism between manifolds:*

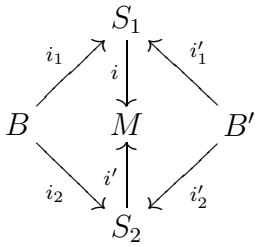
$$B \xrightarrow{i} S \xleftarrow{i'} B' \tag{7.91}$$

*the weak 2-functor assigns a 2-linear map:*

$$Z_G(S) = (p')_* \circ p^* \tag{7.92}$$

*where $p$ and $p'$ are the associated projections for the underlying groupoids of connections weakly modulo gauge transformations.*

- *For a cobordism with corners between two cobordisms with the same source and target:*




(7.93)

*the weak 2-functor assigns a natural transformation $\alpha$, whose components (in the matrix representation) are as in (7.70).*

*The 2-functor $Z_G$ also includes the following:*

- *For each composable pair of cobordisms $S : B_1 \to B_2$ and $S' : B_2 \to B_3$, a natural isomorphism*

$$\beta : Z_G(S' \circ S) \to Z_G(S') \to Z_G(S) \tag{7.94}$$

*, as described in Theorem 10.*

- *For each object $B \in \mathbf{nCob}_2$, the natural transformation*

$$U_B : 1_{Z_G(B)} \tilde{\to} Z_G(1_B) \tag{7.95}$$

*is the natural transformation induced by the equivalence between $[\Pi_1(B), G]$ and $[\Pi_1(1_B), G]$.*

Then we have the following:

**Theorem 13** *For any finite group $G$, there is a weak 2-functor $Z_G : \mathbf{nCob}_2 \to \mathbf{2Vect}$ given by the construction in Definition 23.*

**Proof**: First, we note that by the result of Theorem 8, we know that $Z_G$ assigns a 2-vector space to each object of $\mathbf{nCob}_2$.

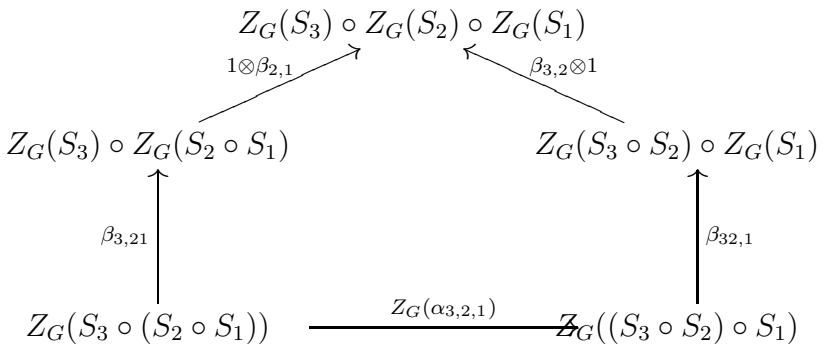
If $S : B \to B'$ is a cobordism between compact manifolds—i.e. a morphism in $\mathbf{nCob}_2$, the map $Z_G(S)$ defined in Definition 21 is a linear functor by the result of Theorem 6, since it is a composite of two linear maps. This respects composition, as shown in Theorem 10.

Next we need to check that our $Z_G$ satisfies the properties of a weak 2-functor: that the isomorphisms from the weak preservation of composition and units satisfy the requisite coherence conditions; and that $Z_G$ strictly preserves horizontal and vertical composition of natural transformations.

The coherence conditions for the compositor morphisms

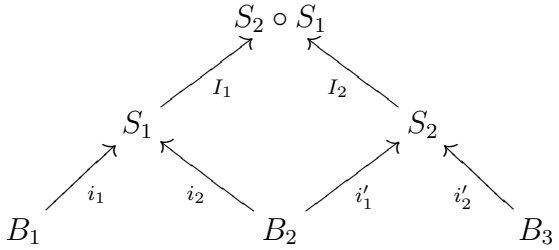
$$\beta_{S,T} : Z_G(T) \circ Z_G(S) \to Z_G(T \circ S) \tag{7.96}$$

and the associator say that these must make the following diagram commute for all composable triples of cobordisms $(S_1, S_2, S_3)$:




(7.97)

We implicitly assume here a trivial associator for the 2-linear maps in the expression $Z_G(S_3) \circ Z_G(S_2) \circ Z_G(S_1)$. This is because each 2-linear map is just a composite of functors, so this composition is associative. But note that we can similarly assume, without loss of generality, that the associator $\alpha$ for composition of cobordisms is trivial. The composite $S_2 \circ S_1$ is a pushout of two spans of manifolds with boundary:




(7.98)

This pushout is only defined up to diffeomorphism, but one candidate is $S_1 \coprod S_2 / \sim$, where $i_1(x) \sim i_2(x)$ for any $x \in B_2$. Any other condidate is diffeomorphic to this one. But then, the associator

$$\alpha_{3,2,1} : Z_G(S_3 \circ (S_2 \circ S_1)) \to Z_G((S_3 \circ S_2) \circ S_1) \tag{7.99}$$

is just the identity. (Choosing different candidates for the pushouts involved would give a non-identity associator).

So it suffices to show that, with this identification,

$$(1 \otimes \beta_{2,1}) \circ \beta_{3,21} = (\beta_{3,2} \otimes 1) = \circ\beta 32, 1 \tag{7.100}$$

This is verified by a computation we leave to the reader.

In general, the coherence conditions for the "unit" isomorphism

$$U_B : 1_{Z_G(B)} \tilde{\to} Z_G(1_B) \tag{7.101}$$

which weakly preserves identities say that it must make the following commute for any cobordism $S : B \to B'$:

$$\begin{array}{ccc} Z_G(S) & & \\ \uparrow{\scriptstyle 1\otimes U_B} & \nwarrow^{Z_G(r_S)} & \\ Z_G(S) \circ Z_G(1_B) & \xleftarrow{\beta_{S,1_B}} & Z_G(S \circ 1_B) \end{array} \tag{7.102}$$

where $r_B$ is the right unitor for $B$. There is also the symmetric condition for the left unitor.

We notice that, as with $Z_G(1_B)$, $Z_G(r_B)$ is equivalent to the identity since we can think of the unitor $r_S : S \circ 1_B \to S$ as a mapping cylinder diffeomorphic to $I \times S$. Since $S \circ 1_B$ and $S$ are diffeomorphic, these are embedded as the ends of the cylinder.

So the condition amounts to the fact that $\beta_{S,1_B} : Z_G(S \circ 1_B) \to Z_G(S) \circ Z_G(1_B) = Z_G(S)$ is equivalent to the identity in such a way that (7.102) commutes. We again leave this to the reader. $\square$

This weak 2-functor is our extended TQFT.

# Chapter 8

# Prospects for Quantum Gravity

The title of this thesis is "Extended TQFT's and Quantum Gravity", but so far we have said much about the former and nearly nothing about the latter. Yet, despite the intrinsic interest extneded TQFT's in themselves, the prospect of applying these results, or very similar ones, to quantum gravity has been one of the major motivations behind this work. The prospects for doing this are good, at least in a low-dimensional toy model. In (2+1) dimensions (two dimensions of space, and one of time), or 3 dimensions (with no time dimension), Einsteinian gravity is a topological theory, whereas in higher dimensions it is not.

So more specifically, the immediate result of extending our results here will be not, in general, quantum gravity, but a topological gauge theory called BF theory. The connection to gravity is that this is the same as Einsteinian gravity in 3 dimensions, and in 4-dimensions it is a limit of Einsteinian gravity as $G \to 0$ (where $G$ is Newton's constant). This is a limitation of our approach, but quantum

gravity is a large and mostly open field (see for instance Rovelli's survey [77] of some of the work to date); so finding a clear framework for certain, fairly simple, cases is a useful project.

In this final chapter, we sketch what kind of extension is needed, and the implications of this work for quantum gravity in the case. This chapter is not intended to be mathematically rigorous. Its role is to describe in an impressionistic way some of the mathematical and physical context for this work, as well as to suggest the directions for its future development.

## 8.1 Extension to Lie Groups

The first thing to consider is the possibility of extending the analysis we have made for extended TQFT's corresponding to finite groups. In particular, we are interested in an analog of the preceding when $G$ is a Lie group. In particular, there is a special case of interest, which is when $G = SU(2)$, and $n = 3$: that is, considering $Z_{SU(2)} : \mathbf{3Cob_2} \to \mathbf{2Vect}$. We will describe here how such a theory, if it is possible to construct it, would be related to a well-studied theory of quantum gravity in three dimensions: the Ponzano-Regge model.

The theorems so far apply only when $G$ is a finite group. However, we have seen in Section 6.3 that there is a notion of an infinite-dimensional 2-vector space

$2L^2X$ for a measure space $(X, \mu)$, consisting of maps from $X$ into **Vect**. This is an infinite dimensional analog of the functor category $[\mathbf{X}, \mathbf{Vect}]$ which was used in constructing an extended TQFT from a finite group (though we must restrict to only "measurable" functors). In particular, it should still make sense to define a 2-vector space $\big[[\Pi_1(B), G], \mathbf{Vect}\big]$ for a manifold $B$. This involves both a generalization and a specialization from the Crane-Yetter 2-vector space **Meas(X)**, since in that case $X$ was a measurable space, wheareas in the case of a Lie group it comes equipped with a standard measure (Haar measure), but we also consider its path groupoid, rather than merely the set. So one would need to extend the theory of categories of measurable fields of Hilbert spaces to a theory of categories of measurable functors into **Vect** from such a measurable groupoid.

Now, the construction used for a finite group used several facts we showed for finite groupoids. For example, Theorem 6 established that the 2-linear map given by pushforward is the adjoint of that given by pullback. However, we only showed this for finite groupoids. In general, if $G$ is not finite, $[\Pi_1(B), G]$ is not an essentially finite groupoid. So this and other theorems would need to be extended to the case of Lie groups. In particular, since 2-vector spaces need not contain arbitrary infinite colimits, the pushfoward we described may not exist. So we need the infiinite-dimensional 2-vector spaces in Crane and Yetter's **Meas**, as discussed in Section 6.3.

So in particular, such an extension should take advantage of the Haar measure on $G$ to define the pushforward of a functor on a space by direct integration, rather than by simply taking a general colimit (which need not exist). This and other such constructions would need to be justified in order to try to imagine constructing an extended TQFT from a Lie group as we have described with a finite group. It seems most clear how this would work in the case where $G$ is compact, since compact Lie groups have finite total Haar measure. If the total measure of the group were infinite, we would not expect the integrals one would use in these definitions to converge, and there would be a problem of well-definedness.

Then in cases where the direct integral exists, we would expect, by analogy with the formula from Definition 21 that the component in some connection $A'$ on $B'$ of $Z(S)$ applied to a "state" 2-vector $\Psi \in Z(B)$ is:

$$(Z(S)\Psi)(A) = \int_{[\Pi_1(B),G]}^{\oplus} \Big( \int_{\overline{[\Pi_1(S),G]}}^{\oplus} \Psi(A) \mathrm{d}\overline{A} \Big) \mathrm{d}A' \tag{8.1}$$

Where $\overline{[\Pi_1(S), G]}$ is the set of connections $\overline{A}$ on $S$ such that $\overline{A}|_B = A'$ and $\overline{A}|'_B = A'$. Both integrals are "direct integrals" of Hilbert spaces. The outer integral, over $B$, uses the principle that $\Psi$ can be represented as a direct integral (though not a finite linear combination) of simple objects in $Z_G(B)$. The direct integral over connections on $S$ stands in for a general colimit. This assumes that we can treat the "pushforward" phase of $Z_G(S)$ as a direct integral (rather than a direct sum) of quotient spaces.

Here we are integrating with respect to a measure on the space of connections. Since this consists of functors from a finitely generated groupoid into $G$, the measure is derived from the Haar measure on $G$.

Presuming that this is justified, it should be possible to extend the main results (somewhat modified) from this discussion of extended TQFT's to the case where $G$ is any compact Lie group (and possibly any Lie group). The groups of major interest to quantum gravity are rotation groups of various signatures, and their double covers (which are used in describing *spin* connections) . For example, connections valued in Euclidean rotation groups $SO(3)$ and $SO(4)$, and their double covers $SU(2)$ and $SU(2) \times SU(2)$, are relevant to 3- and 4-dimensional Euclidean quantum gravity respectively.

More precisely, since what we have discussed are *flat* connections, this remark needs to be qualified. Flat $SU(2)$ connections do indeed describe configurations for 3D quantum gravity, since in that case, gravity is a purely topological theory. (For more background on 3D quantum gravity, particularly in the case of signature $(2,1)$, see work by Steven Carlip [22], [21]).

However, in 4 dimensions, a theory of flat connections does not describe gravity, but rather a limiting case of Einsteinian gravity as Newton's constant $G \to 0$. The subject of this limit, and in general the deformation of gauge theories, is considered extensively by Wise [87]. What is true in 4 dimensions is that the

purely topological theory corresponds to a theory of flat connections on a manifold known as $BF$ theory. and by Freidel, Krasnov and Puzio [38]). To describe a theory of gravity would need something more than what is discussed here. In Section 8.3 we briefly consider some possible approaches to this problem.

## 8.2 Ponzano-Regge with Matter

If $G = SU(2)$, the objects of $\mathcal{A}_0/\!\!/G$, just as for a finite group, are equivariant functors from $[\Pi_1, SU(2)]$ to **Vect**, and can be represented in terms of a basis of irreducible objects. Assuming that the previous results hold when $G$ is a Lie group, an irreducible object amounts to a choice of conjugacy class in $SU(2)$ and action of $SU(2)$ on the associated vector spaces coming from the isomorphism assocated to conjugation by $g$. Let us assume that when we replace finite $G$ by the Lie group $SU(2)$, we retain the classification of Example 7. Then irreducible 2-vector by pairs $([g], \rho)$ of a conjugacy class $[g] \in SU(2)/\operatorname{Ad}(SU(2))$, and representation $\rho$ of $SU(2)$ on some vector space $V$. Now, a conjugacy class in $SU(2)$ amounts to specifying an *angle of rotation* in $[0, 4\pi]$. This is since this is the double cover of the 3D rotation group, and all rotations by the same angle are conjugate to all others by some rotation taking one axis of rotation to the other. This number in $[0, 4\pi]$ represents a *mass* in 3D quantum gravity—which manifests as an angle

deficit when one traces a path around a massive particle, one finds, geometrically, that one has rotated by a certain angle proportional to its mass, which has a maximum total mass allowable of $4\pi$ in Euclidean 3D gravity.

On the other hand, a representation of $SU(2)$ is classified by a half-integer, which is called a *spin* since these label angular momenta for spinning quantum particles. This is exactly the other attribute a particle in the 3D Ponzano-Regge model may have. Mass and spin are the characteristics which determine the effect of a particle on the connection—that is, its gravitational effect. In the Ponzano-Regge model, mass and spin label the edges of a graph describing space. In the case that the mass on an edge is zero, this describes a *spin network*, as described first by Penrose [76]. A spin network is a combinatorial representation of the geometry of space.

Penrose's original idea was that a quantum theory of gravity should describe space in intrinsically discrete terms. The description as a graph is intrinsically discrete. Edges are labelled with spins since these are representations of the symmetry group related to angular momentum. This was chosen because angular momentum is already discrete in quantum mechanics, and is plainly related to the (local) rotational symmetry of space.

Such spin networks are related to the Ponzano-Regge model for 3D quantum gravity. The interpretation in terms of gravity comes from the observation that

a conjugacy class in $SU(2)$ is an angle in $[0, \pi]$, which is a *mass* $m$; In the case $m = 0$ the isomorphism is just a spin—an irrep of $SU(2)$, labelled by an integer (or, for physics purposes, a half-integer). For other $m$, we get a spin when we reduce to a skeletal version of the 2-vector spaces.

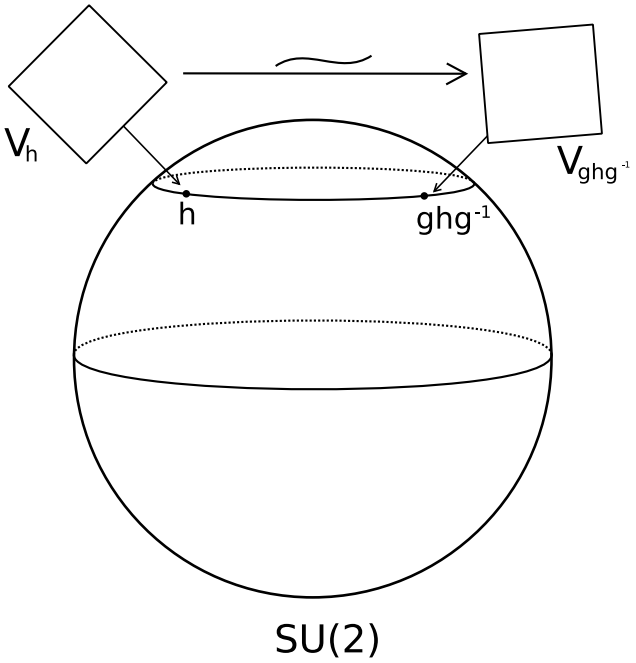



**Figure 8.1:** Irreducible Object in $Z_{SU(2)}(S^1)$

The Ponzano-Regge model is a quantum theory which reproduces classical General Relativity in a suitable limit. Now, in General Relativity, gravity can be thought of as the theory of a connection on a manifold, which is the Levi-Civita connection associated to the metric in the usual formalism. So the Ponzano-

Regge model can be seen as a quantum theory for a connection on space of a given topology.

We can think of a cobordism with corners, as in Figure 1.1 as having boundaries indicating the boundary of the world lines of some system. We can think of this as a Feynman graph for some particles. This interpretation makes the most sense if our group $G$ is a Lorentz group, so that we think of the underlying manifold with corners as "spacetime". However, even if it is only "space", this cobordism can be though of as giving a graph, where the circles represent the boundary in 2D "space" around some system—the "removed" portions of space are the graph. We can think of the edges as particles—by which we only mean some bit of matter. A "fundamental" particle is then an irreducible state on it. This corresponds, as we remarked earlier, to a choice of a pair $([g], \rho)$ consisting of a conjugacy class $[g]$ of $G$ and representation $\rho$ of $G$ on some vector space. Conjugacy classes of rotation or Lorentz groups are "mass shells", corresponding to the mass of the particle. Representations of $G$, at least for $SU(2)$ and similar groups, are labelled by "spins". These determine how a particle interacts with gravity. This is precisely what the Ponzano-Regge model describes: a network of edges labelled with just this data, and with vertex amplitudes at the intersections.

So our extended TQFT gives "particles"—boundaries in space - labelled by a representation of a certain group. Our example was derived from a finite group,

but if $G = SU(2)$ the label is a mass and spin) moving on a background described by Ponzano-Regge quantum gravity (see work by, for example, Freidel, Livine, and Louapré [40] [41][39], discussing the Ponzano-Regge model coupled to matter, by Noui [74], and Noui and Perez [75] on 3D quantum gravity with matter).

Baez, Crans, and Wise [7] describe how conjugacy classes of gauge groups can be construed as "particle types": an "elementary" particle corresponds with an irreducible 2-vector in $Z_G(B)$. This associates to a hole—whose boundary is diffeomorphic to the circle $S^1$—a holonomy in a given conjugacy class $[g]$ of $G$. This is physically indistinguishable from any other corresponding to the same class. But they are distinguishable from particles giving holonomies in some other conjugacy class. So one says these represent different "types" of particle.

Now, we have said that for a 3D extended TQFT, the 2-vector space of states for a circle has a basis in which each object is given by a conjugacy class of $G$ and representation of the stabilizer of that class. Wise [87] describes a way to interpret such conjugacy classes as *particle types* in a topological gauge theory. More generally, in any dimension, given a space with a "puncture" of codimension 2, there can be nontrivial holonomy for a connection around that puncture. In 3D, this is a 1-D puncture, which we think of as the worldline of a point particle. In the framework discussed in this thesis, we think of the particle as a puncture in 2D space, surrounded by a 1D manifold, namely a circle. This is the manifold

$B$ for our extended TQFT. Then the "space" from which the particle is removed is represented as the cobordism $S$ in our setup, and "spacetime"

Just as a conjugacy class in $SU(2)$, as we have seen, can be interpreted as a mass in Ponzano-Regge gravity, similarly, for other gauge groups, conjugacy classes in the group classify "types" of matter particles which may be coupled to the field. A state for the boundary around such a defect in our extended TQFT gives These represent possible holonomies, up to gauge equivalence, around such a defect. These classify the physically distinguishable particles.

The interpretation described here so far is purely kinematical, though in 4D, where these punctures are "strings" (i.e. the punctures in space are 1-dimensional manifolds, namely circles, and in "spacetime" are 2-dimensional, namely "world-sheets") the dynamics for such matter has been studied by Baez and Perez [11]. In terms of our extended TQFT setting, the dynamics are described by the action of $Z_G$ on cobordisms of cobordisms.

In particular, suppose we have a cobordism with corners $M : S \to S'$ for cobordisms $S, S' : B \to B'$, and are given specified "particle types" for the punctures in the initial and final spaces. This amounts to choosing particular basis 2-vectors in $Z_G(B)$ and $Z_G(B')$.

Then on each space—cobordisms having these punctures as boundary - this gives a vector space as the component of $Z_G(S)$ which corresponds to these basis

2-vectors, and similarly for $Z_G(S')$. Then the corresponding component of $Z_G(M)$ is a linear operator between these states. The interpretation is that these components describe the spaces of states for a field coupled to matter of the specified type, and the linear operator which gives its time evolution. This is found, as we saw in (7.70), is given by a certain "sum over histories", where each history is a connection on the "spacetime" $M$. The topology of the punctures in $M$ can be thought of as a Feynman graph for interactions of the matter which is the source of the field.

One should carefully note that to take this interpretation in terms of "histories" and "spacetime" literally requires a noncompact gauge group $G$ such as Loretz groups $SO(2,1)$, $SO(3,1)$, or their double covers $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ respectively. We expect that it would be more difficult to make these concepts precise for noncompact gauge groups.

## 8.3 Further Prospects

The relationship between the extended TQFT's discussed here and BF theory leads one to ask about the relations between this approach and other ways of looking at BF theory which have already been studied. One of these which is particularly relevant involves so-called *spin foam* models. A self-contained de-

scription of such models for BF theory and quantum gravity by Baez [4]. Spin foam models are a generalization of the *spin networks* of Penrose [76].

A spin network is a network in the sense of a *graph*—a collection of nodes, connected by edges. In a spin network, the edges are labelled by spins—representations of $SU(2)$, which are labelled by half-integers. The vertices by intertwining operators—that is, morphisms in the category of representations of $SU(2)$ taking some tensor product of irreducible representations to some other such tensor product. These are taken to be a representation of a "combinatorial spacetime" in which the nodes represent events, and the edges give information about distance between events. In particular, the attitude is that this is the *only* information about distances within this combinatorial model of spacetime.

The idea behind spin foam models is to view spin networks as describing configurations for the geometry of space. Then a spin *foam* is a morphism between spin networks. In fact, it is a structure which contains spin networks as start and end states in much the same way that an $n$-dimensional cobordism has $(n-1)$-manifolds as source and targets. A spin foam is a complex vertices, edges, and faces, with group representations labelling faces, and intertwining operators labelling edges. So, in particular, a generic codimension-1 cross-section of a spin foam

The expected link to the present work is a generalization of the FHK construction described in Section 2.3. In that case, one develops a TQFT by using triangulations of the manifolds and cobordisms on which the TQFT is to define Hilbert spaces and linear maps. We saw, as illustrated in Figure 2.5, that there is a network dual to this triangulation. To the edges in this network one assigns copies of a certain algebra, namely $Z(\mathbb{C}[G])$, and to the nodes one assigns a multiplication operator. As described in Section 2.4, the coherence laws satisfied by these operators are described by tetrahedrons. These are the Pachner moves in 2-D: attaching a tetrahedron to a triangulation along one, two, or three triangular faces gives a move by replacing the attached faces with the remaning faces of the tetrahedron. The way of assigning an operator to a vertex of the dual to a triangulation must have the property that it is invariant under such moves.

We have categorified this picture in order to increase the codimension of the theory - that is, the difference in dimension between the basic manifolds and the highest-dimensional cobordisms. So there should be a categorified equivalent of the FHK construction, in which we begin with triangulated manifolds and cobordisms. In categorifying, we replace the equations given by the Pachner moves with 2-morphisms. Each move gives a 2-morphism between a pair of morphisms in **2Vect**, corresponding to a tetrahedron thought of as a cobordism connecting

two parts of its boundary. Any cobordism can be built of such units, attached together in some triangulation:

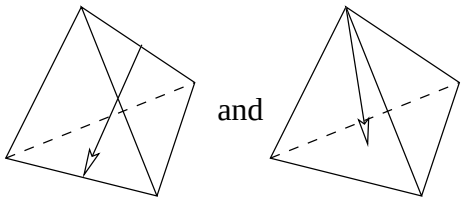



**Figure 8.2:** Tetrahedra Assigned 2-Morphisms

These obey coherence laws (equations) given by the 2-3 and 1-4 Pachner moves:

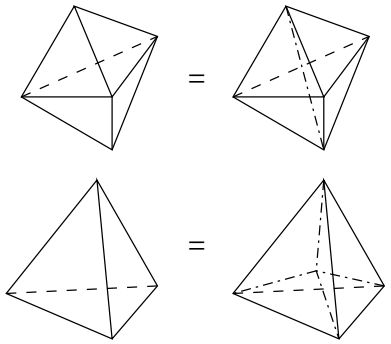



**Figure 8.3:** Coherence Rules as Pachner Moves

As in 2D, where the algebra assigned by the FHK construction to edges is $Z(\mathbb{C}[G])$, the categorified version should assigne $Z(\mathbf{Vect}[G])$, which corresponds to our assignment to a circle of equivariant **Vect**-presheaves on $G$. Assigning these to edges reproduces the Ponzano-Regge model when $G = SU(2)$, since the irreducible objects in this category are, as we have seen, precisely labelled by mass and spin. Analogous results hold for other $G$, giving different field theories. But

notice that this is different from the way we recovered the Ponzano-Regge model above: now we are assigning this data to edges of a triangulation, not a boundary of a "worldline". The relations between these two pictures are close, but more than we can go into in detail here.

However, it is enough to observe that there is a close relation between the extended TQFT we have developed and state-sum (i.e. spin-network and spin-foam) models for BF theory, and 3D quantum gravity. So one avenue for further exploration is to see how the framework described here can be extended to incorporate other theories described by such state sum models.

Our basic result involved the construction of an extended TQFT as a weak 2-functor for any finite group $G$. In Section 8.1, we discussed the possibility of extending the construction to the case where $G$ is a Lie group, and in particular, indicated that this is expected to be more natural when $G$ is a *compact* Lie group. Of course, noncompact groups are also of interest - for example, the Lorentz groups. But there are other directions in which to generalize this. We briefly consider two possibilities here: categorical groups (also known as 2-groups), and quantum groups (by which we mean quasitriangular Hopf algebras).

An extension of the Dijkgraaf-Witten theory to categorical groups is described by Martins and Porter [70]. A categorical group, also known as a 2-group, is a category object in the category **Grp** of groups. That is, it is a structure having a

group of objects and a group of morphisms, satisfying the usual category axioms expressed in terms of morphisms within **Grp**. Any group $G$ is an example of a 2-group, where the group of objects is trivial : this is in fact how we have been thinking of the gauge group $G$ throughout this thesis. But there are many other examples of 2-groups, including, importantly for us, 2-groups which arise from semidirect products of groups $H \rtimes G$. In this case, the group of objects is $G$ and whose group of morphisms is $H \rtimes G$: the group of automorphisms of any given object is isomorphic to $H$. Such a 2-group is called an *automorphic 2-group*.

The category of 2-groups can be shown to be equivalent to the category of *crossed modules*, a concept due to Brown and Spencer [19]. A crossed module consists of a tuple $(G, H, t, \alpha)$, where $G$ and $H$ are groups, $t : H \rightarrow G$ is a homomorphism, and $\alpha : G \rightarrow Aut(H)$ is an action of $G$ on $H$, such that $t$ and $\alpha$ satisfy some compatibility conditions, which turn out to be equivalent to the category axioms in the 2-group described above. The *Poincare 2-group*, introduced by Baez [5], is an example of an automorphic 2-group. It has been a subject of interest as a source of a new class of spin foam models, first suggested by Crane and Sheppeard [25]. Such models are based on the representation theory of 2-groups, which is 2-categorical in nature, since one must consider representations, intertwiners between representations, and 2-intertwiners between intertwiners, which

form a 2-category. A spin foam model based on a 2-group uses these to label faces, edges, and vertices respectively.

The most evident relation of 2-groups to the sort of extended TQFT's we have been discussing is related to gauge theory. The role of the group $G$ in constructing the weak 2-functor $Z_G$ was through the groupoid of *connections on $G$-bundles* on a space $X$. This is $[\Pi_1(X), G]$, the category of functors from the fundamental groupoid of $X$ into $G$ thought of as a category with one object. One might suppose that the natural extension would be to take $G$ to be a 2-group, with a group of objects, and take functors from $\Pi_1(X)$ into this.

This could be done, but perhaps a better approach is in the form of *higher gauge theory*. Discussion of higher gauge theory can be found in work by Baez and Schreiber [12] and Bartels [14]. The principle is that one should assign data from a 2-group to both paths and homotopies of paths, so what one uses is not $\Pi_1(X)$, but $\Pi_2(X)$, the fundamental 2-groupoid of $X$. This is a 2-category whose objects are points in $X$, morphisms are paths, and 2-morphisms are homotopy classes of homotopies of paths. It should be clear that this encodes information not only about the first homotopy group of a space (as does the fundamental group), but also the second homotopy group. In higher gauge theory one studies, in this case, flat "2-connections" (or more generally $n$-connections), which are seen as 2-functors from $\Pi_2(X)$ to a 2-group. In 3D, we have discussed how an

extended TQFT based on a Lie group could possibly describe the evolution of point-particles along worldlines in spacetime. A categorified form of this based on 2-connections could be of interest in 4D, where one could study the behaviour of "strings" as well as point particles (see, for instance, Baez and Perez [11]).

Having begin by categorifying the standard definition of a TQFT, one could then hope to continue the process and find an infinite "tower" of theories, each having one more codimension than the last.

The last possible direction of generalization from our extended TQFT based on a finite group would involve quantum groups. Whereas moving to 2-groups involves "categorifying" the concept of a group, moving to quantum groups, as the name suggests, involves "quantizing". Neither procedure is, in general, a well defined operation, but particular examples are understood. In particular, we could try to generalize from finite groups to "finite quantum groups", by which we mean finite-dimensional quasitriangular Hopf algebras.

The idea behind quantum groups is described by Shahn Majid [68] and also notably by Ross Street [81]. The idea provides a way to speak of deforming topological groups, although there is no way of smoothly deforming the group action of a topological group to a family of other such groups. Instead, one works in a larger category, of "quantum" groups, of which usual groups correspond to special cases. This is done using *Gelfand duality*, which relates commutative

algebra and topological spaces. Specifically, it gives an equivalence saying that each commutative $C^\star$-algebra is the algebra $C(X)$ of continuous complex functions on a compact Hausdorff space $X$.

Continuous functions $f : X \to Y$ give algebra homomorphisms $C(f) : C(Y) \to C(X)$, so that if $X$ is a group as well as a space, the $C^\star$-algebra $C(X)$ gets a comultiplication $C(\cdot) : C(X) \to C(X) \otimes C(X)$, counit $C(1) : C(X) \to \mathbb{C}$ and involution $C(^{-1}) : C(X) \to C(X)$. Since these come from operations on a group, they, along with the (pointwise) multiplication, unit, and inverse in $C(X)$, satisfy certain axioms, and relations. The axioms for a *Hopf* algebra generalize these. In particular, they require that the multiplication be associative, but not necessarily commutative. A quasitriangular, or "braided" Hopf algebra $H$ has a distinguished element $\gamma$, thought of as the image of $1 \otimes 1$ under a "switch" operation $H \otimes H \to H \otimes H$. These Hopf algebras are what are called "quantum groups".

We will not attempt a full explanation of quantum groups here, though see the above references for full details). For our purposes, the interesting point is that the Hopf algebras coming from Lie groups $G$ as $C(G)$ can be deformed to noncommutative and non-cocommutative quantum groups, with a parameter $q$ which is a unit complex number. Given elements $x$ and $y$, the deformation replaces the operations such as multiplication by new ones, given as power series in $q$. When $q$ is a complex root of unity, this has particularly good properties.

In particular, we expected to recover the Ponzano-Regge model of 3D quantum gravity, based on $SU(2)$, as an extended TQFT. Now, the Turaev-Viro model (see [85] and [36]) is based on the $q$-deformed quantum groups $SU(2)_q$, and in some respects is more convenient than the Ponzano-Regge model. In particular, there are infinitely many representations of $SU(2)$, but only finitely many of $SU(2)_q$ when $q$ is an $n^{th}$ root of unity (specifically, $n-1$ of them). This gives better convergent properties when summing over representations. In general, spin foam models involving quantum groups sometimes have such good finiteness properties.

As a first effort to generalize from our situation of an extended TQFT based on a finite group, we may try to develop an extended TQFT from the corresponding class of quantum groups - namely, finite dimensional quasitriangular Hopf algebras.

Finally, it should be possible to combine our different directions of generalization. For example, Crane and Yetter [27] discuss generally a similar family of algebraic and higher-algebraic structures which give rise to TQFTs in various dimensions. In 4D, the relevant structure is a Hopf category - a categorified equivalent of a Hopf algebra. Marco Mackaay [65] shows very explicitly how to construct invariants of 4-manifolds from certain kinds of 2-categories by means of the sort of state-sum model which we have been discussing. It would be useful to

study how much of this can be described in the "geometric" style which we have examined here in the form of groupoids of connections.

All of these directions suggest ways in which our results could be expanded further by future investigation.

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# Part IV

# Appendices

# Appendix A

# Internal Bicategories in Bicat

We rely on the notion of a bicategory *internal* to **Bicat** in our discussion of Verity double bicategories in Chapter 4, and thus in the development of the Verity double bicategory $\mathbf{nCob}_2$ in Chapter 5. Here we present a more precise definition of this concept, and in Lemmas 6 and 7 we use it to show that examples having properties like those of $2\mathrm{Cosp}(\mathbf{C})_0$ (definition 6) give "double bicategories" in the sense of Verity. These lemmas were used in the proofs of Theorems 2 and 3.

To begin with, we remark that the theory of bicategories, **Th(Bicat)** is more complicated than that for categories. However as with **Th(Cat)**, it will be a category with objects Obj, **Mor** and **2Mor**, and having all equalizers and pull-backs. To our knowledge, a model of **Th(Bicat)** in **Bicat** has not been explicitly described as such before. We could treat Obj as a horizontal bicategory, and the objects of Obj, **Mor** and **2Mor** as forming a vertical bicategory, but we note that diagrammatic representation of, for instance, 2-morphisms in **2Mor** would

require a 4-dimensional diagram element. The comparison can be seen by contrasting tables A.1 and A.2.

The axioms satisfied by such a structure are rather more unwieldy than either a bicategory or a double category, but they provide some order to the axioms for a Verity double bicategory, as shown in Definition 5. We note that, although that definition is fairly elaborate, it is simpler than would be a similarly elementary description of a double bicategory.

In particular, where there are compatibility conditions involving equations in this definition, such a structure would have only isomorphisms, themselves satisfying additional coherence laws. In particular, in Verity double bicategories, the action of 2-morphisms on squares is described by strict equations, rather than being given by a definite isomorphism.

Similarly, it is possible (see [86] sec. 1.4) to define categories $\mathbf{Cyl_H}$ (respectively, $\mathbf{Cyl_V}$) of *cylinders* whose objects are squares, and maps are pairs of vertical (respectively, horizontal) 2-morphisms joining the vertical (resp. horizontal) source and targets of pairs of squares which share the other two sides (this is shown in Table A.2, in Section A.3: the cylinders are "thin" versions of higher morphisms appearing there). These are plain categories, with strict associativity and unit laws. These conditions would be weakened in a double bicategory (in which maps would include not just pairs of 2-morphisms, but also a 3-dimensional

interior of the cylinder, which is a morphism in 2 Mor, or 2-morphism in Mor, satisfying properties only up to a 4-dimensional 2-morphism in 2 Mor).

We start to see all this by describing how to obtain a double bicategory.

## A.1 The Theory of Bicategories

We described in Section 3.4 how a double category may be seen as a category internal to **Cat**. To put it another way, it a model of **Th(Cat)**, the theory of categories, in **Cat**, which is a limit-preserving functor from **Th(Cat)** into **Cat**. We did not make a special point of the fact, but this is a *strict* model. A weak model would satisfy the category axioms such as composition only up to a 2-morphism in **Cat**, namely up to natural transformation. So, for instance, the pullback (3.16) would be a weak pullback, so that instead of satisfying $t \circ c_1 = s \circ c_2$, there would only be a natural transformation relating $t \circ c_1$ and $s \circ c_2$. Such a weak model is the most general kind of model available in **Cat**, but double categories arise as strict models.

So here we note that we are thinking of **Bicat** as a mere category, and that we are speaking of *strict* internal bicategories. In particular, the most natural structure for **Bicat** is that of a tricategory: it has objects which are bicategories, morphisms which are weak 2-functors between bicategories, 2-morphisms which

are natural transformations between such weak 2-functors, and 3-morphisms which are "modifications" of such transformations. Indeed, **Bicat** is the standard example of a tricategory, just as **Cat** is the standard example of a bicategory. But we ignore the tricategorical structure for our purposes.

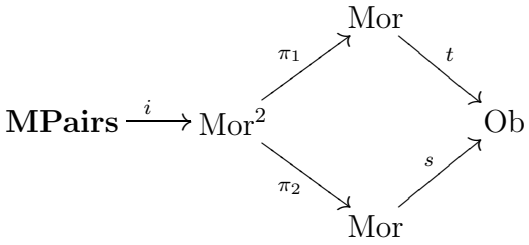
So as with double categories, we only consider strict models of the theory of bicategories, **Th(Bicat)** in **Bicat**. That is, functors from the category **Th(Bicat)** into **Bicat** (seen as a category). Equations in a model are mapped to equations (not isomorphisms) in **Bicat**. We call these models double bicategories.

Before we can say explicitly what this means, we must describe **Th(Bicat)** as we did for **Th(Cat)** in Section 3.4.

**Definition 24** *The theory of bicategories is the category (with finite limits)* **Th(Bicat)** *given by the following data:*

- *Objects* Ob*,* Mor*,* 2Mor
- *Morphisms* $s, t : \mathrm{Ob} \to \mathrm{Mor}$ *and* $s, t : \mathrm{Mor} \to \mathrm{2Mor}$
- ***composition*** *maps* $\circ : \mathrm{MPairs} \to \mathbf{Mor}$ *and* $\cdot : \mathrm{BPairs} \to \mathbf{2Mor}$*, satisfying the interchange law (3.2), where* $\mathrm{MPairs} = \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$ *and* $\mathbf{BPairs} =$

$2\mathrm{Mor} \times_{\mathrm{Mor}} 2\mathrm{Mor}$ *are equalizers of diagrams of the form:*




(A.1)

*and similarly for opname$BPairs$.*

- *the* ***associator*** *map* $a : \mathrm{Triples} \to \mathbf{2Mor}$*, where* $\mathrm{Triples} = \times_{\mathrm{Ob}} \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$ *is the equalizer of a similar diagram for involving* $\mathrm{Mor}^3$*, such that* $a$ *satisfies* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$

- ***unitors*** $l, r : \mathrm{Ob} \to \mathrm{Mor}$ *with* $s \circ l = t \circ l = \mathrm{id}_{\mathrm{Ob}}$ *and* $s \circ r = t \circ r = \mathrm{id}_{\mathrm{Ob}}$

*This data is subject to the conditions that the associator is subject to the Pentagon identity, and the unitors obey certain unitor laws.*

**Remark 17** The Pentagon identity is shown in (3.6) for a model of **Th(Bicat)** in **Sets**), where we can specify elements of Mor, but the formal relations—that the composites on each side of the diagram are equal—hold in general. These are built from composable quadruples of morphisms and composition as indicated in the labels. Similar remarks apply to the unitor laws shown in (3.7).

So we have the following:

**Definition 25** *A double bicategory consists of:*

- *bicategories* $\mathbf{Obj}$ *of* ***objects****,* $\mathbf{Mor}$ *of* ***morphisms****,* $\mathbf{2Mor}$ *of* ***2-morphisms***
- ***source*** *and* ***target*** *maps* $s, t : \mathbf{Mor} \to \mathbf{Obj}$ *and* $s, t : \mathbf{2Mor} \to \mathbf{Mor}$
- *partially defined* ***composition*** *functors* $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$ *and* $\cdot : \mathbf{2Mor}^2 \to \mathbf{2Mor}$*, satisfying the interchange law (3.2)*
- *partially defined* ***associator*** $a : \mathbf{Mor}^3 \to \mathbf{2Mor}$ *with* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$
- *partially defined* ***unitors*** $l, r : \mathbf{Obj} \to \mathbf{Mor}$ *with* $s(l(x)) = t(l(x)) = x$ *and* $s(r(x)) = t(r(x)) = x$

*All the partially defined functors are defined for composable pairs or triples, for which source and target maps coincide in the obvious ways. The associator should satisfy the pentagon identity (3.6), and the unitors should satisfy the unitor laws (3.7).*

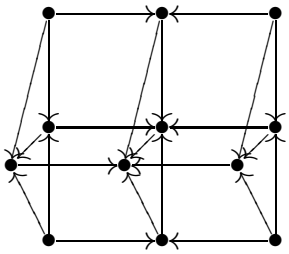
With this definition in mind, we recall Bénabou's classic example of a bicategory, that of spans, revieweed in Section 3.3. There is an analogous example here, namely double spans, or in our case *double cospans*.

## A.2 The Double Cospan Example

In Section 4.3, we described a Verity double bicategory of "double cospans", $2\mathrm{Cosp}(\mathbf{C})_0$. This notation is intended to suggest it derives from a larger structure, $2\mathrm{Cosp}(\mathbf{C})$, which is a double bicategory, as we shall show shortly. It is analogous to the "profunctor-based examples" of pseudo-double categories described by Grandis and Paré [46]. The Verity double bicategory described above is derived from it. To see these facts, we first define $2\mathrm{Cosp}(\mathbf{C})$ explicitly:

**Definition 26** $2\mathrm{Cosp}(\mathbf{C})$ *is a double bicategory of* ***double cospans*** *in* $\mathbf{C}$*, consisting of the following:*

- *the bicategory of objects is* $\mathbf{Obj} = \mathrm{Cosp}(\mathbf{C})$
- *the bicategory of morphisms* **Mor** *has: as objects, cospans in* $\mathbf{C}$*; as morphisms, commuting diagrams of the form 4.21 (in subsequent diagrams we suppress the labels for clarity)*
- *as 2-morphisms, cospans of cospan maps, namely commuting diagrams of the following shape:*

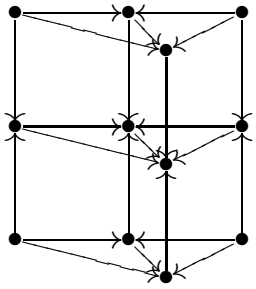


(A.2)

- *the bicategory of 2-morphisms has:*

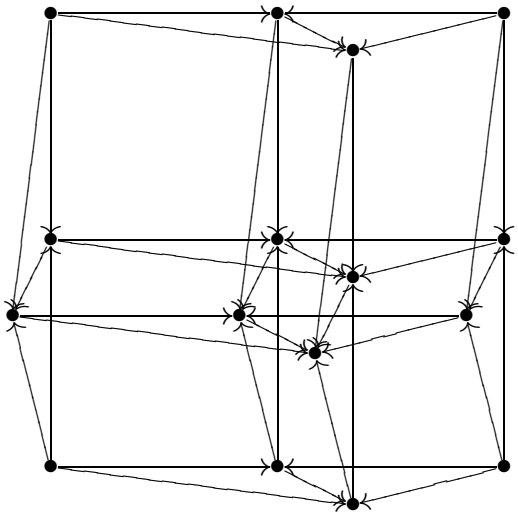
  - *as objects, cospan maps in* $\mathbf{C}$ *as in (3.9)*

  - *as morphisms, cospan maps of cospans:*



(A.3)

  - *as 2-morphisms, cospan maps of cospan maps:*



(A.4)

*All composition operations are by pushout; source and target operations are the same as those for cospans.*

Note that we could of course make the dual definition for spans, which may be more natural (but is not what we need for the cobordism case).

**Remark 18** Just as 2-morphisms in **Mor** and morphisms in **2Mor** can be seen as diagrams which are "products" of a cospan with a map of cospans, 2-morphisms in **2Mor** are given by diagrams which are "products" of horizontal and vertical cospan maps. These have, in either direction, four maps of cospans, with objects joined by maps of cospans. Composition again is by pushout in composable pairs of diagrams.

The next lemma shows how this is really an example of a double bicategory:

**Lemma 6** *For any category* $\mathbf{C}$ *with pushouts,* $2\mathrm{Cosp}(\mathbf{C})$ *forms a double bicategory.*

**Proof**: **Mor** and **2Mor** are bicategories since the composition functors act just like composition in $\mathrm{Cosp}(\mathbf{C})$ in each column, and therefore satisfies the same axioms.

Since the horizontal and vertical directions are symmetric, we can construct functors between **Obj**, **Mor**, and **2Mor** with the properties of a bicategory simply by using the same constructions that turn each into a bicategory. In particular, the source and target maps from **Mor** to **Obj** and from **2Mor** to **Mor** are the obvious maps giving the ranges of the projection maps in (4.21). The partially defined (horizontal) composition maps $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$ and $\otimes_H : \mathbf{2Mor}^2 \to \mathbf{2Mor}$ are defined by taking pushouts of diagrams in $\mathbf{C}$, which exist for any compos-

able pairs of diagrams because **C** has pushouts. They are functorial since they are independent of composition in the horizontal direction. The associator for composition of morphisms is given in the pushout construction.

To see that this construction gives a double bicategory, we note that **Obj**, **Mor**, and **2Mor** as defined above are indeed bicategories. **Obj**, because Cosp(**C**) is a bicategory. **Mor** and **2Mor** because the morphism and 2-morphism maps from the composition, associator, and other functors required for an double bicategory give these the structure of bicategories as well.

Moreover, the composition functors satisfy the properties of a bicategory for just the same reason that composition of cospans (and spans) does, since each of the three maps involved are given by this construction. Thus, we have a double bicategory. □

## A.3 Decategorification

Our motivation for showing Lemma 6 is to get show that cobordisms with corners form a special example of a Verity double bicategory of double cospans in some suitable category **C**. We have described how to get a double bicategory of such structures, so to get what we want, we need to show how a Verity double bicategory can be a special kind of double bicategory. In particular, we need to

define conditions which allow us to speak of the action of 2-cells upon squares. It is helpful, in trying to understand what these are, to consider a "lower dimensional" example of a similar process.

In a double category, thought of as an internal category in **Cat**, we have data of four sorts, as shown in Table A.1.

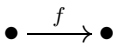
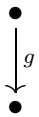
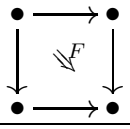
| | **Obj** | **Mor** |
|---|---|---|
| Objects |  |  |
| Morphisms |  |  |

**Table A.1:** Data of a Double Category

That is, a double category **DC** has categories **Obj** of objects and **Mor** of morphisms. The first column of the table shows the data of **Obj**: its objects are the objects of **DC**; its morphisms are the *vertical* morphisms. The second column shows the data of **Mor**: its objects are the *horizontal* morphisms of **DC**; its morphisms are the squares of **DC**.

**Remark 19** The kind of "decategorification" we will want to do to obtain Verity double bicategories has an analog in the case of double categories. Namely, there is a condition we can impose which effectively turns the double category into a category, where the horizontal and vertical morphisms are composable, and the

squares can be ignored. The sort of condition involved is similar to the *horn-filling conditions* introduced by Ross Street [80] in his first introduction of the idea of weak $\omega$-categories. In that case, all morphisms correspond to simplicial sets, and a horn filling condition is one which says that, for a given hollow simplex with just one face (morphism) missing from the boundary, there will be a morphism to fill that face, and a "filler" for the inside of the simplex, making the whole commute. A restricted horn-filling condition demands that this is possible for some class of candidate simplices.

For a double category, morphisms can be edges or squares, rather than $n$-simplices, but we can define the following "filler" condition: given any pair $(f, g)$ of a horizontal and vertical morphism where the target object of $f$ is the source object of $g$, there will be a unique pair $(h, \star)$ consisting of a unique horizontal morphism $h$ and unique invertible square $\star$ making the following diagram commute:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle h}\downarrow & \swarrow^{*} & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z \end{array} \tag{A.5}$$

and similarly when the source of $f$ is the target of $g$. Notice that taking $f$ or $g$ to be the identity in these cases implies $F$ is the identity.

If, furthermore, there are no other interesting squares, then this double category can be seen as just a category. In that case, the unique $h$ can just be

interpreted as the composite of $f$ and $g$ and $\star$ as the process of composition. So we will use the notation $g \circ f$ instead of $h$ in this situation.

To see that this defines a composition operation, we need to observe that composition defined using these fillers agrees with the usual composition in the horizontal or vertical categories, is associative, etc. For example, given morphisms as in the diagram:

$$\begin{array}{ccccc} w & \xrightarrow{f} & x & \xrightarrow{f'} & y \\ & & & & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z & \xrightarrow[1_z]{} & z \end{array} \tag{A.6}$$

there are two ways to use the unique-filler principle to fill this rectangle. One way is to first compose the pairs of horizontal morphisms on the top and bottom, then fill the resulting square. The square we get is unique, and the morphism is denoted $g \circ (f' \circ f)$. The second way is to first fill the right-hand square, and then using the unique morphism we call $g \circ f'$, we get another square on the left hand side, which our principle allows us to fill as well. The square is unique, and the resulting morphism is called $(g \circ f') \circ f$. Composing the two squares obtained this way must give the square obtained the other way, since both make the diagram commute, and both are unique. So we have:

$$\begin{array}{ccccc} w & \xrightarrow{f} & x & \xrightarrow{f'} & y \\ {\scriptstyle (g\circ f')\circ f}\downarrow & \swarrow^{*} & \downarrow{\scriptstyle g\circ f'} & \swarrow^{*} & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z & \xrightarrow[1_z]{} & z \end{array} \quad = \quad \begin{array}{ccc} w & \xrightarrow{f'\circ f} & y \\ {\scriptstyle g\circ(f'\circ f)}\downarrow & \swarrow^{*} & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z \end{array} \tag{A.7}$$

So in fact we can "decategorify" a double category satisfying the unique filler condition, and treat it as if it were a mere category with horizontal and vertical morphisms equivalent. The composition between horizontal and vertical morphisms is given by the filler: given one of each, we can find a square of the required kind, by taking the third side to be an identity.

**Remark 20** Note that our condition does not give a square for every possible combination of morphisms which might form its sources and targets. In particular, there must be an identity morphism—on the bottom in the example shown. If that identity could be any morphism $h$, then by choosing $f$ and $g$ to be identities, this would imply that every morphism must be invertible (at least weakly), since there must then be an $h^{-1}$ with $h^{-1} \circ h$ isomorphic to the identity. When a filler square does exist, and we consider **DB** as a category **C**, the filler square indicates there is a commuting square in **C**: we think of it as the identity between the composites along the upper right and lower left.

The decategorification of a double bicategory to give a Verity double bicategory is similar, except that whereas with a double category we were cutting down only the squares (the lower-right quadrant of Table A.1. We need to do more with a double bicategory, since there are more sorts of data, but they fall into a similar arrangement, as shown in Table A.2.

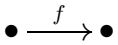
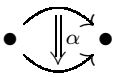
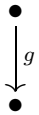
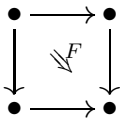
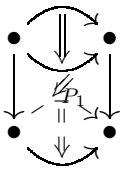
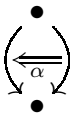
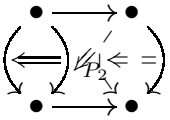
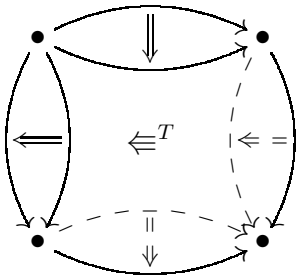
| | **Obj** | **Mor** | **2Mor** |
|---|---|---|---|
| Objects |   |   |   |
| Morphisms |   |   |   |
| 2-Cells |   |   |   |

**Table A.2:** The data of a double bicategory

**Remark 21** This shows the data of the bicategories **Obj**, **Mor**, and **2Mor**, each of which has objects, morphisms, and 2-cells. Note that the morphisms in the three entries in the lower right hand corner—2-cells in **Mor**, and morphisms and 2-cells in **2Mor**—are not 2-dimensional. The 2-cells in **Mor** and morphisms in **2Mor** are the three-dimensional "filling" inside the illustrated cylinders, which each have two square faces and two bigonal faces.

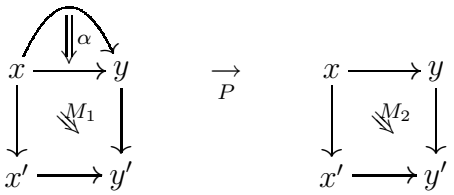
The 2-cells in **2Mor** should be drawn 4-dimensionally. The picture illustrated can be thought of as taking both square faces of one cylinder $P_1$ to those of

another, $P_2$, by means of two other cylinders ($S_1$ and $S_2$, say), in such a way that $P_1$ and $P_2$ share their bigonal faces. This description works whether we consider the $P_i$ to be horizontal and the $S_j$ vertical, or vice versa. These describe the "frame" of this sort of morphism: the "filling" is the 4-dimensional track taking $P_1$ to $P_2$, or equivalently, $S_1$ to $S_2$ (just as a square in a double category can be read horizontally or vertically). Not all relevant parts of the diagram have been labelled here, for clarity.

Next we want to describe a condition similar to the one we gave which made it possible to think of a double category as a category. In that case, we got a condition which effectively allowed us to treat any square as an identity, so that we only had objects and morphisms. Here, we want a condition which lets us throw away the three entries of table A.2 in the bottom right. This condition, when satisfied, should allow us to treat a double bicategory as a Verity double bicategory. It comes in two parts:

**Definition 27** *We say that a double bicategory satisfies the* ***vertical action condition*** *if, for any morphism $M_1 \in$* **Mor** *and 2-morphism $\alpha \in$* **Obj** *such that $s(M_1) = t(\alpha)$, there is a morphism $M_2 \in$* **Mor** *and 2-morphism $P \in$* **Mor** *such*

*that $P$ fills the "pillow diagram":*




(A.8)

*where $M_2$ is the back face of this diagram, and the 2-morphism in* **Obj** *at the bottom is the identity.*

*An double bicategory satisfies the* ***horizontal action condition*** *if for any morphism $M_1 \in$* **Mor** *and object $\alpha$ in* **2Mor** *with $s(M_1) = t(\alpha$ there is a morphism $M_2 \in$* **Mor** *and morphism $P \in$* **2Mor** *such that $P$ fill the pillow diagram which is the same as (A.8) turned sideways.*

Here, $M_2$ is the square which will eventually be named $M_1 \star_V \alpha$ when we define an action of 2-cells on squares.

**Remark 22** One can easily this condition is analogous to our filler condition (A.5) in a double category by turning the diagram (A.8) on its side. What the diagram says is that when we have a square with two bigons—the top one arbitrary and the bottom one the identity—there is another square $M_2$ (the back face of a pillow diagram) and a filler 2-morphism $P \in$ **2Mor** which fills the diagram. If one imagines turning this diagram on its side and viewing it obliquely, one sees precisely (A.5), as a dimension has been suppressed. What is a square in (A.5) is

a cylinder (2-morphism in **2Mor**); the roles of both squares and bigons in (A.8) are played by arrows in (A.5); arrows in (A.8) become pointlike objects in (A.5).

However, to get the compatibility between horizontal and vertical actions, we need something more than this. In particular, since these involve both horizontal and vertical cylinders (3-dimensional morphisms in the general sense), the compatibility condition must correspond to the 4-dimensional 2-cells in **2Mor**, shown in the lower right corner of Table A.2.

To draw necessary condition is difficult, since the necessary diagram is four-dimensional, but we can describe it as follows:

**Definition 28** *We say a double bicategory satisfies the* ***action compatibility condition*** *if the following holds. Suppose we are given*

- *a morphism* $F \in$ **Mor**
- *an object* $\alpha \in$ **2Mor** *whose target in* **Mor** *is a source of* $F$
- *a 2-cell* $\beta \in$ **Obj** *whose target morphism is a source of* $F$
- *an invertible morphism* $P_1 \in$ **2Mor** *with* $F$ *as source, and the objects* $\alpha$ *and* id *in* **2Mor** *as source and target*
- *an invertible 2-cell* $P_2 \in$ **Mor** *with* $F$ *as source, and the 2-cells* $\beta$ *and* id *in* **Mor** *as source and target*

*where $P_1$ and $P_2$ have, as targets, morphisms in* **Mor** *we call $\alpha \star F$ and $\beta \star F$ respectively. Then there is a unique morphism $\hat{F}$ in* **Mor** *and 2-cell $T$ in* **2Mor** *having all of the above as sources and targets.*

Geometrically, we can think of the unique 2-cell in **2Mor** as resembling the structure in the bottom right corner of Table A.2. This can be seen as taking one horizontal cylinder to another in a way that fixes the (vertical) bigons on its sides, by means of a translation which acts on the front and back faces with a pair of vertical cylinders (which share the top and bottom bigonal faces). Alternatively, it can be seen as taking one vertical cylinder to another, acting on the faces with a pair of horizontal cylinders. In either case, the cylinders involved in the translation act on the faces, but the four-dimensional interior, $T$, acts on the original cylinder to give another. The simplest interpretation of this condition is that it is precisely the condition needed to give the compatibility condition (4.13).

**Remark 23** Notice that the two conditions given imply the existence of unique data of three different sorts in our double bicategory. If these are the only data of these kinds, we can effectively omit them (since it suffices to know information about their sources and targets. This omission is part of a decategorification of the same kind we saw for the double category **DC**.

In particular, we use the above conditions to show the following:

**Lemma 7** *Suppose* **D** *is a double bicategory which has at most a unique morphism or 2-morphisms in* **2Mor**, *and at most a unique 2-morphism in* **Mor**, *having any specified sources and targets; and* **D** *satisfies the horizontal and vertical action conditions and the action compatibility condition; then* **D** *gives a Verity double bicategory in the sense of Verity.*

**Proof**: **D** consists of bicategories $(\mathbf{Obj}, \mathbf{Mor}, \mathbf{2Mor})$ together with all required maps (three kinds of source and target maps, two kinds of identity, three partially-defined compositions, left and right unitors, and the associator), satisfying the usual properties. To begin with, we describe how the elements of a Verity double bicategory **V** (definition 5) arise from this.

The horizontal bicategory **Hor** of **V** is simply **Obj**. The vertical bicategory **Ver** consists of the objects of each of **Obj**, **Mor**, and **2Mor**, where the required source, target and composition maps for **Ver** are just the object maps from those for **D**, which are all functors. We next check that this is a bicategory.

The source and target maps for **Ver** satisfy all the usual rules for a bicategory since the corresponding functors in **D** do. Similarly, the composition maps satisfy (3.3), (3.4) and (3.5) up to natural isomorphisms: they are just object maps of functors which satisfy corresponding conditions. We next illustrate this for composition.

In **D**, there is an associator 2-natural transformation. That is, a partially defined weak 2-functor $\alpha : \mathbf{Mor}^3 \to \mathbf{2Mor}$ satisfying the pentagon identity (strictly, since we are considering a *strict model* of the theory of bicategories). Among the data for $\alpha$ are the object maps, which give the maps for the associator in **Ver**. Since the associator 2-natural transformation satisfies the pentagon identity, so do these object maps. The other properties are shown similarly, so that **Ver** is a bicategory.

Next, we declare that the squares of **V** are the morphisms of **Mor**. Their vertical source and target maps are the morphism maps from the source and target functors from **Mor** to **Obj**. Their horizontal source and target maps are the internal ones in **Mor**. These satisfy equations (4.3) because the source and target maps of **D** are functors (in our special example of cospans, this amounts to the fact that (4.21) commutes).

The horizontal composition of squares (4.5) is just the composition of morphisms in **Mor**. Now, by assumption, **Mor** is a bicategory with at most unique 2-morphisms having any given source and target. If we declare these are identities (that is, identify their source and target morphisms), we get that horizontal composition is exactly associative and has exact identities.

The vertical composition of squares (4.4) is given by the morphism maps for the partially defined functor $\circ$ for **Mor**, and so composition here satisfies the

axioms for a bicategory. In particular, it has an associator and a unitor: but these must be morphisms in **2Mor** since we take the morphism maps from the associator and unitor functors (and the theory of bicategories says that these give 2-morphisms). But again, we can declare that there are only identity morphisms in **2Mor**, and this composition is exactly associative.

The interchange rule (4.6) follows again from functoriality of the composition functors.

The action of the 2-morphisms (bigons) on squares is guaranteed by the horizontal and vertical action conditions. In particular, by composition of in **Mor** or **2Mor**, we guarantee the existence of the categories of horizontal and vertical cylinders $\mathbf{Cyl_H}$ and $\mathbf{Cyl_V}$, respectively. These come from the 2-morphisms in **Mor** or morphisms in **2Mor** respectively which those conditions demand must exist. Taking these to be identities, the cylinders consist of commuting cylindrical diagrams with two bigons and two squares.

In the case where one bigon is the identity, and the other is any bigon $\alpha$, the conditions guarantee the existence of a cylinder, which we have declared to be the identity. This defines the effect of the action of $\alpha$ on the square whose source is the target of $\alpha$. If this square is $F$, we denote the other square $\alpha \star_H F$ or $\alpha \star_V F$ as appropriate.

The condition (4.11) guaranteeing independence of the horizontal and vertical actions follows from the action compatibility condition. For suppose we have a square $F$ whose horizontal and vertical source arrows are the targets of 2-cells $\alpha$ and $\beta$, and attach to its opposite faces two identity 2-cells. Then the horizontal and vertical action conditions mean that there will be a square $\alpha \star_H F$ and a square $\beta \star_V F$). Then the action compatibility condition applies (the $P_i$ are the identities we get from the action condition), and there is a morphism in **Mor**, namely a square in **V** and a 2-cell $T \in$ **2Mor**. Consider the remaining face, which the action condition suggests we call $\alpha \star_H (\beta \star_V F)$ or $\beta \star_V (\alpha \star_H F)$, depending on the order in which we apply them. The compatibility condition says that there is a unique square which fills this spot so the two must be equal.

Now suppose we have three composable squares—that is, morphisms $F$, $G$, and $H$ in **Mor**, which are composable along shared source and target objects in **Mor**. The associator functor has an object map, giving objects in **2Mor** at the "top" and "bottom" of the squares. It also has a morphism map, giving morphisms in **2Mor**. But by assumption there is only a unique such map between , these associators must be the unique morphism in **2Mor** with source $(H \circ G) \circ F$ and target $H \circ (G \circ F)$. Then by the vertical action condition, we have a filler 2-morphism in **Mor** for the action on the composite square by the top associator, and then, taking the result and composing with the bottom associator, we get

another filler. This must be the unique map between the two composites—which is the identity, since they have the same sources and targets. So we get a commuting cylinder. Composing squares along source and target morphisms in **Obj** works the same way by a symmetric argument.

The condition (4.15) is similar—the unitor functor will give the unique morphism in **2Mor**, and the action compatibility condition gives the commuting cylinder for unitors on the composite of squares.

So from any such double bicategory we get a Verity double bicategory. □

**Remark 24** It is interesting to note how these arguments apply to the case when we are looking at constructions in 2Cosp(**C**), as will be the case in *nCob*.

In particular, the interchange rules hold because the middle objects in the four squares being composed form the vertices of a new square. The pushouts in the vertical and horizontal direction form the middle objects of vertical and horizontal cospans over these. The interchange law means that the pushout (in the horizontal direction) of the objects from the vertical cospans is in the same isomorphism class as the pushout (in the vertical direction) of the objects from the horizontal cospans. This is true because of the universal property of the pushout.

The horizontal and vertical 2-morphisms are maps of cospans, and act on the squares by composition of morphisms in **C**: given a square $M$ with four maps $P_i$ and $\Pi_i$ to the edges as in (4.21); and a morphism of cospans on any edge

(for definiteness, say the top), where the **C**-morphism in the middle is $S \xrightarrow{f} \tilde{S}$. Then the composite $f \circ P_1 : M \to \tilde{S}$ is a source (or target) map to the cospan $X \xrightarrow{\iota_1} \tilde{S} \xleftarrow{\iota_2} Y$. The result is again a square. In particular, composition of internal maps in horizontal and vertical morphism of cospans with the projections in a square are independent.