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Higher Gauge Theory: 2-Bundles

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by

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Tobias Keith Bartels  
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The Dissertation of Tobias Keith Bartels is approved:

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ABSTRACT OF THE DISSERTATION

Higher Gauge Theory: 2-Bundles

by

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Dr. John C. Baez, Chairperson

I categorify the definition of fibre bundle, replacing smooth manifolds with differentiable categories, Lie groups with coherent Lie 2-groups, and bundles with a suitable notion of 2-bundle. To link this with previous work, I show that certain 2-categories of principal 2-bundles are equivalent to certain 2-categories of (nonabelian) gerbes. This relationship can be (and has been) extended to connections on 2-bundles and gerbes.

The main theorem, from a perspective internal to this paper, is that the 2-category of 2-bundles over a given 2-space under a given 2-group is (up to equivalence) independent of the fibre and can be expressed in terms of cohomological data (called 2-transitions). From the perspective of linking to previous work on gerbes, the main theorem is that when the 2-space is the 2-space corresponding to a given space and the 2-group is the automorphism 2-group of a given group, then this 2-category is equivalent to the 2-category of gerbes over that space under that group (being described by the same cohomological data).

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# Introduction

Gauge theory, built on the theory of fibre bundles, has underlain all verified theories of basic physics for half a century. Geometrically, there is a direct relationship between a connection on a principal bundle under a group —which (at least relative to local coordinates) assigns group elements to paths in spacetime— and the effects of a physical field on particles that (in the Feynman interpretation) travel paths in spacetime. On a more purely mathematical side, studying bundles over a given space  $B$  under a given group  $G$  allows one to approach the  $G$ -valued degree-1 cohomology of  $B$ ; this can be also be expressed sheaf-theoretically, which can be generalised, ultimately, to degree-1 cohomology valued in sheaves of groups. Indeed, everything done with bundles can be done with sheaves.

More recently, much of this has been categorified (see below) with gerbes. This began on the mathematical side, with degree-2 cohomology valued in (nonabelian) groups (or sheaves thereof); but there is also hope for physical applications. A categorified theory of bundles could describe physical theories of strings, which are (in spacetime) 2-dimensional. Before this work began, several authors [BM, Att, GP]

have begun discussing connections on gerbes. However, gerbes are special stacks, which in turn are a categorification of sheaves; they are not directly a categorification of bundles, and bundles are more familiar to working physicists. The goal of this paper is to develop, analogously to a direct development of fibre bundles, a theory of categorified fibre bundles. Rather than invent a new term, I call these *2-bundles*.

As sheaves are more general than bundles, so gerbes are more general than 2-bundles. However, I also have the opportunity here to generalise from groups to 2-groups. 2-Groups [HDA5] are a categorification of groups, although any group may be made into a 2-group, the *automorphism 2-group* of the group. Previous work on gerbes implicitly uses 2-groups, but usually only automorphism 2-groups of groups. Abelian gerbes use a different method (that applies only to abelian groups) of turning a group into a 2-group. Lawrence Breen [Bre90] and Branislav Jurčo [Jur] have discussed using crossed modules, which include the above as special cases. In a theory of 2-bundles, however, it is natural to use general 2-groups from the beginning; indeed, I will use the most general (properly weakened) *coherent 2-groups*. (Crossed modules are equivalent to ‘strict’ 2-groups.) Thus, this paper is at once both more general and less general than work on gerbes; however, it is to be hoped that gerbes themselves will also be generalised to (stacks of) coherent 2-groups in the future.

Categorifying the first paragraph of this Introduction, one expects to say that higher gauge theory, built on a theory of 2-bundles, can underlie stringy theories of physics:

Geometrically, there is a direct relationship between a connection on a principal 2-bundle under a 2-group—which (at least relative to local coordinates) assigns 2-group object-elements to paths in spacetime and 2-group morphism-elements to surfaces in spacetime—and the effects of a physical field on (open) strings that travel surfaces between paths in spacetime.

Although this paper does not go into connections on 2-bundles, John Baez and Urs Schreiber have begun that theory (based on earlier versions of this paper) in [HGTc] and [HGT]. Schreiber has even applied this to physics in his doctoral dissertation [Sch].

There is another approach to gerbes that, like this one, comes closer to the traditional view of bundles: the concept of bundle gerbe (introduced in [Mur], generalised to arbitrary groups in [ACJ], and generalised to crossed modules in [Jur]). This approach removes all (explicit) category theory, both the category theory involved in extending bundles to sheaves and the categorification involved in raising to a higher dimension. Like them, I also downplay the first aspect, but the second is my prime motivator, a concept of central importance to this exposition.

In general, *categorification* [CF, BD] is a project in which mathematical structures are enriched by replacing equations between elements of a set with specific isomorphisms between objects in a category. The resulting categorified concept is generally richer, because a given pair of isomorphic objects might have many isomorphisms

between them (whose differences were previously ignored), and other interesting morphisms (previously invisible) might not be invertible at all. The classic historical example is the revolution in homology theory, begun by Leopold Vietoris and Emmy Noether at the beginning of the 20th century (see [ML] for a brief history), in which they replaced Betti numbers (elements of a set) with homology groups (objects of a category) and studied induced group homomorphisms.

Previous work using only *strict* Lie 2-groups (as by Breen [Bre90] in the guise of crossed modules and by Baez [Baez] explicitly) violates the spirit of categorification, since the axioms for a strict 2-group require *equations* between certain objects; a more natural definition from the perspective of categorification should replace these with *isomorphisms*, and then enrich the theory by positing coherence laws between those isomorphisms. The *coherent* 2-groups do precisely this; I use them throughout. Although not directly relevant to this paper, Baez and Alissa Crans [HDA6] have developed a corresponding theory of Lie 2-algebras; here one finds a canonical 1-parameter family of categorifications of a Lie algebra— but these categorifications must be so called ‘weak’ Lie 2-algebras (which correspond to coherent Lie 2-groups).

# Part 1

## Uncategorified bundles

To begin with, I review the theory of bundles with a structure group, which is what I intend to categorify. Some of this exposition covers rather elementary material; my purpose is to summarise that material in purely arrow-theoretic terms, to make clear how it is to be categorified in part 2.

### 1.1 Categorical preliminaries

Here, I review the necessary category theory, which will become 2-category theory in section 2.1.

In theory, I work with a specific category  $\mathbf{C}$ , to be described in section 1.2. (It's no secret; it's just the category of smooth manifolds and smooth functions. But I don't want to focus attention on that fact yet.) However, I will not normally make reference to the specific features of this category; I will just refer to it as the category

$\mathbf{C}$ . Thus the ideas in this paper generalise immediately to other categories, such as categories of topological spaces, algebraic varieties, and the like. All that matters is that  $\mathbf{C}$  support the categorical operations described below.

### 1.1.1 Notation and terminology

A **space** is an object in  $\mathbf{C}$ , and a **map** is a morphism in  $\mathbf{C}$ . Uppercase italic letters like ‘ $X$ ’ denote spaces, and lowercase italic letters like ‘ $x$ ’ denote maps. However, the identity morphism on  $X$  will be denoted ‘ $\underline{X}$ ’.

In general, the names of maps will label arrows directly, perhaps in a small inline diagram like  $X \xrightarrow{x} Y \xrightarrow{y} Z$ , which denotes a composition of maps, or perhaps in a huge commutative diagram. I will endeavour to make such diagrams easy to read by orienting maps to the right when convenient, or if not then downwards. Sometimes I will place a lowercase Greek letter like ‘ $\psi$ ’ inside a commutative diagram, which at this stage is merely a convenient label for the diagram. (When a small commutative diagram, previously discussed, appears inside of a large, complicated commutative diagram later on, these labels can help you keep track of what’s what.) When I categorify the situation in part 2, however, the meaning of these labels will be upgraded to natural transformations.

Except in section 1.2 (at least within part 1), my discussion is in purely arrow-theoretic terms, making no direct reference to points, open subsets, and the like. However, to make the connection to more set-theoretic presentations, there is a useful

generalised arrow-theoretic concept of element. Specifically, an **element**  $x$  of a space  $Y$  is simply a space  $X$  together with a map  $X \xrightarrow{x} Y$ . In certain contexts, the map  $x$  really has the same information in it as an ordinary (set-theoretic) element of (the underlying set of)  $Y$ . On the other hand, if  $X$  is chosen differently, then the map  $x$  may describe more complicated features, like curves. The most general element is in fact the identity map  $X \xrightarrow{X} X$ . You (as reader) may imagine the elements as set-theoretic points if that helps, but their power in proofs lies in their complete generality.

### 1.1.2 Products

Given spaces  $X$  and  $Y$ , there must be a space  $X \times Y$ , the **Cartesian product** of  $X$  and  $Y$ . One can also form more general Cartesian products, like  $X \times Y \times Z$  and so on. In particular, there is a **singleton space**  $1$ , which is the Cartesian product of no spaces.

The generalised elements of a Cartesian product may be taken to be ordered pairs; that is, given maps  $X \xrightarrow{x} Y$  and  $X \xrightarrow{x'} Y'$ , there is a **pairing map**  $X \xrightarrow{\langle x, x' \rangle} Y \times Y'$ ; conversely, given a map  $X \xrightarrow{y} Y \times Y'$ , there is a unique pair of maps  $X \xrightarrow{x} Y$  and  $X \xrightarrow{x'} Y'$  such that  $y = \langle x, x' \rangle$ . Similarly, given maps  $X \xrightarrow{x} Y$ ,  $X \xrightarrow{x'} Y'$ , and  $X \xrightarrow{x''} Y''$ , there is a map  $X \xrightarrow{\langle x, x', x'' \rangle} Y \times Y' \times Y''$ ; conversely, given a map  $X \xrightarrow{y} Y \times Y' \times Y''$ , there is a unique triple of maps  $X \xrightarrow{x} Y$ ,  $X \xrightarrow{x'} Y'$ , and  $X \xrightarrow{x''} Y''$  such that  $y = \langle x, x', x'' \rangle$ . And so on. Additionally, given any space  $X$ ,



there is a **trivial map**  $X \xrightarrow{\hat{X}} 1$ ; conversely, every map  $X \xrightarrow{y} 1$  must be equal to  $\hat{X}$ . (That is, the singleton space has a unique generalised element, once you fix the source  $X$ .) Inverting this notation, it's sometimes convenient to abbreviate the map  $X \xrightarrow{\langle X, X \rangle} X \times X$  as  $X \xrightarrow{\hat{X}} X \times X$ .

I will only occasionally want to use the pairing maps directly; more often, I'll be working with product maps. Specifically, given maps  $X \xrightarrow{x} Y$  and  $X' \xrightarrow{x'} Y'$ , there is a **product map**  $X \times X' \xrightarrow{x \times x'} Y \times Y'$ . This respects composition; given  $X \xrightarrow{x} Y \xrightarrow{y} Z$  and  $X' \xrightarrow{x'} Y' \xrightarrow{y'} Z'$ , the product map  $X \times X' \longrightarrow Z \times Z'$  is  $X \times X' \xrightarrow{x \times x'} Y \times Y' \xrightarrow{y \times y'} Z \times Z'$ . This also respects identities; the product map of  $X \xrightarrow{X} X$  and  $X' \xrightarrow{X'} X$  is the identity map  $X \times X' \xrightarrow{X \times X'} X \times X'$ . Similarly, given maps  $X \xrightarrow{x} Y$ ,  $X' \xrightarrow{x'} Y'$ , and  $X'' \xrightarrow{x''} Y''$ , there is a product map  $X \times X' \times X'' \xrightarrow{x \times x' \times x''} Y \times Y' \times Y''$  with similar nice properties; and so on. The product maps and pairing maps fit together nicely: given  $X \xrightarrow{\langle x, x' \rangle} Y \times Y'$ ,  $x$  and  $x'$  may be reconstructed as (respectively)  $X \xrightarrow{\langle x, x' \rangle} Y \times Y' \xrightarrow{Y \times \hat{Y}'} Y$  and  $X \xrightarrow{\langle x, x' \rangle} Y \times Y' \xrightarrow{\hat{Y} \times Y'} Y'$ ; and the product map  $X \times X' \xrightarrow{x \times x'} Y \times Y'$  is equal to  $\langle x \times \hat{X}', \hat{X} \times x' \rangle$ .

$\mathbf{C}$  is a monoidal category under  $\times$ , so the Mac Lane Coherence Theorem [CWM, VII.2] applies, allowing me to treat  $X \times 1$  and  $X$  as the same space,  $X \xrightarrow{\hat{X}} X \times X \xrightarrow{X \times \hat{X}} X \times X \times X$  and  $X \xrightarrow{\hat{X}} X \times X \xrightarrow{\hat{X} \times X} X \times X \times X$  as the same well defined map, and so on. By Mac Lane's theorem, there will always be a canonical functor that links any identified pair; and these will all fit together properly. I will make such identifications

in the future without further comment. (In fact, I already did so towards the end of the previous paragraph.)

### 1.1.3 Pullbacks

Unlike products, pullbacks [CWM, III.4] need not always exist in  $\mathbf{C}$ . Thus, I will have to treat this in more detail, so that I can discuss exactly what it means for a pullback to exist.

A **pullback diagram** consists of spaces  $X$ ,  $Y$ , and  $Z$ , and maps  $X \xrightarrow{x} Z$  and  $Y \xrightarrow{y} Z$ :

$$\begin{array}{ccc} & X & \\ & \downarrow x & \\ Y & \xrightarrow{y} & Z \end{array} \quad (1.1)$$

Given a pullback diagram, a **pullback cone** is a space  $C$  together with maps  $C \xrightarrow{z} X$  and  $C \xrightarrow{w} Y$  that make the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{z} & X \\ w \downarrow & \omega & \downarrow x \\ Y & \xrightarrow{y} & Z \end{array}$$

(Recall that here  $\omega$  is just a name for the commutative diagram, to help keep track of it.) Given pullback cones  $C$  and  $C'$ , a **pullback cone morphism** from  $C$  to  $C'$  is a map  $C \xrightarrow{u} C'$  that makes the following diagrams commute:

$$\begin{array}{ccc} C & & \\ w \downarrow & \searrow u & \\ & C' & \\ \psi \downarrow & & \downarrow w' \\ & Y & \end{array} \quad \begin{array}{ccc} C & \xrightarrow{z} & X \\ u \searrow & \chi & \searrow \\ & C' & \xrightarrow{z'} & X \end{array}$$

A **pullback** of the given pullback diagram is a pullback cone  $P$  that is *universal* in the sense that, given any other pullback cone  $C$ , there is a unique pullback cone morphism from  $C$  to  $P$ . Pullbacks always exist in, for example, the category of sets (as the subset of  $X \times Y$  consisting of those pairs whose left and right components are mapped to the same element of  $Z$ ). However, they don't always exist for the category  $\mathbf{C}$ , so I will need to make sure that they exist in the relevant cases.

In the rest of this paper, I will want to define certain spaces as pullbacks, when they exist. If such definitions are to be sensible, then I must show that it doesn't matter which pullback one uses.

**Proposition 1** *Given the pullback diagram (1.1) and pullbacks  $P$  and  $P'$ , the spaces  $P$  and  $P'$  are isomorphic in  $\mathbf{C}$ .*

**Proof:** Since  $P$  is universal, there is a pullback cone morphism  $P' \xrightarrow{u} P$ :

$$\begin{array}{ccccc}
 P' & \xrightarrow{z'} & & & \\
 \downarrow u & \searrow \chi & & & \\
 P & \xrightarrow{z} & X & & \\
 \downarrow \psi & \downarrow w & \downarrow \omega & & \downarrow x \\
 Y & \xrightarrow{y} & Z & & \\
 \uparrow w' & & & & 
 \end{array}$$

Since  $P'$  is universal, there is also a pullback cone morphism  $P \xrightarrow{\bar{u}} P'$ :

$$\begin{array}{ccccc}
 P & \xrightarrow{z} & & & \\
 \downarrow \bar{u} & \searrow \bar{\chi} & & & \\
 P' & \xrightarrow{z'} & X & & \\
 \downarrow \bar{\psi} & \downarrow w' & \downarrow \omega' & & \downarrow x \\
 Y & \xrightarrow{y} & Z & & \\
 \uparrow w & & & & 
 \end{array}$$

Composing these one way, I get a pullback cone morphism from  $P$  to itself:

$$\begin{array}{ccccc}
 P & & & & \\
 \downarrow \bar{u} & \xrightarrow{z} & & & \\
 P' & \xrightarrow{z'} & \bar{\chi} & & \\
 \downarrow u & \searrow \chi & & & \\
 P & \xrightarrow{z} & X & & \\
 \downarrow \psi & \searrow \omega & \downarrow x & & \\
 Y & \xrightarrow{y} & Z & & \\
 \uparrow \bar{\psi} & & & & \\
 w & & & & 
 \end{array}
 \tag{1.2}$$

But since the identity on  $P$  is also a pullback cone morphism, the universal property shows that  $P \xrightarrow{\bar{u}} P' \xrightarrow{u} P$  is the identity on  $P$ . Similarly,  $P' \xrightarrow{u} P \xrightarrow{\bar{u}} P'$  is the identity on  $P'$ . Therefore,  $P$  and  $P'$  are isomorphic. ■

### 1.1.4 Equivalence relations

Given a space  $U$ , a **binary relation** on  $U$  is a space  $R^{[2]}$  equipped with maps  $R^{[2]} \xrightarrow{j_{[0]}} U$  and  $R^{[2]} \xrightarrow{j_{[1]}} U$  that are *jointly monic*; this means that given generalised elements  $X \xrightarrow{x} R^{[2]}$  and  $X \xrightarrow{y} R^{[2]}$  of  $R^{[2]}$ , if these diagrams commute:

$$\begin{array}{ccc}
 X & \xrightarrow{x} & R^{[2]} \\
 \downarrow y & \searrow \chi_{[0]} & \downarrow j_{[0]} \\
 R^{[2]} & & U \\
 & \searrow j_{[0]} & \\
 & & U
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{x} & R^{[2]} \\
 \downarrow y & \searrow \chi_{[1]} & \downarrow j_{[1]} \\
 R^{[2]} & \xrightarrow{j_{[1]}} & U
 \end{array}
 \tag{1.3}$$

then  $x = y$ . The upshot of this is that an element of  $R^{[2]}$  is determined by two elements of  $U$ , or equivalently by a single element of  $U \times U$ ; in other words, the set of elements (with given domain) of  $R^{[2]}$  is a subset of the set of elements of  $U \times U$ . (Thus the link with the set-theoretic notion of binary relation.)

A binary relation is **reflexive** if it is equipped with a *reflexivity map*  $U \xrightarrow{j_{[00]}} R^{[2]}$  that is a section of both  $j_{[0]}$  and  $j_{[1]}$ ; that is, the following diagrams commute:

$$\begin{array}{ccc}
 U & \xrightarrow{j_{[00]}} & R^{[2]} \\
 \omega_{[0]} \downarrow & & \downarrow j_{[0]} \\
 U & & U
 \end{array}
 \qquad
 \begin{array}{ccc}
 U & \xrightarrow{\underline{U}} & U \\
 \omega_{[1]} \downarrow & & \downarrow j_{[1]} \\
 R^{[2]} & \xrightarrow{j_{[1]}} & U
 \end{array}$$

I say ‘equipped with’ rather than ‘such that there exists’, but a reflexivity map (if extant) is unique by the relation’s joint monicity. In terms of generalised elements, given any element  $X \xrightarrow{x} U$  of  $U$ , composing with  $j_{[00]}$  gives an element of  $R^{[2]}$ ; the commutative diagrams state that this corresponds to the element  $\langle x, x \rangle$  of  $U \times U$ . (Thus the link with the set-theoretic notion of reflexivity.)

Assume that the kernel pair of  $j_{[0]}$  exists, and let it be the space  $R^{[3]}$ :

$$\begin{array}{ccc}
 R^{[3]} & \xrightarrow{j_{[02]}} & R^{[2]} \\
 j_{[01]} \downarrow & \omega_{[00]} & \downarrow j_{[0]} \\
 R^{[2]} & \xrightarrow{j_{[0]}} & U
 \end{array}$$

A binary relation is **right Euclidean** if it is equipped with a (right) *Euclideaness map*  $R^{[3]} \xrightarrow{j_{[12]}} R^{[2]}$  making these diagrams commute:

$$\begin{array}{ccc}
 R^{[3]} & \xrightarrow{j_{[12]}} & R^{[2]} \\
 j_{[01]} \downarrow & \omega_{[01]} & \downarrow j_{[1]} \\
 R^{[2]} & \xrightarrow{j_{[0]}} & U
 \end{array}
 \qquad
 \begin{array}{ccc}
 R^{[3]} & \xrightarrow{j_{[12]}} & R^{[2]} \\
 j_{[02]} \downarrow & \omega_{[11]} & \downarrow j_{[1]} \\
 R^{[2]} & \xrightarrow{j_{[1]}} & U
 \end{array}$$

Like the reflexivity map, any Euclideaness map is unique. In terms of elements, each

element of  $R^{[3]}$  is uniquely defined by elements  $x$ ,  $y$ , and  $z$  of  $U$  such that  $\langle x, y \rangle$  and  $\langle x, z \rangle$  give elements of  $R^{[2]}$ . The Euclidean condition states precisely that under these circumstances,  $\langle y, z \rangle$  also gives an element of  $R^{[2]}$ . As Euclid [Euc, CN1] put it (in Heath’s translation), referring to magnitudes of geometric figures, ‘Things which are equal to the same thing are also equal to one another.’; hence the name.

Putting these together, an **equivalence relation** is a binary relation that is both reflexive and Euclidean. (The usual definition uses symmetry and transitivity instead of Euclidean; in the presence of reflexivity, however, these are equivalent. Using Euclidean —more to the point, avoiding explicit mention of symmetry— helps me to maintain a consistent orientation on all diagrams; it also privileges  $j_{[0]}$ , which seems bad at first but is actually desirable in light of the axioms for covers in section 1.1.5.)

Given an equivalence relation as above, a **quotient** of the equivalence relation is a space  $R^{[0]}$  and a map  $U \xrightarrow{j} R^{[0]}$  that *coequalises*  $j_{[0]}$  and  $j_{[1]}$ . This means, first, that this diagram commutes:

$$\begin{array}{ccc} R^{[2]} & \xrightarrow{j_{[1]}} & U \\ j_{[0]} \downarrow & \omega & \downarrow j \\ U & \xrightarrow{j} & R^{[0]} \end{array}$$

and also that, given any map  $U \xrightarrow{x} R^{[0]}$  such that this diagram commutes:

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{j_{[1]}} & U \\ j_{[0]} \downarrow & \omega_x & \downarrow x \\ U & \xrightarrow{x} & X \end{array} \tag{1.4}$$

there is a *unique* map  $R^{[0]} \xrightarrow{\tilde{x}} X$  such that this diagram commutes:

$$\begin{array}{ccc}
 U & \xrightarrow{x} & X \\
 \searrow j & \tilde{\omega}_x & \searrow \\
 R^{[0]} & \xrightarrow{\tilde{x}} & X
 \end{array} \tag{1.5}$$

Thus if an equivalence relation has a quotient, then I can define a map out of this quotient by defining a map (with certain properties) out of the relation's base space.

### 1.1.5 Covers

I will require a notion of (open) cover expressed in a purely arrow-theoretic way. For me, **covers** will be certain maps  $U \xrightarrow{j} B$  in the category  $\mathbf{C}$ . I'll describe which maps these are in section 1.2.2 (although again it's no secret: they're the surjective local diffeomorphisms); for now, I list and examine the category-theoretic properties that covers must have.

Here are the properties of covers that I will need:

- All isomorphisms are covers;
  - A composite of covers is a cover;
  - The pullback of a cover along any map exists and is a cover;
  - The quotient of every equivalence relation involving a cover exists and is a cover;
- and
- Every cover is the quotient of its kernel pair.

(Compare [Ele1, B.1.5.7].)

I will be particularly interested in the pullback of  $U$  along itself, that is the pullback of this diagram:

$$\begin{array}{ccc} & & U \\ & & \downarrow j \\ U & \xrightarrow{j} & B \end{array}$$

This pullback,  $U^{[2]}$ , is the **kernel pair** of  $U$ ; it amounts to an open cover of  $B$  by the pairwise intersections of the original open subsets. In the pullback diagram:

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{j_{[0]}} & U \\ j_{[1]} \downarrow & \omega & \downarrow j \\ U & \xrightarrow{j} & B \end{array} \quad (1.6)$$

$j_{[0]}$  may be interpreted as the (collective) inclusion of  $U_k \cap U_{k'}$  into  $U_k$ , while  $j_{[1]}$  is the inclusion of  $U_k \cap U_{k'}$  into  $U_{k'}$ . But I will not refer to these individual intersections, only to the kernel pair  $U^{[2]}$  as a whole.

Notice that this commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\underline{U}} & U \\ \underline{U} \downarrow & \not\parallel & \downarrow j \\ U & \xrightarrow{j} & B \end{array}$$

is a pullback cone for (1.6), so the universal property of that pullback defines a map  $U \xrightarrow{j_{[00]}} U^{[2]}$ . This map may be thought of as the inclusion of  $U_k$  into  $U_k \cap U_{k'}$ . (But don't let this trick you into thinking that  $j_{[00]}$  is an isomorphism; in general, most elements of  $U^{[2]}$  are not of that form.) This universal map comes with the following



commutative diagrams:

$$\begin{array}{ccc}
 U & & \\
 \omega_{[1]} \downarrow & \searrow^{j_{[00]}} & \\
 U & \xrightarrow{\quad} & U \\
 \omega_{[1]} \downarrow & & \omega_{[0]} \downarrow \\
 U & & U
 \end{array}
 \qquad
 \begin{array}{ccc}
 U & \xrightarrow{\quad} & U \\
 \omega_{[0]} \downarrow & \searrow & \\
 U & \xrightarrow{j_{[0]}} & U
 \end{array}$$

In one instance, I will even need the space  $U^{[3]}$ , built out of the triple intersections  $U_k \cap U_{k'} \cap U_{k''}$ . This space may also be defined as a pullback:

$$\begin{array}{ccc}
 U^{[3]} & \xrightarrow{j_{[01]}} & U^{[2]} \\
 j_{[12]} \downarrow & \omega_{[01]} & \downarrow j_{[1]} \\
 U^{[2]} & \xrightarrow{j_{[0]}} & U
 \end{array}$$

Notice that this commutative diagram:

$$\begin{array}{ccccc}
 U^{[3]} & \xrightarrow{j_{[01]}} & U^{[2]} & & \\
 j_{[12]} \downarrow & \omega_{[01]} & \downarrow j_{[1]} & \searrow^{j_{[0]}} & \\
 U^{[2]} & \xrightarrow{j_{[0]}} & U & \xrightarrow{\omega} & U \\
 j_{[1]} \searrow & \omega & \searrow^j & & \downarrow j \\
 & U & \xrightarrow{j} & & B
 \end{array}$$

is a pullback cone for (1.6), so the universal property of that pullback defines a map  $U^{[3]} \xrightarrow{j_{[02]}} U^{[2]}$ . This map may be thought of as the inclusion of  $U_k \cap U_{k'} \cap U_{k''}$  into  $U_k \cap U_{k''}$ . This universal map comes with the following commutative diagrams:

$$\begin{array}{ccc}
 U^{[3]} & \xrightarrow{j_{[02]}} & U^{[2]} \\
 j_{[12]} \downarrow & \omega_{[11]} & \downarrow j_{[1]} \\
 U^{[2]} & \xrightarrow{j_{[1]}} & U
 \end{array}
 \qquad
 \begin{array}{ccc}
 U^{[3]} & \xrightarrow{j_{[01]}} & U^{[2]} \\
 j_{[02]} \downarrow & \omega_{[00]} & \downarrow j_{[0]} \\
 U^{[2]} & \xrightarrow{j_{[0]}} & U
 \end{array}$$

These squares are in fact pullbacks themselves.

## 1.2 The category of spaces

Here I want to explain how the category of smooth manifolds has the properties needed from section 1.1. This section (and this section alone) is not analogous to the corresponding section in part 2.

A **space** (object) is a smooth manifold, and a **map** (morphism in  $\mathbf{C}$ ) is a smooth function. A composite of smooth functions is smooth, and the identity function on any smooth manifold is smooth, so this forms a category  $\mathbf{C}$ . An isomorphism in this category is a diffeomorphism.

If the space  $X$  consists of a single point, then the map  $X \xrightarrow{x} Y$  has the same information in it as an ordinary (set-theoretic) element of  $Y$ . Furthermore, if two maps from  $Y$  to  $Z$  agree on each of these elements, then they are equal. Thus when I speak of generalised elements, you (as reader) may think of set-theoretic elements; everything will make perfect sense.

### 1.2.1 Products and pullbacks

The singleton space  $1$  is the connected 0-dimensional manifold, which consists of a single point. Every function to this space is smooth, and there is always a unique function from any space to  $1$ . Similarly, the product of smooth manifolds  $X$  and  $Y$  exists, although I won't spell it in so much detail.

Pullbacks are trickier, because they don't always exist. For example, in the pullback diagram (1.1), take  $X$  to be the singleton space,  $Z$  to be the real line, and  $Y$  to be the plane  $Z \times Z$ ; let  $x$  be the constant map to the real number 0, and let  $y$  be multiplication. Then the pullback of this diagram, if it existed, would be the union of the two axes in  $Y$  —but that is not a manifold. (It is possible to generalise the notion of manifold; Baez & Schreiber try this in [HGT, Definition 4ff]. However, it's enough here that certain pullbacks exist.)

When pullbacks do exist, they are the spaces of solutions to equations. That is why the counterexample above should have been the set of solutions to the equation  $(xy) = 0$  (where  $x$  and  $y$  are elements of  $Z$ , so that  $\langle x, y \rangle$  is an element of  $Y \cong Y \times 1$ ). A useful example is where one of the maps in the pullback is the inclusion of a subset; then the pullback is the preimage of that subset along the other map. If the subset is open, then this always exists (is a manifold) and always is itself an open subset. This is the first part of the idea of cover; but we need to modify this so that covers are regular epimorphisms (which inclusion of a proper open subset never is).

### 1.2.2 Open covers

The usual notion of open cover of the space  $B$  is an index set  $K$  and a family  $(U_k : k \in K)$  of open subsets of  $B$ , with union  $\bigcup_k U_k = B$ . This family can be combined into the disjoint union  $U := \{(k, x) \in K \times B : x \in U_k\}$  and the single map

$U \xrightarrow{j} B$  given by  $(k, x) \mapsto x$ . Notice that this map  $j$  is a surjective local diffeomorphism.

Conversely, consider any surjective local diffeomorphism  $U \xrightarrow{j} B$ . Since  $j$  is a *local* diffeomorphism, there exists an open cover  $(U_k : k \in K)$  of  $U$  such that  $j$  becomes an open embedding when restricted to any  $U_k$ . Since  $j$  is surjective, the family  $(j(U_k) : k \in K)$  is therefore an open cover of  $B$ . Also note that if I start with an open cover of  $B$ , turn it into a surjective local diffeomorphism, and turn that back into an open cover, then I get back the same open cover that I started with.

Thus, I will define a **cover** to be any surjective local diffeomorphism  $U \xrightarrow{j} B$ . It seems that [JST, 7.C] was the first to use this generality as a specification for  $j$ . (Note that their ‘ $E$ ’ and ‘ $p$ ’ refer to my  $U$  and  $j$ ; while they call my  $E$  and  $p$ —to be introduced in section 1.3 below—‘ $A$ ’ and ‘ $\alpha$ ’.)

You could reinterpret this paper with slightly different notions of cover. For example, you could insist that covers always be the form given above, a map from (a space diffeomorphic to) a disjoint union of open subsets; but that wouldn’t make any significant difference. A more interesting possibility might be to take all surjective submersions (without requiring them to be immersions as well). Compare the uses of these in [Met].

## 1.3 Bundles

The most general notion of bundle is quite simple; if  $B$  is a space, then a **bundle** over  $B$  is simply a space  $E$  together with a map  $E \xrightarrow{p} B$ . Before long, of course, I will restrict attention to bundles with more structure than this— eventually, to locally trivial bundles with a structure group.

I will generally denote the bundle by the same name as the space  $E$ , and the default name for the map is always ‘ $p$ ’— with subscripts, primes, or other decorations as needed. In diagrams, I will try to orient  $p$  at least partially downward, to appeal to the common intuition of  $E$  as lying ‘over’  $B$ .

### 1.3.1 The category of bundles

For a proper notion of equivalence of bundles, I should define the category  $\mathbf{C}/B$  of bundles over  $B$ .

Given bundles  $E$  and  $E'$  over  $B$ , a **bundle morphism** from  $E$  to  $E'$  is a map  $E \xrightarrow{f} E'$  making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & & B \end{array}$$

Given bundle morphisms  $f$  from  $E$  to  $E'$  and  $f'$  from  $E'$  to  $E''$ , the composite bundle morphism is simply the composite map  $E \xrightarrow{f} E' \xrightarrow{f'} E''$ ; notice that this diagram

commutes:

$$\begin{array}{ccccc}
 E & \xrightarrow{f} & E' & \xrightarrow{f'} & E'' \\
 & \searrow p & \downarrow \pi & \downarrow \pi' & \swarrow p'' \\
 & & B & & 
 \end{array}$$

Also, the identity bundle morphism on  $E$  is the identity map on  $E$ .

**Proposition 2** *Given a space  $B$ , bundles over  $B$  and their bundle morphisms form a category.*

**Proof:**  $\mathbf{C}/B$  is just a slice category [CWM, II.6] of the category  $\mathbf{C}$  of spaces. ■

I now know what it means for bundles  $E$  and  $E'$  to be **equivalent bundles**: isomorphic objects in the category  $\mathbf{C}/B$ . There must be a bundle morphism from  $E$  to  $E'$  and a bundle morphism from  $E'$  to  $E$  whose composite bundle morphisms, in either order, are identity bundle morphisms. What this amounts to, then, are maps  $E \xrightarrow{f} E'$  and  $E' \xrightarrow{\bar{f}} E$  making the following diagrams commute:

$$\begin{array}{ccc}
 \begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \downarrow \pi & \downarrow \pi' & \swarrow p' \\ & & B & & \end{array} & 
 \begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ & \searrow p' & \downarrow \bar{\pi} & \downarrow \pi & \swarrow p \\ & & B & & \end{array} & 
 \begin{array}{ccc} E & \xrightarrow{\underline{E}} & E \\ & \searrow f & \downarrow \kappa & \downarrow \bar{f} & \swarrow f \\ & & E' & & \end{array} & 
 \begin{array}{ccc} E' & \xrightarrow{\underline{E'}} & E' \\ & \searrow \bar{f} & \downarrow \bar{\kappa} & \downarrow f & \swarrow f \\ & & E & & \end{array}
 \end{array}$$

In particular,  $E$  and  $E'$  are equivalent as spaces (diffeomorphic).

### 1.3.2 Trivial bundles

If  $B$  and  $F$  are spaces, then the Cartesian product  $F \times B$  is automatically a bundle over  $B$ . Just let the map be  $F \times B \xrightarrow{\hat{F} \times B} B$ , the projection that forgets about  $F$ .

Of this bundle, it can rightly be said that  $F$  is its fibre. I'll want to define a more general notion, however, of bundle over  $B$  with fibre  $F$ . To begin, I'll generalise this example to the notion of a trivial bundle. Specifically, a **trivial bundle** over  $B$  with fibre  $F$  is simply a bundle over  $B$  equipped with a bundle equivalence to it from  $F \times B$ .

In more detail, this is a space  $E$  equipped with maps  $E \xrightarrow{p} B$ ,  $F \times B \xrightarrow{f} E$ , and  $E \xrightarrow{\bar{f}} F \times B$  making the following diagrams commute:

$$\begin{array}{ccc}
 F \times B & \xrightarrow{f} & E \\
 \hat{F} \times \underline{B} \searrow & \pi & \nearrow p \\
 & B &
 \end{array}
 \qquad
 \begin{array}{ccc}
 E & \xrightarrow{\bar{f}} & F \times B \\
 p \searrow & \bar{\pi} & \nearrow \hat{F} \times \underline{B} \\
 & B &
 \end{array}
 \qquad
 \begin{array}{ccc}
 F \times B & \xrightarrow{\underline{E \times B}} & F \times B \\
 f \searrow & \kappa & \nearrow \bar{f} \\
 & E &
 \end{array}
 \qquad
 \begin{array}{ccc}
 E & \xrightarrow{\underline{E}} & E \\
 \bar{f} \searrow & \bar{\kappa} & \nearrow f \\
 & F \times B &
 \end{array}$$

### 1.3.3 Restrictions of bundles

Ultimately, I'll want to deal with *locally* trivial bundles, so I need a notion of restriction to an open cover. In this section, I'll work quite generally, where the open cover is modelled by *any* map  $U \xrightarrow{j} B$ . I will call such a map an (unqualified) **subspace** of  $B$ ; this is just a radical generalisation of terminology like 'immersed submanifold', which can be seen sometimes in differential geometry [Spi]. (Thus logically, there is no difference between a subspace of  $B$  and a bundle over  $B$ ; but one does different things with them, and they will be refined in different ways.)

Given a bundle  $E$  and a subspace  $U$  (equipped with the map  $j$  as above), I get a

pullback diagram:

$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ U & \xrightarrow{j} & B \end{array}$$

Then the **restriction**  $E|_U$  of  $E$  to  $U$  is *any* pullback of this diagram, if any exists. I

name the associated maps as in this commutative diagram:

$$\begin{array}{ccc} E|_U & \xrightarrow{\tilde{j}} & E \\ p|_U \downarrow & \tilde{\omega} & \downarrow p \\ U & \xrightarrow{j} & B \end{array}$$

Notice that  $E|_U$  becomes both a subspace of  $E$  and a bundle over  $U$ .

By the unicity of pullbacks, the restriction  $E|_U$ , if it exists, is well defined up to diffeomorphism; but is it well defined up to equivalence of bundles over  $U$ ? The answer is yes, because the pullback cone morphisms in diagram (1.2) become bundle morphisms when applied to this situation.

### 1.3.4 Locally trivial bundles

I can now combine the preceding ideas to define a locally trivial bundle with a given fibre, which is the general notion of fibre bundle without a fixed structure group (compare [Ste, I.1.1, I.2.3]).

Given spaces  $B$  and  $F$ , suppose that  $B$  has been supplied with a subspace  $U \xrightarrow{j} B$ , and let this subspace be a *cover* in the sense of section 1.1.5. In this case, the restriction  $E|_U$  (like any pullback along a cover map) must exist. A **locally trivial bundle** over  $B$  with fibre  $F$  subordinate to  $U$  is a bundle  $E$  over  $B$  such that the



trivial bundle  $F \times U$  over  $U$  is isomorphic to  $E|_U$ . In fact, since the restriction is defined to be *any* pullback, I can simply take  $F \times U$  to be  $E|_U$ , by specifying the appropriate  $\tilde{j}$ .

So to sum up, a locally trivial bundle consists of the following items:

- a **base space**  $B$ ;
- a **cover space**  $U$ ;
- a **fibre space**  $F$ ;
- a **total space**  $E$ ;
- a **cover map**  $U \xrightarrow{j} B$ ;
- a **projection map**  $E \xrightarrow{p} B$ ; and
- a **pulled-back map**  $F \times U \xrightarrow{\tilde{j}} E$ ;

such that this diagram is a pullback:

$$\begin{array}{ccc}
 F \times U & \xrightarrow{\tilde{j}} & E \\
 \hat{F} \times \underline{U} \downarrow & \tilde{\omega} & \downarrow p \\
 U & \xrightarrow{j} & B
 \end{array} \tag{1.7}$$

In studying fibre bundles, I regard  $B$  as a fixed structure on which the bundle is defined. The fibre  $F$  (like the group  $G$  in the next section) indicates the type of problem that one is considering; while the cover  $U$  (together with  $j$ ) is subsidiary structure that is not preserved by bundle morphisms. Accordingly,  $E$  (together with  $p$ ) ‘is’ the bundle; I may use superscripts or primes on  $p$  (as well as  $U$  and  $j$ ) if I’m studying more than one bundle.

If  $X$  is any space,  $X \xrightarrow{w} F$  is a generalised element of  $F$ , and  $X \xrightarrow{x} U$  is a generalised element of  $U$ , then let  $w/x$  be the composite  $X \xrightarrow{\langle w, x \rangle} F \times U \xrightarrow{\tilde{j}} E$ . (The mnemonic is that  $w/x$  is the interpretation of the point  $w$  in the fibre *over* the point given in local coordinates as  $x$ .) This notation will be convenient in the sequel.

## 1.4 $G$ -spaces

Although the theory of groups acting on sets is well known [Lang, I.5], I will review it in strictly arrow-theoretic terms to make the categorification clearer.

### 1.4.1 Groups

The arrow-theoretic definition of group may be found for example in [CWM, page 2f].

First, a **monoid** is a space  $G$  together with maps  $1 \xrightarrow{e} G$  (the identity, or unit, element) and  $G \times G \xrightarrow{m} G$  (multiplication), making the following diagrams commute:

$$\begin{array}{ccc}
 G \times G \times G \xrightarrow{m \times \underline{G}} G \times G & & G \xrightarrow{e \times \underline{G}} G \times G \\
 \underline{G} \times m \downarrow & \alpha & \downarrow m \\
 G \times G \xrightarrow{m} G & & G \\
 & & \searrow \underline{G} \quad \lambda \\
 & & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G \xrightarrow{\underline{G} \times e} G \times G & & G \xrightarrow{\underline{G} \times e} G \times G \\
 \downarrow m & \rho & \downarrow m \\
 G & & G
 \end{array}$$

These are (respectively) the *associative law* and the *left and right unit laws* (hence the letters chosen to stand for the diagrams).

In terms of generalised elements, if  $x$  and  $y$  are elements of  $G$  (that is,  $X \xrightarrow{x} G$  and  $X \xrightarrow{y} G$  are maps for some arbitrary space  $X$ ), then I will write  $(xy)$  for the composite  $X \xrightarrow{\langle x, y \rangle} G \times G \xrightarrow{m} G$ . Also, I'll write  $1$  for  $X \xrightarrow{\hat{X}} 1 \xrightarrow{e} G$  (suppressing

$X$ ). Using this notation, if  $x$ ,  $y$ , and  $z$  are elements of  $G$ , then the associative law says that  $((xy)z) = (x(yz))$ , and the unit laws say (respectively) that  $(1x) = x$  and  $(x1) = x$ .

Then a **group** is a monoid  $G$  together with a map  $G \xrightarrow{i} G$  (the inverse operator) such that the following diagrams commute:

$$\begin{array}{ccc}
 G & \xrightarrow{\langle i, G \rangle} & G \times G \\
 \hat{G} \downarrow & \epsilon & \downarrow m \\
 1 & \xrightarrow{e} & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xrightarrow{\hat{G}} & 1 \\
 \langle G, i \rangle \downarrow & \iota & \downarrow e \\
 G \times G & \xrightarrow{m} & G
 \end{array}$$

These are (respectively) the *left and right inverse laws*.

In terms of elements, if  $X \xrightarrow{x} G$  is an element of  $G$ , then I'll write  $X \xrightarrow{x} G \xrightarrow{i} G$  as  $x^{-1}$ . Then the inverse laws say (respectively) that  $(x^{-1}x) = 1$  and  $1 = (xx^{-1})$ .

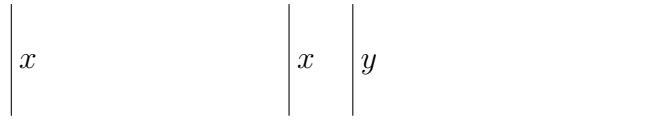
Note that since maps in the category  $\mathbf{C}$  are smooth functions, a group in that category is automatically a *Lie group*. At this point, I could define (internal) homomorphisms to get the category  $\mathbf{C}^{\mathbf{Grp}}$  of internal groups. But in the following, I'm only interested in a single fixed group. For the rest of part 1, let  $G$  be this fixed group.

### 1.4.2 String diagrams for groups

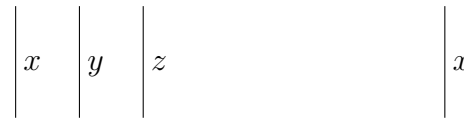
In part 2, I'll make extensive use of string diagrams ([Str]) to perform calculations in 2-groups. But in fact, these can already be used in groups, and (although they're not essential) I'll make reference to them throughout the rest of part 1 for practice.

If  $X \xrightarrow{x} G$  is a generalised element of the group (or monoid)  $G$ , then this map is drawn as in the diagram on the left below. Multiplication is shown by juxtaposition;

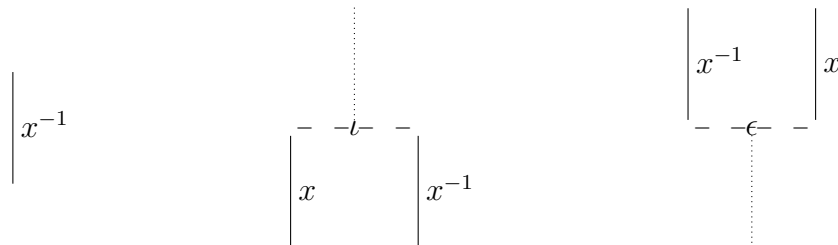
given elements  $x$  and  $y$ ,  $(xy)$  is in the middle below. The identity element  $1$  is generally invisible, but it may be shown as on the right below to stress its presence.



String diagrams reflect the associativity of monoids, so there is no distinction between  $((xy)z)$  and  $(x(yz))$ , both of which are shown on the left below. Similarly, the picture on the right below may be interpreted as  $x$ ,  $(1x)$ , or  $(x1)$ ;  $1$  is properly invisible here. You can also consider these maps to be pictures of the associative and unit laws; these laws are invisible to string diagrams.



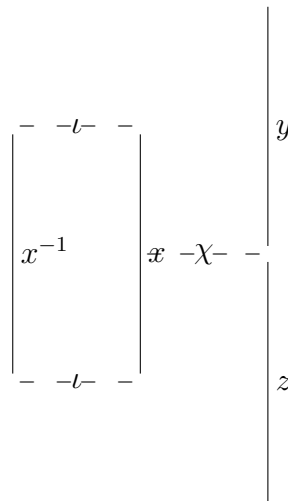
String diagrams are boring for monoids, but the inverse operation is more interesting. I will denote  $x^{-1}$  simply by the diagram at the left, but the inverse laws are now interesting:



The point of such diagrams is to say that the expression at the top of the diagram

is equal to the expression at the bottom, for the reasons given in the middle. (Of course, since these equations go both ways, they can also appear upside down.)

I can use string diagrams to prove equations in elementary string theory. For example, to prove that  $(xy) = (xz)$  implies that  $y = z$ , you could use this string diagram:



where  $\chi$  is the given fact  $(xy) = (xz)$ . This argument is essentially this:  $y = (1y) = ((x^{-1}x)y) = (x^{-1}(xy)) = (x^{-1}xz) = ((x^{-1}x)z) = (1z) = z$ , which uses (in turn) the left unit law  $\lambda$ , the right inverse law  $\iota$ , the associative law  $\alpha$ , the given fact  $\chi$ , the associative law again, the right inverse law again, and finally the left unit law again. Notice that the associative and unit laws are invisible in the string diagram, while the others appear explicitly; this is typical.

The rest of part 1 will have several more examples, with each string diagram also interpreted as an equation. This is perhaps overkill; but in part 2, these equations will become specific isomorphisms, and the string diagrams will be quite helpful in keeping track of them.

### 1.4.3 Action on a space

I want  $G$  to act on the fibre of a bundle. The inverse map  $i$  will play no role in this section, which could be applied when  $G$  is just a monoid (as was explicitly recognised at least as early as [Weyl, III.A.1]).

A **right  $G$ -space** is a space  $F$  equipped with a map  $F \times G \xrightarrow{r} F$  that makes the following diagrams commute:

$$\begin{array}{ccc}
 F \times G \times G & \xrightarrow{r \times G} & F \times G \\
 \underline{E} \times m \downarrow & \mu & \downarrow r \\
 F \times G & \xrightarrow{r} & F
 \end{array}
 \qquad
 \begin{array}{ccc}
 F & \xrightarrow{F \times e} & F \times G \\
 \underline{E} \searrow & v & \downarrow r \\
 & & F
 \end{array}
 \tag{1.8}$$

If more than one  $G$ -space is around at a time, then I'll use subscripts or primes on  $r$  to keep things straight.

In terms of generalised elements, if  $X \xrightarrow{w} F$  is a generalised element of  $F$  and  $X \xrightarrow{x} G$  is a generalised element of  $G$ , then I will write  $(xz)$  for the composite  $X \xrightarrow{\langle w, x \rangle} F \times G \xrightarrow{r} F$ , overloading the notation that introduced in section 1.4.1 for  $m$ . Then if  $w$  is an element of  $F$  and  $x$  and  $y$  are elements of  $G$ , then the laws (1.8) say (respectively) that  $((wx)y) = (w(xy))$  and that  $(w1) = w$ .

The axioms  $\mu$  and  $v$  are the associative and right unit laws of the right action. These fit in with the laws for the group itself:

**Proposition 3** *The group  $G$  is itself a  $G$ -space.*

**Proof:** Set  $r_G := m$ . The requirements (1.8) then reduce to (some of) the require-

ments for a group.

■

I'll use this to define  $G$ -torsors and principal  $G$ -bundles.

### 1.4.4 The category of $G$ -spaces

Just as spaces form a category  $\mathbf{C}$ , so  $G$ -spaces form a category  $\mathbf{C}^G$ .

Given right  $G$ -spaces  $F$  and  $F'$ , a  $G$ -**map** from  $F$  to  $F'$  is a map  $F \xrightarrow{t} F'$  making the following diagram commute:

$$\begin{array}{ccc} F \times G & \xrightarrow{t \times G} & F' \times G \\ r \downarrow & \phi & \downarrow r' \\ F & \xrightarrow{t} & F' \end{array}$$

Note that a composition of  $G$ -maps is a  $G$ -map, since both squares below commute:

$$\begin{array}{ccccc} F \times G & \xrightarrow{t \times G} & F' \times G & \xrightarrow{t' \times G} & F'' \times G \\ r \downarrow & \phi & r' \downarrow & \phi' & \downarrow r'' \\ F & \xrightarrow{t} & F' & \xrightarrow{t'} & F'' \end{array}$$

The identity map on  $F$  is also a  $G$ -map. Indeed,  $G$ -spaces and  $G$ -maps form a category  $\mathbf{C}^G$ .

Now I know what it means for  $G$ -spaces  $F$  and  $F'$  to be **equivalent**: There must be maps  $F \xrightarrow{t} F'$  and  $F' \xrightarrow{\bar{t}} F$  such that the following diagrams commute:

$$\begin{array}{ccc} F \times G & \xrightarrow{t \times G} & F' \times G \\ r \downarrow & \phi & \downarrow r' \\ F & \xrightarrow{t} & F' \end{array} \quad \begin{array}{ccc} F' \times G & \xrightarrow{\bar{t} \times G} & F \times G \\ r' \downarrow & \bar{\phi} & \downarrow r \\ F' & \xrightarrow{\bar{t}} & F \end{array}$$

$$\begin{array}{ccc}
F & \xrightarrow{E} & F \\
& \searrow t & \nearrow \bar{t} \\
& & F'
\end{array}
\quad
\begin{array}{ccc}
F' & \xrightarrow{E'} & F' \\
& \searrow \bar{t} & \nearrow t \\
& & F
\end{array}$$

### 1.4.5 $G$ -torsors

When I define principal bundles, I will want the fibre of the bundle to be  $G$ . However, it should be good enough if the fibre is only *equivalent* to  $G$ . Such a  $G$ -space is sometimes called ‘homogeneous’, but I prefer the noun ‘torsor’. Thus, a **right  $G$ -torsor** is any right  $G$ -space that is equivalent, as a  $G$ -space, to  $G$  itself.

In more detail, this is a space  $F$  equipped with maps  $F \times G \xrightarrow{r} F$ ,  $F \xrightarrow{t} G$ , and  $G \xrightarrow{\bar{t}} F$ , such that the following diagrams commute:

$$\begin{array}{ccc}
F \times G \times G \xrightarrow{r \times G} F \times G & F \xrightarrow{E \times e} F \times G & F \times G \xrightarrow{t \times G} G \times G \\
\downarrow E \times m & \searrow v & \downarrow r \\
F \times G & \xrightarrow{E} & F \\
\downarrow r & & \downarrow r \\
F & & F
\end{array}
\quad
\begin{array}{ccc}
F \times G \times G \xrightarrow{r \times G} F \times G & F \times G \xrightarrow{t \times G} G \times G & \\
\downarrow E \times m & \searrow v & \downarrow r \\
F \times G & \xrightarrow{E} & F \\
\downarrow r & & \downarrow r \\
F & & F
\end{array}$$

$$\begin{array}{ccc}
F \times G \times G \xrightarrow{r \times G} F \times G & F \times G \xrightarrow{t \times G} G \times G & \\
\downarrow E \times m & \searrow v & \downarrow r \\
F \times G & \xrightarrow{E} & F \\
\downarrow r & & \downarrow r \\
F & & F
\end{array}$$

$$\begin{array}{ccc}
G \times G \xrightarrow{\bar{t} \times G} F \times G & F \xrightarrow{E} F & G \xrightarrow{G} G \\
\downarrow m & \searrow t & \downarrow m \\
G & \xrightarrow{\bar{t}} & F \\
\downarrow \bar{t} & & \downarrow t \\
F & & F
\end{array}$$

## 1.5 $G$ -bundles

I can now put the above ideas together to get the concept of  $G$ -bundle.

### 1.5.1 Definition of $G$ -bundle

Suppose that I have a space  $B$  with a cover  $U \xrightarrow{j} B$ , as well as a group  $G$ . In these circumstances, a  **$G$ -transition** of the cover  $U$  is a map  $U^{[2]} \xrightarrow{g} G$  such that the



following diagrams commute:

$$\begin{array}{ccc}
 U^{[3]} & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & U^{[2]} \times U^{[2]} \xrightarrow{g \times g} G \times G \\
 & \searrow^{j_{[02]}} & \downarrow m \\
 & & U^{[2]} \xrightarrow{g} G
 \end{array}
 \qquad
 \begin{array}{ccc}
 U & \xrightarrow{\hat{U}} & 1 \\
 j_{[00]} \downarrow & \eta & \downarrow e \\
 U^{[2]} & \xrightarrow{g} & G
 \end{array}
 \tag{1.9}$$

Notice that if  $U$  appears directly as a disjoint union, then  $g$  breaks down into a family of maps on each disjunct; thus one normally speaks of *transition maps* in the plural.

In terms of elements, if  $\langle x, y \rangle$  defines an element of  $U^{[2]}$ , then let  $g_{xy}$  be the composite  $X \xrightarrow{\langle x, y \rangle} U^{[2]} \xrightarrow{g} G$ . Then the law  $\gamma$  says that  $(g_{xy}g_{yz}) = g_{xz}$  (for an element  $\langle x, y, z \rangle$  of  $U^{[3]}$ ), while the law  $\eta$  says that  $1 = g_{xx}$  (for an element  $X \xrightarrow{x} U$  of  $U$ ).

The string diagrams for  $g_{xy}$ , for  $\gamma_{xyz}$ , and for  $\eta_x$  are (respectively):

$$\begin{array}{c}
 x \quad \left| \begin{array}{c} g \\ y \end{array} \right. \\
 \\
 \begin{array}{ccc}
 \left| \begin{array}{c} g \\ y \end{array} \right. & \text{---} \gamma \text{---} & \left| \begin{array}{c} g \\ z \end{array} \right. \\
 x & & z \\
 & & \left| \begin{array}{c} g \\ \phantom{z} \end{array} \right.
 \end{array}
 \end{array}
 \qquad
 \begin{array}{c}
 x \\
 \left| \begin{array}{c} \eta \\ g \end{array} \right.
 \end{array}$$

(Here I've drawn in the invisible identity string diagram to clarify the height of the diagram for  $\eta$ .) I should also point out that the string diagrams of this form cover only some of the meaningful expressions that one can build using  $g$ ; for example, there is no diagram for  $(g_{wx}g_{yz})$ . However, I will never need to use this expression; the string diagrams can handle those expressions that I want.

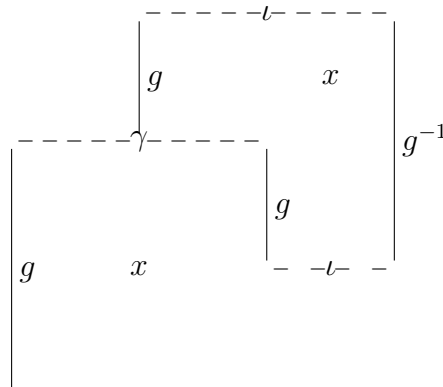
As you will see, there is an analogy to be made between a  $G$ -transition and a group. In this analogy,  $g$  corresponds to the underlying set of the group, the commu-

tative diagram  $\gamma$  corresponds to the operation of multiplication, and the commutative diagram  $\eta$  corresponds to the identity element. (This analogy will be even more fruitful when categorified.) Another possible analogy is with a group *homomorphism*; in fact, the following proposition (and its proof) are analogous to the result that a function between groups, if it preserves multiplication, must also preserve the identity. (The *truth* is that a  $G$ -transition is a *groupoid* homomorphism; see the discussion in section 2.2.4, where groupoid homomorphisms appear in the guise of 2-maps.)

**Proposition 4** *If a map  $U^{[2]} \xrightarrow{g} G$  satisfies the law  $\gamma$ , then it also satisfies  $\eta$ .*

**Proof:**  $\eta$  may be given in terms of generalised elements:  $1 = (g_{xx}g_{xx}^{-1}) = ((g_{xx}g_{xx})g_{xx}^{-1}) = (g_{xx}(g_{xx}g_{xx}^{-1})) = (g_{xx}1) = g_{xx}$ ; using (in order) the inverse law  $\iota$ , the assumed law  $\gamma$ , the associative law  $\alpha$ ,  $\iota$  again, and the unit law  $\rho$ . ■

I think that these proofs are easier to read in terms of elements, so I'll normally give them that way, without drawing a commutative diagram. But it's important to know that such a diagram can be drawn; the reasoning does not depend on  $\mathbf{C}$ 's being a concrete category with literal points as elements. This proof can also be written as a string diagram:





## 1.5.2 The category of $G$ -transitions

To define a morphism of  $G$ -bundles properly, I first need the notion of morphism of  $G$ -transitions.

Given covers  $U$  and  $U'$  of  $B$ , a  $G$ -transition  $g$  on  $U$ , and a  $G$ -transition  $g'$  on  $U'$ , a  **$G$ -transition morphism** from  $g$  to  $g'$  is a map  $U \cap U' \xrightarrow{b} G$  such that these diagrams commute:

$$\begin{array}{ccc}
 U^{[2]} \cap U' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & U^{[2]} \times U \cap U' \xrightarrow{g \times b} G \times G \\
 \searrow j_{[02]} & & \downarrow m \\
 & & U \cap U' \xrightarrow{b} G
 \end{array} \quad (1.12)$$

and:

$$\begin{array}{ccc}
 U \cap U'^{[2]} & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & U \cap U' \times U'^{[2]} \xrightarrow{b \times g'} G \times G \\
 \searrow j_{[02]} & & \downarrow m \\
 & & U \cap U' \xrightarrow{b} G
 \end{array} \quad (1.13)$$

In terms of generalised elements, if  $\langle x, x' \rangle$  defines an element of  $U \cap U'$ , then let  $b_{xx'}$  be  $X \xrightarrow{\langle x, x' \rangle} U \cap U' \xrightarrow{b} G$ . Then if  $\langle x, y, x' \rangle$  defines an element of  $U^{[2]} \cap U'$ , then the law  $\sigma$  says that  $(g_{xy} b_{yx'}) = b_{xx'}$ ; and if  $\langle x, x', y' \rangle$  defines an element of  $U \cap U'^{[2]}$ , then the law  $\delta$  says that  $(b_{xx'} g'_{x'y'}) = b_{xy'}$ . Here are the string diagrams for  $b_{xx'}$ , for  $\sigma_{xyx'}$ , and for  $\delta_{xx'y'}$ :

$$\begin{array}{c}
 x \quad \left| \begin{array}{c} b \\ \hline \end{array} \right| \quad x' \\
 \\
 \begin{array}{ccc}
 \left| \begin{array}{c} g \\ \hline \end{array} \right| & y & \left| \begin{array}{c} b \\ \hline \end{array} \right| \\
 \hline \text{---} \sigma \text{---} \hline \\
 x & \left| \begin{array}{c} b \\ \hline \end{array} \right| & x'
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 \left| \begin{array}{c} b \\ \hline \end{array} \right| & x' & \left| \begin{array}{c} g' \\ \hline \end{array} \right| \\
 \hline \text{---} \delta \text{---} \hline \\
 x & \left| \begin{array}{c} b \\ \hline \end{array} \right| & y'
 \end{array}$$

In the analogy where the  $G$ -transitions  $g, g'$  are analogous to groups, the transition morphism  $b$  is analogous to a space that is acted on by  $g$  on the left and by  $g'$  on the right. (This is sometimes called a *bihomogeneous space* or a *bitorsor*, hence the letter ‘ $b$ ’ in my notation.) If  $\gamma$  and  $\eta$  in (1.9) are analogous to the multiplication and identity in the groups, then  $\sigma$  and  $\delta$  here are analogous (respectively) to the left and right actions. (The notation recalls the Latin words ‘sinister’ and ‘dexter’.)

As a simple example, take  $U'$  be the same cover as  $U$ , and take  $g, g'$ , and  $b$  to be all the same map  $g$ ; then diagrams (1.12) and (1.13) both reduce to the first diagram in (1.9). In this way, the  $G$ -transition  $g$  serves as its own identity  $G$ -transition morphism.

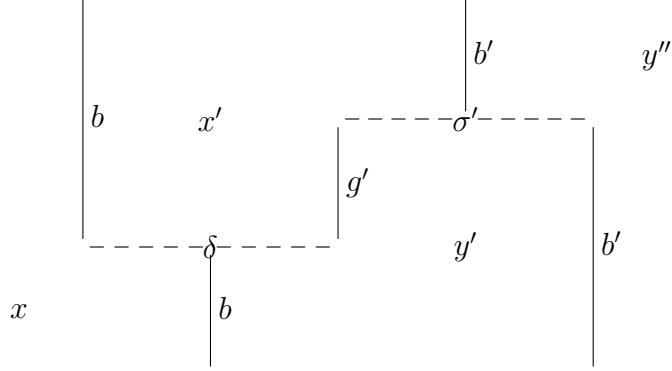
**Proposition 5** *Suppose that  $g, g', g''$  are all  $G$ -transitions on (respectively) covers  $U, U', U''$ , and suppose that  $b$  is a  $G$ -transition morphism from  $g$  to  $g'$  while  $b'$  is a  $G$ -transition morphism from  $g'$  to  $g''$ . Then I can define a  $G$ -transition morphism  $b; b'$  from  $g$  to  $g''$  such that this diagram commutes:*

$$\begin{array}{ccc}
 U \cap U' \cap U'' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & U \cap U' \times U' \cap U'' \xrightarrow{b \times b'} G \times G \\
 & \searrow^{j_{[02]}} & \beta \downarrow m \\
 & & U \cap U'' \xrightarrow{b; b'} G
 \end{array} \tag{1.14}$$

Given an element  $\langle x, x'' \rangle$  of  $U \cap U''$ , I will denote its composite with  $b; b'$  as  $(b; b')_{xx''}$ . Then given an element  $\langle x, x', x'' \rangle$  of  $U \cap U' \cap U''$ , the property (1.14) of  $b; b'$  states that  $(b_{xx'} b'_{x'x''}) = (b; b')_{xx''}$ .

**Proof:** This is probably most clearly expressed in terms of an element  $\langle x, x', y', y'' \rangle$

of  $U \cap U'^{[2]} \cap U''$ . Applying the laws  $\sigma'$ ,  $\alpha$ , and  $\delta$  in turn, I find that  $(b_{xx'}b'_{x'y''}) = (b_{xx'}(g'_{x'y'}b'_{y'y''})) = ((b_{xx'}g'_{x'y'})b'_{y'y''}) = (b_{xy'}b'_{y'y''})$ . As a string diagram:



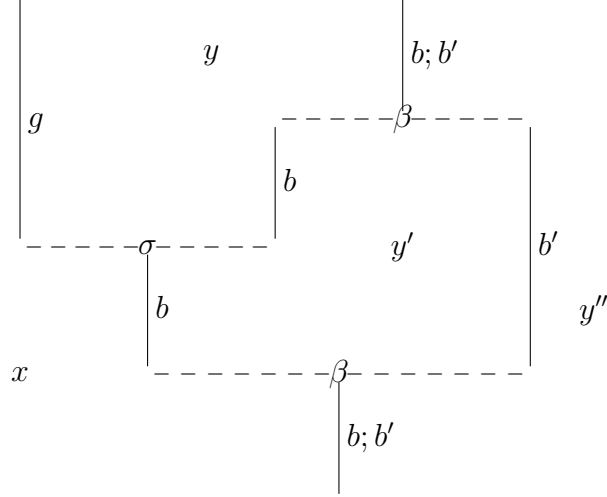
Thus, this diagram commutes:

$$\begin{array}{ccc}
 U \cap U'^{[2]} \cap U'' & \xrightarrow{j_{[013]}} & U \cap U' \cap U'' \\
 \downarrow j_{[023]} & & \downarrow \langle j_{[01]}, j_{[12]} \rangle \\
 & & U \cap U' \times U' \cap U'' \\
 & & \downarrow b \times b' \\
 & & G \times G \\
 & & \downarrow m \\
 U \cap U' \cap U'' & \xrightarrow[\langle j_{[01]}, j_{[12]} \rangle]{} U \cap U' \times U' \cap U'' \xrightarrow[b \times b']{} G \times G \xrightarrow{m} G & \quad (1.15)
 \end{array}$$

This diagram is an example of (1.4) for the cover  $U \cap U' \cap U'' \xrightarrow{j_{[02]}} U \cap U''$ , so (1.5) defines a map  $U \cap U'' \xrightarrow{b; b'} G$  satisfying (1.14).

Now given an element  $\langle x, y, y', y'' \rangle$  of  $U^{[2]} \cap U' \cap U''$ , I use (in turn) the laws  $\beta$ ,  $\alpha$ ,  $\sigma$ , and  $\beta$  again to deduce that  $(g_{xy}(b; b')_{yy''}) = (g_{xy}(b_{yy'}b'_{y'y''})) = ((g_{xy}b_{yy'})b'_{y'y''}) =$

$(b_{xy'}b'_{y'y''}) = (b;b')_{xy''}$ . As a string diagram:



Thus, this diagram commutes:

$$\begin{array}{ccc}
 U^{[2]} \cap U' \cap U'' & \xrightarrow{j_{[013]}} & U^{[2]} \cap U'' \\
 \downarrow j_{[013]} & & \downarrow \langle j_{[01]}, j_{[12]} \rangle \\
 & & U^{[2]} \times U \cap U'' \\
 & & \downarrow g \times b; b' \\
 & & G \times G \\
 & & \downarrow m \\
 U^{[2]} \cap U'' & \xrightarrow{j_{[02]}} & U \cap U'' \xrightarrow{b; b'} G
 \end{array}$$

Because the cover  $U^{[2]} \cap U' \cap U'' \xrightarrow{j_{[013]}} U \cap U' \cap U''$  is an epimorphism, it follows that

this diagram commutes:

$$\begin{array}{ccccc}
 U^{[2]} \cap U'' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & U^{[2]} \times U \cap U'' & \xrightarrow{g \times b; b'} & G \times G \\
 \searrow j_{[02]} & & & & \downarrow m \\
 & & U \cap U'' & \xrightarrow{b; b'} & G
 \end{array}$$

This is precisely the law  $\sigma$  for  $b; b'$ . (In terms of an element  $\langle x, y, y'' \rangle$  of  $U^{[2]} \cap U''$ , this is the same equation  $(g_{xy}(b; b')_{yy''}) = (b; b')_{xy''}$  as above; it is because  $j_{[013]}$  is an epimorphism that I don't need to posit  $y'$  anymore. In more set-theoretic terms, since  $j_{[013]}$  is onto, some  $y'$  must always exist.)

The law  $\delta$  for  $b; b'$  follows by an analogous argument; thus,  $b; b'$  is a  $G$ -transition morphism, as desired. ■

This  $b; b'$  is the composite of  $b$  and  $b'$  in the category  $G^B$  of  $G$ -transitions in  $B$ .

**Proposition 6**  $G^B$  is a category.

**Proof:** I must check that composition of  $G$ -transitions is associative and has identities. The key to this is that the diagram (1.15) not only shows (as in the previous proof) that a map  $b; b'$  satisfying (1.14) exists, but that  $b; b'$  is the *unique* map satisfying (1.14). Associativity follows. ■

I now know what it means to say that the  $G$ -transitions  $g$  and  $g'$  are equivalent: there are  $G$ -transition morphisms  $b$  from  $g$  to  $g'$  and  $\bar{b}$  from  $g'$  to  $g$ , such that  $b; \bar{b}$  equals  $g$  (which is the identity bundle morphism on  $g$ ) and  $\bar{b}; b$  equals  $g'$  (the identity on  $g'$ ).

### 1.5.3 The category of $G$ -bundles

To classify  $G$ -2-bundles, I need a proper notion of equivalence of  $G$ -bundles. For this, I should define the category  $\mathbf{Bun}_G(G, B, F)$  of  $G$ -bundles over  $B$  with fibre  $F$ .

Assume  $G$ -bundles  $E$  and  $E'$  over  $B$ , both with the given fibre  $F$ , and associated with the  $G$ -transitions  $g$  and  $g'$  (respectively). Then a  $G$ -bundle morphism from



$E$  to  $E'$  is a bundle morphism  $f$  from  $E$  to  $E'$  together with a  $G$ -transition morphism  $b$  from  $g$  to  $g'$ , such that this diagram commutes:

$$\begin{array}{ccc}
 F \times U \cap U' & \xrightarrow{F \times \langle b, j_{[1]} \rangle} & F \times G \times U' \\
 \underline{E} \times j_{[0]} \downarrow & & \downarrow r \times \underline{U}' \\
 F \times U & \xrightarrow{\zeta} & F \times U' \\
 & \searrow \tilde{j} & \downarrow \tilde{j} \\
 & & E \xrightarrow{f} E'
 \end{array} \tag{1.16}$$

In terms of generalised elements, if  $w$  is an element of  $F$  and  $\langle x, x' \rangle$  defines an element of  $U \cap U'$ , then  $(wb_{xx'})/_{x'}$  is equal to the composite of  $w/_x$  with  $f$ ; this also can be written as a string diagram:

$$\begin{array}{ccc}
 \parallel & & \parallel \\
 w \parallel & & \begin{array}{c} b \quad x' \\ \text{---} \text{---} \text{---} \zeta \end{array} \parallel \\
 \parallel & & \parallel \\
 & x &
 \end{array} \tag{1.17}$$

(In this string diagram, the transition labelled  $\zeta$  does not —like usual— indicate equality of the top and bottom sides, but instead that the top is the composite of the bottom with  $f$ . In other words,  $f$  is applied to the portion in  $E$  so that the whole diagram may be interpreted in  $E'$ .) I think of  $f$  as *being* the  $G$ -bundle morphism and say that  $f$  is **associated** with the transition morphism  $b$ .

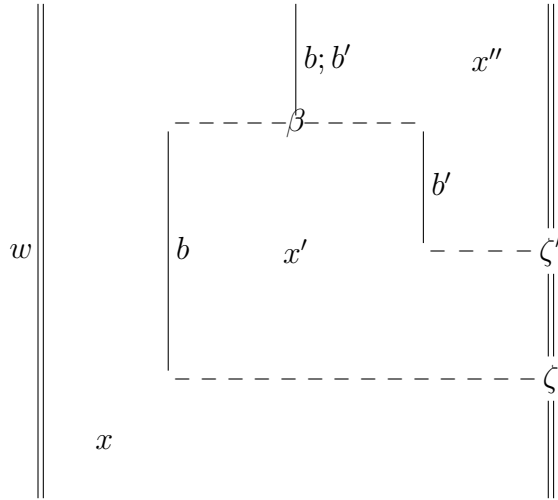
If  $E$  and  $E'$  are the same  $G$ -bundle (so associated with the same  $G$ -transition map) and  $f$  is the identity map  $\underline{E}$ , then these diagrams (1.16, 1.17) are a special case of the

diagrams (1.10, 1.11), with  $b$  taken to be the identity  $G$ -transition morphism on  $g$ , which is  $g$  itself. In this way, every  $G$ -bundle has an identity  $G$ -bundle automorphism.

Given  $G$ -bundle morphisms  $f$  from  $E$  to  $E'$  and  $f'$  from  $E'$  to  $E''$ , the composite  $G$ -bundle morphism  $f; f'$  is simply the composite bundle morphism  $E \xrightarrow{f} E' \xrightarrow{f'} E''$  together with the composite  $G$ -transition  $b; b'$  as described in section 1.5.2.

**Proposition 7** *This  $f; f'$  really is a  $G$ -bundle morphism.*

**Proof:** Let  $w$  be an element of  $F$ , and let  $\langle x, x', x'' \rangle$  define an element of  $U \cap U' \cap U''$ . Then using (in turn) the laws  $\zeta$ ,  $\zeta'$ ,  $\mu$ , and  $\beta$ , I see that  $(w(b; b')_{x''x})/_{x''} = (w(b_{xx'}b'_{x'x''}))/_{x''} = ((wb_{xx'})b'_{x'x''})/_{x''}$ , which is the composite of  $f'$  with  $(wb_{xx'})/_{x'}$ , which is the composite of  $f$  with  $w/x$ ; or as a string diagram:



Because  $U[3] \xrightarrow{J[02]} U[2]$  is epic, I can ignore  $x'$  (just as I ignored  $y'$  in the proof of

Proposition 5); thus this diagram commutes:

$$\begin{array}{ccccc}
 F \times U \cap U'' & \xrightarrow{\underline{E} \times \langle b; b', j_{[1]} \rangle} & F \times G \times U'' & & \\
 \downarrow \underline{E} \times j_{[0]} & & \downarrow r \times U'' & & \\
 F \times U & \xrightarrow{\zeta; \zeta'} & F \times U'' & & \\
 \searrow \tilde{j} & & \downarrow \tilde{j}'' & & \\
 E & \xrightarrow{f} & E' & \xrightarrow{f'} & E''
 \end{array}$$

This is simply the diagram (1.16) for the  $G$ -bundle morphism  $f; f'$ . Therefore,  $f; f'$  really is a  $G$ -bundle morphism. ■

**Proposition 8** *Given a space  $B$ , bundles over  $B$  and their bundle morphisms form a category.*

**Proof:** Composition of bundle morphisms is just composition of maps, which I know to be associative; and Proposition 6 proves that composition of the associated  $G$ -transition morphisms is associative; thus, composition of  $G$ -bundle morphisms is associative. Similarly, this composition has identities; therefore,  $G$ -bundles form a category  $\mathbf{Bun}_{\mathbf{C}}(G, B, F)$ . ■

I now know what it means for  $G$ -bundles  $E$  and  $E'$  to be **equivalent  $G$ -bundles**: isomorphic objects in the category  $\mathbf{Bun}_{\mathbf{C}}(G, B, F)$ . There must be a  $G$ -bundle morphism from  $E$  to  $E'$  and a  $G$ -bundle morphism from  $E'$  to  $E$  whose composite  $G$ -bundle morphisms, in either order, are identity  $G$ -bundle morphisms. In particular,  $E$  and  $E'$  are equivalent as bundles.

I should make a remark about when the action of  $G$  on  $F$  is unfaithful. Many

references will define a bundle morphism to be a bundle morphism  $f$  such that there exists a transition morphism  $b$ , while I've defined it to be  $f$  equipped with  $b$ . The difference affects the notion of equality of bundle morphisms: whether  $f = f'$  as maps is enough, or if  $b = b'$  (as maps) must also hold. This is relevant to the next section 1.5.4 as well; the propositions there are fine as far as they go, but the final theorem requires a functor from  $\mathbf{Bun}_G(G, B, F)$  to the category  $B^G$  of  $G$ -transitions, so each  $G$ -bundle morphism must be associated with a unique  $G$ -transition morphism. You can fix this either by passing from  $G$  to  $G/N$ , where  $N$  is the kernel of the action of  $G$  on  $F$ ; or even by requiring this action to be faithful in the definition of  $G$ -bundle (as is done, for example, in [Ste, 2.3]). I find the theory cleaner without these restrictions; in any case, there is no problem for principal bundles, since  $G$  acts on itself faithfully.

### 1.5.4 Associated bundles

The local data given in terms of  $G$ -transitions is in fact sufficient to recreate the associated bundle.

**Proposition 9** *Given a cover  $U \xrightarrow{j} B$ , a  $G$ -transition  $U^{[2]} \xrightarrow{g} G$ , and a  $G$ -space  $F$ , there is a  $G$ -bundle  $E$  over  $B$  with fibre  $F$  associated with the transition  $g$ .*

**Proof:** I will construct  $E$  as the quotient of an equivalence relation from  $F \times U^{[2]}$  to  $F \times U$ . One of the maps in the equivalence relation is  $\underline{F} \times j_{[0]}$ ; the other is  $F \times U^{[2]} \xrightarrow{F \times \langle g, j_{[1]} \rangle} F \times G \times U \xrightarrow{r \times U} F \times U$ . (You can see these maps in diagram (1.10), where

they appear with the quotient map  $F \times U \xrightarrow{\tilde{j}} E$  that this proof will construct.) Since  $\underline{F} \times j_{[0]}$  is a cover and every equivalence relation involving a cover has a quotient, the desired quotient does exist, satisfying (1.10)—at least, if this really is an equivalence relation!

To begin with, it is a relation; that is, the two maps are jointly monic. Given two elements  $\langle w, \langle x, y \rangle \rangle$  and  $\langle w', \langle x', y' \rangle \rangle$  of  $F \times U^{[2]}$ , the monicity diagrams (1.3) say that  $\langle w, x \rangle = \langle w', x' \rangle$  and  $\langle (wg_{xy}), y \rangle = \langle (w'g_{x'y'}), y' \rangle$ ; if these hold, then certainly  $w = w'$ ,  $x = x'$ , and  $y = y'$ , which is what I need.

The reflexivity map of the equivalence relation is  $F \times U \xrightarrow{E \times j_{[00]}} F \times U^{[2]}$ ; this is obviously a section of  $\underline{F} \times j_{[0]}$ . It's a section of  $F \times U^{[2]} \xrightarrow{E \times \langle g, j_{[1]} \rangle} F \times G \times U \xrightarrow{r \times U} F \times U$  using the laws  $v$  and  $\eta$ ; in terms of elements,  $\langle w, x \rangle = \langle (w1), x \rangle = \langle (wg_{xx}), x \rangle$  for an element  $\langle w, x \rangle$  of  $F \times U$ ; or in a string diagram:

$$\begin{array}{ccc} & & x \\ & & \eta \\ w & & | \\ & & g \\ & & \end{array}$$

The kernel pair of  $\underline{F} \times j_{[0]}$  is  $F \times U^{[3]}$ :

$$\begin{array}{ccc} F \times U^{[3]} & \xrightarrow{E \times j_{[02]}} & F \times U^{[2]} \\ \underline{E} \times j_{[01]} \downarrow & & \downarrow \underline{E} \times j_{[0]} \\ F \times U^{[2]} & \xrightarrow{\underline{E} \times j_{[0]}} & F \times U \end{array}$$

Let the Euclideaness map be  $F \times U^{[3]} \xrightarrow{E \times \langle j_{[01]}, j_{[12]} \rangle} F \times U^{[2]} \times U^{[2]} \xrightarrow{E \times g \times U^{[2]}} F \times G \times U$

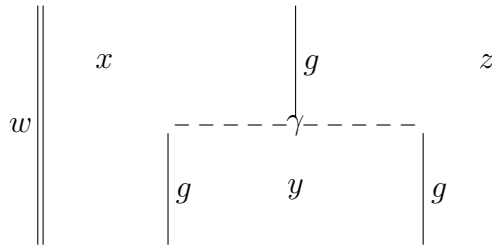
$U^{[2]} \xrightarrow{r \times U^{[2]}} F \times U^{[2]}$ . Thus, to prove that I have an equivalence relation, I need to show that these diagrams commute:

$$\begin{array}{ccccc}
 F \times U^{[3]} & \xrightarrow{\underline{E} \times \langle j_{[01]}, j_{[12]} \rangle} & F \times U^{[2]} \times U^{[2]} & \xrightarrow{\underline{F} \times g \times U^{[2]}} & F \times G \times U^{[2]} & \xrightarrow{r \times U^{[2]}} & F \times U^{[2]} \\
 \downarrow \underline{F} \times j_{[01]} & & & & & & \downarrow \underline{F} \times j_{[0]} \\
 F \times U^{[2]} & \xrightarrow{\underline{E} \times \langle g, j_{[1]} \rangle} & & & F \times G \times U & \xrightarrow{r \times U} & F \times U
 \end{array}$$

and:

$$\begin{array}{ccc}
 F \times U^{[3]} & \xrightarrow{\underline{E} \times j_{[02]}} & F \times U^{[2]} \\
 \downarrow \underline{F} \times \langle j_{[01]}, j_{[12]} \rangle & & \downarrow \underline{F} \times \langle g, j_{[1]} \rangle \\
 F \times U^{[2]} \times U^{[2]} & & F \times G \times U \\
 \downarrow \underline{F} \times g \times U^{[2]} & & \downarrow r \times U \\
 F \times G \times U^{[2]} & & F \times G \times U \\
 \downarrow r \times U^{[2]} & & \downarrow r \times U \\
 F \times U^{[2]} & \xrightarrow{\underline{E} \times \langle g, j_{[1]} \rangle} & F \times G \times U \xrightarrow{r \times U} F \times U
 \end{array}$$

The first of these diagrams is essentially one of the structural diagrams of the cover  $U$  (all that business with  $g$  and  $r$  is just along for the ride); in terms of elements, both sides describe  $\langle (wg_{xy}), y \rangle$  for an element  $\langle w, \langle x, y, z \rangle \rangle$  of  $F \times U^{[3]}$ . The second diagram holds using the laws  $\mu$  and  $\gamma$ ; in terms of elements,  $\langle (wg_{xz}), z \rangle = \langle (w(g_{xy}g_{yz})), z \rangle = \langle ((wg_{xy})g_{yz}), z \rangle$ ; in a string diagram:



Therefore, I really do have an equivalence relation, so some quotient  $F \times U \xrightarrow{\tilde{j}} E$

must exist satisfying (1.10). To define the bundle map  $E \xrightarrow{p} B$ , first note that this diagram commutes:

$$\begin{array}{ccccc}
 F \times U^{[2]} & \xrightarrow{\underline{E} \times \langle g, j_{[1]} \rangle} & F \times G \times U \xrightarrow{\underline{r} \times \underline{U}} & F \times U & \\
 \downarrow \underline{E} \times j_{[0]} & & & \downarrow \hat{F} \times \underline{U} & \\
 F \times U & \xrightarrow{\hat{F} \times \underline{U}} & U & \xrightarrow{j} & B \\
 & & & & \downarrow j \\
 & & & & U \\
 & & & & \downarrow j \\
 & & & & B
 \end{array}$$

in fact, both sides are simply  $\hat{F} \times j^{[2]}$ . Since  $E$  is a quotient, this defines a map  $E \xrightarrow{p} B$  such that (1.7) commutes.

Therefore,  $E$  is a  $G$ -bundle over  $B$  with fibre  $F$  associated with the transition  $g$ .

■

Now, I would like to say that  $E$  is *the* associated  $G$ -bundle, but this terminology is appropriate only if  $E$  is unique— that is, unique up to isomorphism of  $G$ -bundles. This is true; in fact, it is a special case of this result:

**Proposition 10** *Given  $G$ -bundles  $E$  and  $E'$  (both over  $B$  and with fibre  $F$ ) associated with  $G$ -transition morphisms  $g$  and  $g'$  (respectively), and given a  $G$ -transition morphism  $b$  from  $g$  to  $g'$ , there is a unique  $G$ -bundle morphism from  $E$  to  $E'$  associated with  $b$ .*

**Proof:** First note that this diagram commutes:

$$\begin{array}{ccc}
 F \times U \cap U'^{[2]} & \xrightarrow{E \times j_{[02]}} & F \times U \cap U' \\
 \downarrow E \times j_{[01]} & & \downarrow E \times \langle b, j_{[1]} \rangle \\
 & & F \times G \times U' \\
 & & \downarrow r \times \underline{U} \\
 & & F \times U \\
 & & \downarrow \tilde{j}' \\
 F \times U \cap U' & \xrightarrow{E \times \langle b, j_{[1]} \rangle} F \times G \times U' \xrightarrow{r \times \underline{U}'} F \times U' \xrightarrow{\tilde{j}'} E'
 \end{array}$$

because  $(wb_{xy'})/y' = (w(b_{xx'}b_{x'y'}))/y' = ((wb_{xx'})b_{x'y'})/y' = (wb_{xx'})/x'$ , as also seen in this string diagram:

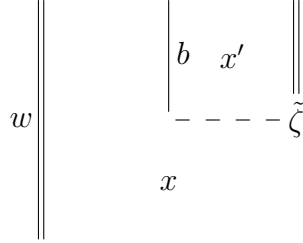
Since  $\underline{E} \times j_{[0]}$ , being a cover, is a quotient of its kernel pair, I can construct from this

a map  $F \times U \xrightarrow{\tilde{f}} E$  making this diagram commute:

$$\begin{array}{ccc}
 F \times U \cap U' & \xrightarrow{E \times \langle b, j_{[1]} \rangle} & F \times G \times U' \\
 \downarrow E \times j_{[0]} & & \downarrow r \times \underline{U}' \\
 & \tilde{\zeta} & F \times U \\
 & & \downarrow \tilde{j}' \\
 F \times U & \xrightarrow{\tilde{f}} & E'
 \end{array}$$



In terms of elements  $w$  of  $F$  and  $\langle x, x' \rangle$  of  $U \cap U'$ , this result states that  $(wb_{xx'})/_{x'}$  is equal to  $\langle w, x \rangle$  composed with  $\tilde{f}$ , which can be drawn as this string diagram:

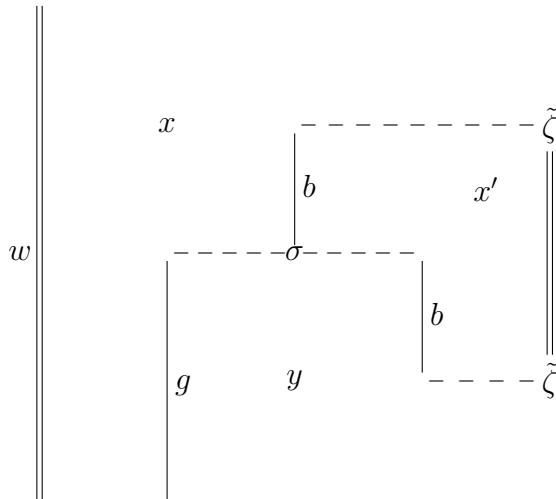


where  $\tilde{f}$  is applied to the portion of the diagram open on the right.

Next, notice that this diagram commutes:

$$\begin{array}{ccc}
 F \times U^{[2]} & \xrightarrow{\underline{F} \times \langle g, j_{[1]} \rangle} & F \times G \times U \xrightarrow{j \times U} F \times U \\
 \underline{E} \times j_{[0]} \downarrow & & \downarrow \tilde{f} \\
 F \times U & \xrightarrow{\tilde{f}} & E'
 \end{array}$$

because  $\langle (wg_{xy}), y \rangle$  composed with  $\tilde{f}$  is  $((wg_{xy})b_{yx'})/_{x'} = (w(g_{xy}b_{yx'}))/_{x'} = (wb_{xx'})/_{x'}$ , which is the composite with  $\tilde{f}$  of  $\langle w, x \rangle$ ; as also seen in this string diagram:



(Notice that I can drop  $x'$  because  $F \times U^{[2]} \cap U' \xrightarrow{j_{[01]}} F \times U^{[2]}$  is epic.) Since  $\tilde{j}$ , being the pullback of a cover, is a cover and hence the quotient of its kernel pair, I can construct from this a map  $E \xrightarrow{f} E'$  making the diagram (1.16) commute.

In short, I've constructed a  $G$ -bundle morphism  $E \xrightarrow{f} E'$  associated with  $b$ . Given any other  $G$ -bundle morphism  $E \xrightarrow{f'} E'$  associated with  $b$ , let  $\tilde{f}'$  be the composite  $F \times U \xrightarrow{\tilde{j}} E \xrightarrow{f'} E'$ . Then  $\tilde{f}' = \tilde{f}$  by the unicity of maps out of quotients, so  $f' = f$  by that same principle. ■

In particular, the identity  $G$ -transition morphism on  $g$  becomes a  $G$ -bundle morphism between any  $G$ -bundles associated with  $g$ . Also, notice that the composite of  $G$ -bundle morphisms associated with a composable pair of  $G$ -transition morphisms is itself associated with the composite  $G$ -transition morphism. By the unicity clause of the previous proposition, it is the same as the associated  $G$ -bundle morphism constructed by the existence clause. This shows, in particular, that any  $G$ -bundle morphism associated with an identity  $G$ -transition morphism is a  $G$ -bundle isomorphism (an equivalence of  $G$ -bundles).

In fact, there is a functor from the category  $\mathbf{Bun}_{\mathbf{C}}(G, B, F)$  of  $G$ -bundles over  $B$  with fibre  $F$  to the category  $B^G$  of  $G$ -transitions over  $B$ , defined by simply forgetting the total space  $E$ . The propositions above state (respectively) that this functor is surjective and fully faithful. In other words, I have proved this theorem:

**Theorem 1** *Given any  $G$ -space  $F$ , the category  $\mathbf{Bun}_{\mathbf{C}}(G, B, F)$  is equivalent to the category  $B^G$ .*

So depending on the point of view desired, you can think of  $G$ -transitions over  $B$  (a local view), principal  $G$ -bundles over  $B$  (a global view), or  $G$ -bundles over  $B$  with some convenient fibre  $F$  (such as when  $G$  is defined as a group of transformations of  $F$ , as is particularly common for linear groups); the concepts are all equivalent.

It remains to explain my use of the terminology ‘associated’. Normally, one begins with a principal bundle (a bundle with fibre  $G$ )  $E_G$  and asks for the bundle  $E_F$  with fibre  $F$  associated with the given  $E_G$ . But this is constructed by looking at the transition map of  $E_G$  and building  $E_F$  out of that as a certain quotient space. So I’ve adapted the terminology to say that the bundle is associated with the transition map *directly* rather than merely associated with the principal bundle *through* the transition map— not because I thought that this concept deserves the term better, but because I needed *some* terminology to refer to it! But really, all of the bundles associated with a given transition map should be considered to be associated with one another; that is the only really fair way to look at it.

Theorem 1 is not a new theorem. Although I’ve found no reference that expresses it just like this, it may be found implicitly in references on fibre bundles, such as [Ste, 3.2, 8.2, 9.1]. However, its categorification, Theorem 2, is the central internal result of this paper. Accordingly, I now turn to part 2 with a categorification of Theorem 1 as my goal.

## Part 2

# Categorified bundles

Now I categorify the above to construct the theory of 2-bundles with a structure 2-group.

### 2.1 2-Categorical preliminaries

First, I will turn the category theory from section 1.1 into 2-category theory.

Just as part 1 was about a specific category  $\mathbf{C}$ , so this part 2 is about a specific 2-category  $\mathfrak{C}$ . But just as the details of  $\mathbf{C}$  were largely irrelevant in part 1, so the details of  $\mathfrak{C}$  will be largely irrelevant here. I will explain what  $\mathfrak{C}$  is in section 2.2, but all that matters outside that section is that  $\mathfrak{C}$  supports the structures described here.

I should mention, however, that the 2-category to be described in section 2.2 is (unlike in [HDA5], [HDA6], and some earlier versions of this paper) a *weak* 2-category (that is a bicategory), while I will mostly treat it as if it were a *strict*

2-category (the more familiar sort of 2-category). This is all right, however, since a generalisation of the Mac Lane Coherence Theorem (see [Bén]) shows that every bicategory is equivalent to a strict 2-category.

### 2.1.1 Notation and terminology

A **2-space** is an object in  $\mathfrak{C}$ , and a **2-map** is a morphism in  $\mathfrak{C}$ , and a **natural transformation** is a 2-morphism in  $\mathfrak{C}$ . Uppercase calligraphic letters like  $\mathcal{X}$  denote 2-spaces, lowercase Fraktur letters like  $\mathfrak{x}$  denote 2-maps, and lowercase Greek letters like  $\psi$  denote natural transformations. However, the identity 2-map on  $\mathcal{X}$  will be denoted  $\underline{\mathcal{X}}$ , and the identity natural transformation on  $\mathfrak{x}$  will be denoted  $\underline{\mathfrak{x}}$ . (Combining these, even  $\underline{\underline{\mathcal{X}}}$  is possible.)

In general, the names of 2-maps and natural transformations will label arrows directly, perhaps in a small inline diagram like  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{Y} \xrightarrow{\mathfrak{y}} \mathcal{Z}$ , which denotes a composition of 2-maps, or perhaps in a huge 2-cell diagram which describes the composition of several natural transformations. I will endeavour to make such diagrams easy to read by orienting 2-maps to the right when convenient, or if not then downwards; natural transformations will always go down and/or to the left.

As in part 1, my discussion is in almost purely arrow-theoretic terms. However, I will again use the generalised arrow-theoretic concept of element. Specifically, an **element**  $\mathfrak{x}$  of a 2-space  $\mathcal{Y}$  is a 2-space  $\mathcal{X}$  together with a 2-map  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{Y}$ . In certain contexts, the 2-map  $\mathfrak{x}$  has the same information in it as a point in the space of objects

of  $\mathcal{Y}$ . On the other hand, if  $\mathcal{X}$  is chosen differently, then the map  $\mathfrak{x}$  may be described with more complicated features, such as a curve of objects, a point in the space of arrows, or even (say) a surface of arrows between two curves of objects. The most general element is in fact the identity 2-map  $\mathcal{X} \xrightarrow{\text{id}} \mathcal{X}$ . You (as reader) may imagine the elements as set-theoretic points if that helps, but their power in proofs lies in their complete generality. I will also use morphism-elements, or **arrows**, that is natural transformations between these generalised elements.

### 2.1.2 2-Products

Given 2-spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , there is a 2-space  $\mathcal{X} \times \mathcal{Y}$ , the **Cartesian 2-product** of  $\mathcal{X}$  and  $\mathcal{Y}$ . One can also form more general Cartesian 2-products, like  $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$  and so on. There is also a trivial 2-space  $1$ , which is the Cartesian 2-product of no 2-spaces.

The generalised elements of  $\mathcal{X} \times \mathcal{Y}$  may be taken to be ordered pairs; that is, given 2-maps  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{Y}$  and  $\mathcal{X} \xrightarrow{\mathfrak{x}'} \mathcal{Y}'$ , there is a **pairing 2-map**  $\mathcal{X} \xrightarrow{\langle \mathfrak{x}, \mathfrak{x}' \rangle} \mathcal{Y} \times \mathcal{Y}'$ ; conversely, given a 2-map  $\mathcal{X} \xrightarrow{\mathfrak{h}} \mathcal{Y} \times \mathcal{Y}'$ , there is a pair of 2-maps  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{Y}$  and  $\mathcal{X} \xrightarrow{\mathfrak{x}'} \mathcal{Y}'$  such that  $\mathfrak{h}$  is naturally isomorphic to  $\langle \mathfrak{x}, \mathfrak{x}' \rangle$ . (That is, there is an invertible natural transformation between  $\mathfrak{h}$  and  $\langle \mathfrak{x}, \mathfrak{x}' \rangle$ .) Furthermore, this pair is unique up to natural isomorphism; and the automorphisms of such a pair are in bijective correspondence with the automorphisms of  $\mathfrak{h}$ . This extends to 2-products of multiple 2-spaces. Additionally, given any 2-space  $\mathcal{X}$ , there is a **trivial 2-map**  $\mathcal{X} \xrightarrow{\hat{\text{id}}} 1$ ; it's unique in

the sense that every map  $\mathcal{X} \xrightarrow{\eta} 1$  is isomorphic to  $\hat{\mathcal{X}}$ , and this isomorphism is itself unique.

There are also product 2-maps; given 2-maps  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{Y}$  and  $\mathcal{X}' \xrightarrow{\mathfrak{r}'} \mathcal{Y}'$ , the **product 2-map** is  $\mathcal{X} \times \mathcal{X}' \xrightarrow{\mathfrak{r} \times \mathfrak{r}'} \mathcal{Y} \times \mathcal{Y}'$ . If furthermore there are 2-maps  $\mathcal{X} \xrightarrow{\eta} \mathcal{Y}$  and  $\mathcal{X}' \xrightarrow{\eta'} \mathcal{Y}'$ , with natural transformations from  $\mathfrak{r}$  to  $\eta$  and from  $\mathfrak{r}'$  to  $\eta'$ , then there is a product natural transformation from  $\mathfrak{r} \times \mathfrak{r}'$  to  $\eta \times \eta'$ . This respects all of the 2-category operations. The Mac Lane Coherence Theorem even applies, allowing me to make all of the same abuses of notation that appeared in part 1.

### 2.1.3 2-Pullbacks

Unlike products, pullbacks do not always exist in  $\mathbf{C}$ , hence neither do 2-pullbacks in  $\mathfrak{C}$ . Thus, I will have to treat this in more detail, so that I can discuss exactly what it means for a 2-pullback to exist.

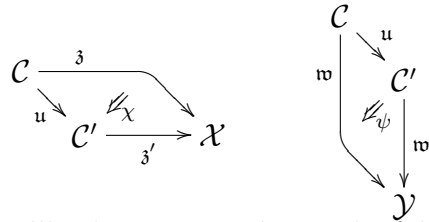
A **2-pullback diagram** consists of 2-spaces  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$ , and 2-maps  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{Z}$  and  $\mathcal{Y} \xrightarrow{\eta} \mathcal{Z}$ :

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow \mathfrak{r} \\ \mathcal{Y} & \xrightarrow{\eta} & \mathcal{Z} \end{array} \quad (2.1)$$

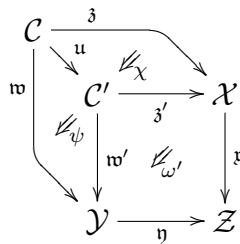
Given a 2-pullback diagram, a **2-pullback cone** is a 2-space  $\mathcal{C}$  together with 2-maps  $\mathcal{C} \xrightarrow{\mathfrak{j}} \mathcal{X}$  and  $\mathcal{C} \xrightarrow{\mathfrak{w}} \mathcal{Y}$  and a natural isomorphism  $\omega$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathfrak{j}} & \mathcal{X} \\ \mathfrak{w} \downarrow & \swarrow \omega & \downarrow \mathfrak{r} \\ \mathcal{Y} & \xrightarrow{\eta} & \mathcal{Z} \end{array}$$

(So while in section 1.1.3,  $\omega$  merely labelled a commutative diagram, here it refers to a specific natural transformation!) Given 2-pullback cones  $\mathcal{C}$  and  $\mathcal{C}'$ , a **2-pullback cone morphism** from  $\mathcal{C}$  to  $\mathcal{C}'$  is a 2-map  $\mathcal{C} \xrightarrow{u} \mathcal{C}'$  and natural isomorphisms  $\chi$  and  $\psi$ :



Furthermore, to have a 2-pullback cone morphism, the following composite of natural transformations must be equal to  $\omega$ :



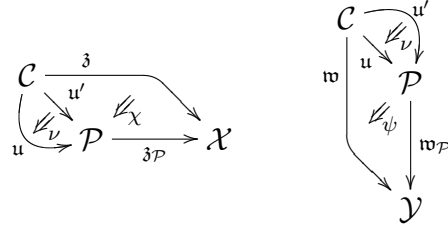
This is a typical example of a *coherence law*; it's the specification of coherence laws that makes a categorified theory logically richer than the original theory.

A **2-pullback** of the given 2-pullback diagram is a 2-pullback cone  $\mathcal{P}$  that is *universal* in the sense that, given any other 2-pullback cone  $\mathcal{C}$ , there is a 2-pullback cone morphism  $\mathcal{C} \xrightarrow{u} \mathcal{P}$  such that, given any other 2-pullback cone morphism  $\mathcal{C} \xrightarrow{u'} \mathcal{P}$ , there is a unique natural isomorphism  $\nu$ :





such that the following natural transformations:



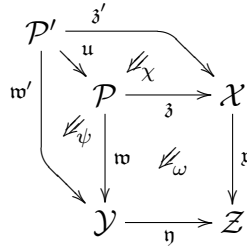
are equal (respectively) to  $\chi'$  and  $\psi'$ .

As with pullbacks in  $\mathbf{C}$ , so 2-pullbacks in  $\mathfrak{C}$  needn't always exist.

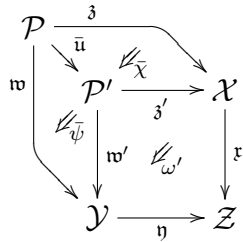
In the rest of this paper, I will want to define certain spaces as pullbacks, when they exist. If such definitions are to be sensible, then I must show that it doesn't matter which pullback one uses.

**Proposition 11** *Given the pullback diagram (2.1) and pullbacks  $\mathcal{P}$  and  $\mathcal{P}'$ , the 2-spaces  $\mathcal{P}$  and  $\mathcal{P}'$  are equivalent in  $\mathfrak{C}$ .*

**Proof:** Since  $\mathcal{P}$  is universal, there is a pullback cone morphism  $\mathcal{P}' \xrightarrow{u} \mathcal{P}$ :



Since  $\mathcal{P}'$  is universal, there is also a pullback cone morphism  $\mathcal{P} \xrightarrow{\bar{u}} \mathcal{P}'$ :



Composing these one way, I get a pullback cone morphism from  $\mathcal{P}$  to itself:

$$\begin{array}{ccccc}
 \mathcal{P} & & \delta & & \\
 \downarrow \bar{u} & \searrow & \downarrow \delta' & \swarrow \chi & \\
 \mathcal{P}' & \xrightarrow{u} & \mathcal{P} & \xrightarrow{\delta} & \mathcal{X} \\
 \downarrow w' & \swarrow \psi & \downarrow w & \swarrow \omega & \downarrow r \\
 \mathcal{Y} & \xrightarrow{\eta} & \mathcal{Z} & & \\
 \uparrow w & & & & \\
 \mathcal{P} & & & & 
 \end{array}
 \quad (2.2)$$

But since the identity on  $\mathcal{P}$  is also a pullback cone morphism, the universal property shows that  $\mathcal{P} \xrightarrow{\bar{u}} \mathcal{P}' \xrightarrow{u} \mathcal{P}$  is naturally isomorphic to the identity on  $\mathcal{P}$ . Similarly,  $\mathcal{P}' \xrightarrow{u} \mathcal{P} \xrightarrow{\bar{u}} \mathcal{P}'$  is naturally isomorphic to the identity on  $\mathcal{P}'$ . Therefore,  $\mathcal{P}$  and  $\mathcal{P}'$  are equivalent. ■

### 2.1.4 Equivalence 2-relations

Given a 2-space  $\mathcal{U}$ , a **binary 2-relation** on  $\mathcal{U}$  is a 2-space  $\mathcal{R}^{[2]}$  equipped with 2-maps  $\mathcal{R}^{[2]} \xrightarrow{j_{[0]}} \mathcal{U}$  and  $\mathcal{R}^{[2]} \xrightarrow{j_{[1]}} \mathcal{U}$  that are *jointly 2-monic*; this means that given generalised elements  $\mathcal{X} \xrightarrow{r} \mathcal{R}^{[2]}$  and  $\mathcal{X} \xrightarrow{\eta} \mathcal{R}^{[2]}$  of  $\mathcal{R}^{[2]}$  and natural isomorphisms  $\chi_{[0]}$  and  $\chi_{[1]}$  as follows:

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{r} & \mathcal{R}^{[2]} \\
 \eta \downarrow & & \downarrow \chi_{[0]} \\
 \mathcal{R}^{[2]} & \xrightarrow{j_{[0]}} & \mathcal{U}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{X} & \xrightarrow{r} & \mathcal{R}^{[2]} \\
 \eta \downarrow & & \downarrow \chi_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U}
 \end{array}
 \quad (2.3)$$

then there is a unique natural isomorphism  $\mathfrak{r} \xrightarrow{\chi} \mathfrak{r}$  such that the following composites are equal, respectively, to  $\chi_{[0]}$  and  $\chi_{[1]}$ :

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[2]} \\
 \eta \downarrow & \swarrow \chi & \downarrow j_{[0]} \\
 \mathcal{R}^{[2]} & \xrightarrow{\mathcal{R}^{[2]}} & \mathcal{U} \\
 & \searrow j_{[0]} & \\
 & & \mathcal{U}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{X} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[2]} \\
 \eta \downarrow & \swarrow \chi & \downarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{\mathcal{R}^{[2]}} & \mathcal{U} \\
 & \searrow j_{[1]} & \\
 & & \mathcal{U}
 \end{array}
 \tag{2.4}$$

The upshot of this is that an element of  $\mathcal{R}^{[2]}$  is determined, up to unique isomorphism, by two elements of  $\mathcal{U}$ .

A binary 2-relation is **reflexive** if it is equipped with a *reflexivity 2-map*  $\mathcal{U} \xrightarrow{j_{[00]}} \mathcal{R}^{[2]}$  and natural isomorphisms  $\omega_{[0]}$  and  $\omega_{[1]}$ :

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{j_{[00]}} & \mathcal{R}^{[2]} \\
 \omega_{[0]} \downarrow & \swarrow \omega_{[0]} & \downarrow j_{[0]} \\
 \mathcal{U} & \xrightarrow{\mathcal{U}} & \mathcal{U} \\
 & \searrow j_{[0]} & \\
 & & \mathcal{U}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\mathcal{U}} & \mathcal{U} \\
 j_{[00]} \downarrow & \swarrow \omega_{[1]} & \downarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{\mathcal{R}^{[2]}} & \mathcal{U} \\
 & \searrow j_{[1]} & \\
 & & \mathcal{U}
 \end{array}
 \tag{2.5}$$

In terms of generalised elements, given any element  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{U}$  of  $\mathcal{U}$ , composing with  $j_{[00]}$  gives an element of  $\mathcal{R}^{[2]}$  that corresponds to  $\langle \mathfrak{r}, \mathfrak{r} \rangle$  in  $\mathcal{U} \times \mathcal{U}$ . By 2-monicity, the reflexivity 2-map is unique up to unique isomorphism.

Assume that the 2-kernel pair of  $j_{[0]}$  exists, and let it be the 2-space  $\mathcal{R}^{[3]}$ :

$$\begin{array}{ccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[02]}} & \mathcal{R}^{[2]} \\
 j_{[01]} \downarrow & \swarrow \omega_{[00]} & \downarrow j_{[0]} \\
 \mathcal{R}^{[2]} & \xrightarrow{\mathcal{R}^{[2]}} & \mathcal{U} \\
 & \searrow j_{[0]} & \\
 & & \mathcal{U}
 \end{array}$$

A binary 2-relation is **right Euclidean** if it is equipped with a (right) *Euclideaness* 2-map  $\mathcal{R}^{[3]} \xrightarrow{j_{[12]}} \mathcal{R}^{[2]}$  and natural isomorphisms  $\omega_{[01]}$  and  $\omega_{[11]}$ :

$$\begin{array}{ccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} \\
 j_{[01]} \downarrow & \Downarrow \omega_{[01]} & \downarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{j_{[0]}} & \mathcal{U}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} \\
 j_{[02]} \searrow & \Downarrow \omega_{[11]} & \swarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U}
 \end{array}
 \tag{2.6}$$

Like the reflexivity 2-map, any Euclideaness 2-map is unique up to unique isomorphism. An element of  $\mathcal{R}^{[3]}$  is given by elements  $\mathfrak{x}$ ,  $\mathfrak{y}$ , and  $\mathfrak{z}$  of  $\mathcal{U}$  such that  $\langle \mathfrak{x}, \mathfrak{y} \rangle$  and  $\langle \mathfrak{x}, \mathfrak{z} \rangle$  give elements of  $\mathcal{R}^{[2]}$ ; then by the Euclideaness 2-map,  $\langle \mathfrak{y}, \mathfrak{z} \rangle$  also gives an element of  $\mathcal{R}^{[2]}$ .

Putting these together, an **equivalence 2-relation** is a binary 2-relation that is both reflexive and Euclidean.

Given an equivalence 2-relation as above, a **2-quotient** of the equivalence 2-relation is a 2-space  $\mathcal{R}^{[0]}$  and a map  $\mathcal{U} \xrightarrow{j} \mathcal{R}^{[0]}$  that 2-coequalises  $j_{[0]}$  and  $j_{[1]}$ . This means, first, a natural isomorphism  $\omega$ :

$$\begin{array}{ccc}
 \mathcal{R}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 j_{[0]} \downarrow & \Downarrow \omega & \downarrow j \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{R}^{[0]}
 \end{array}
 \tag{2.7}$$

such that these two composite isomorphisms are equal:

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow j_{[00]} & \searrow \omega_{[1]} & \downarrow \underline{u} \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow \underline{u} & \searrow \omega_{[0]} & \downarrow j_{[1]} \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \omega & \downarrow j \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{R}^{[0]}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow \underline{u} & \searrow \omega & \downarrow j \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{R}^{[0]}
 \end{array}
 \quad (2.8)$$

and also these two composite isomorphisms are equal:

$$\begin{array}{ccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} \\
 \downarrow j_{[01]} & \searrow j_{[02]} & \downarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{j_{[11]}} & \mathcal{R}^{[2]} \\
 \downarrow j_{[0]} & \searrow j_{[0]} & \downarrow j \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{R}^{[0]}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} \\
 \downarrow j_{[01]} & \searrow \omega_{[01]} & \downarrow j_{[1]} \\
 \mathcal{R}^{[2]} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \underline{u} & \downarrow j \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{R}^{[0]}
 \end{array}
 \quad (2.9)$$

Secondly, given any 2-map  $\mathcal{U} \xrightarrow{\mathfrak{r}} \mathcal{R}^{[0]}$  and natural isomorphism  $\omega_{\mathfrak{r}}$ :

$$\begin{array}{ccc}
 \mathcal{U}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \omega_{\mathfrak{r}} & \downarrow \mathfrak{r} \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X}
 \end{array}
 \quad (2.10)$$

if these two composite isomorphisms are equal:

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow j_{[00]} & \searrow \omega_{[1]} & \downarrow \underline{u} \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow \underline{u} & \searrow \omega_{[0]} & \downarrow j_{[1]} \\
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \omega_{\mathfrak{r}} & \downarrow \mathfrak{r} \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[0]}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\
 \downarrow \underline{u} & \searrow \omega & \downarrow \mathfrak{r} \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[0]}
 \end{array}$$

and also these two composite isomorphisms are equal:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} & & \\
 \downarrow j_{[01]} & \searrow j_{[02]} & \swarrow \omega_{[00]} & \searrow j_{[1]} & \\
 \mathcal{R}^{[2]} & \swarrow \omega_{[11]} & \mathcal{R}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow j_{[0]} & \swarrow \omega_x & \downarrow \mathfrak{r} & \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[0]} & & \\
 \end{array} & = & 
 \begin{array}{ccccc}
 \mathcal{R}^{[3]} & \xrightarrow{j_{[12]}} & \mathcal{R}^{[2]} & & \\
 \downarrow j_{[01]} & & \swarrow \omega_{[01]} & \searrow j_{[1]} & \\
 \mathcal{R}^{[2]} & & \mathcal{U} & & \\
 \downarrow j_{[0]} & \searrow \omega & \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{R}^{[0]} \\
 \mathcal{U} & & \mathcal{U} & & \\
 \end{array}
 \end{array}$$

then there is a 2-map  $\mathcal{R}^{[0]} \xrightarrow{\bar{\mathfrak{r}}} \mathcal{X}$  and a natural isomorphism  $\tilde{\omega}_x$ :

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X} \\
 \downarrow j & \searrow \omega_x & \\
 \mathcal{R}^{[0]} & \xrightarrow{\bar{\mathfrak{r}}} & \mathcal{X}
 \end{array} \quad (2.11)$$

such that these composite isomorphisms are equal:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{U}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \omega_x & \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X} \\
 \downarrow j & \searrow \omega_x & \\
 \mathcal{R}^{[0]} & \xrightarrow{\bar{\mathfrak{r}}} & \mathcal{X}
 \end{array} & = & 
 \begin{array}{ccc}
 \mathcal{U}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & \searrow \omega & \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{R}^{[0]} \\
 \downarrow j & \searrow \omega_x & \\
 \mathcal{R}^{[0]} & \xrightarrow{\bar{\mathfrak{r}}} & \mathcal{X}
 \end{array}
 \end{array} \quad (2.12)$$

Finally, given any alternative 2-map  $\mathcal{R}^{[0]} \xrightarrow{\bar{\mathfrak{r}}'} \mathcal{X}$  and natural isomorphism  $\tilde{\omega}'_x$ :

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X} \\
 \downarrow j & \searrow \omega'_x & \\
 \mathcal{R}^{[0]} & \xrightarrow{\bar{\mathfrak{r}}'} & \mathcal{X}
 \end{array}$$

if these composite isomorphisms are equal:

$$\begin{array}{ccc}
 \mathcal{U}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & & \swarrow \omega_{\mathfrak{r}} \\
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X} \\
 \searrow j & & \swarrow \omega'_{\mathfrak{r}} \\
 \mathcal{R}^{[0]} & \xrightarrow{\mathfrak{r}'} & \mathcal{X}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{U}^{[2]} & \xrightarrow{j_{[1]}} & \mathcal{U} \\
 \downarrow j_{[0]} & & \downarrow j \\
 \mathcal{U} & & \mathcal{R}^{[0]} \\
 \searrow j & & \swarrow \omega'_{\mathfrak{r}} \\
 \mathcal{R}^{[0]} & \xrightarrow{\mathfrak{r}'} & \mathcal{X}
 \end{array}$$

then there exists a unique isomorphism  $\mathfrak{r} \xrightarrow{\nu} \mathfrak{r}'$  such that this isomorphism:

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\mathfrak{r}} & \mathcal{X} \\
 \downarrow j & & \swarrow \omega_{\mathfrak{r}} \\
 \mathcal{R}^{[0]} & \xrightarrow{\mathfrak{r}'} & \mathcal{X} \\
 \downarrow \nu & & \swarrow \omega'_{\mathfrak{r}} \\
 \mathcal{R}^{[0]} & \xrightarrow{\mathfrak{r}'} & \mathcal{X}
 \end{array}$$

is equal to  $\tilde{\omega}'_{\mathfrak{r}}$ .

Thus if an equivalence 2-relation has a 2-quotient, then I can define a 2-map out of this 2-quotient, up to unique natural isomorphism, by defining a 2-map out of the relation's base space with certain attendant isomorphisms satisfying appropriate coherence relations.

### 2.1.5 2-Covers

Analogously to section 1.1.5, here are the axioms for 2-covers:

- All equivalences are 2-covers;
- A composite of 2-covers is a 2-cover;
- The 2-pullback of a 2-cover along any 2-map exists and is a 2-cover;

- The 2-quotient of every equivalence 2-relation involving a 2-cover exists and is a 2-cover; and
- Every 2-cover is the 2-quotient of its 2-kernel pair, with the same natural isomorphism  $\omega$  involved in each.

I will be particularly interested in the 2-pullback of  $\mathcal{U}$  along itself, that is the 2-pullback of this diagram:

$$\begin{array}{ccc} & & \mathcal{U} \\ & & \downarrow j \\ \mathcal{U} & \xrightarrow{j} & \mathcal{B} \end{array}$$

This 2-pullback,  $\mathcal{U}^{[2]}$ , is the **2-kernel pair** of  $\mathcal{U}$ , giving the 2-pullback diagram:

$$\begin{array}{ccc} \mathcal{U}^{[2]} & \xrightarrow{j_{[0]}} & \mathcal{U} \\ j_{[1]} \downarrow & \Downarrow \omega & \downarrow j \\ \mathcal{U} & \xrightarrow{j} & \mathcal{B} \end{array}$$

As before, the 2-space  $\mathcal{U}^{[3]}$  may also be defined as a 2-pullback:

$$\begin{array}{ccc} \mathcal{U}^{[3]} & \xrightarrow{j_{[01]}} & \mathcal{U}^{[2]} \\ j_{[12]} \downarrow & \Downarrow & \downarrow j_{[1]} \\ \mathcal{U}^{[2]} & \xrightarrow{j_{[0]}} & \mathcal{U} \end{array}$$

And the 2-space  $\mathcal{U}^{[4]}$ :

$$\begin{array}{ccc} \mathcal{U}^{[4]} & \xrightarrow{j_{[012]}} & \mathcal{U}^{[3]} \\ j_{[123]} \downarrow & \Downarrow & \downarrow j_{[12]} \\ \mathcal{U}^{[3]} & \xrightarrow{j_{[01]}} & \mathcal{U}^{[2]} \end{array}$$

And so forth, very much as in section 1.1.5.



In fact, all of these natural isomorphisms based on  $\omega$  can be ignored using (again) the Mac Lane Coherence Theorem, applied to the 2-products in the slice 2-category  $\mathfrak{C}/\mathcal{B}$ .

## 2.2 The 2-category of 2-spaces

To define my 2-category  $\mathfrak{C}$ , I need to categorify the notion of space to get a notion of 2-space. If a space is a smooth manifold, then a 2-space is a *Lie category*, or *smooth category*, a category whose objects and morphisms form smooth manifolds. Then 2-spaces (together with 2-maps and a new feature, natural transformations) will form my 2-category  $\mathfrak{C}$ .

### 2.2.1 2-Spaces

It was Charles Ehresmann [Ehr] who first defined the notion of differentiable category; today, one understands this as a special case of the general notion of *internal category* [HCA1, chapter 8].

Following this, a **2-space** consists of a space  $\mathcal{X}^1$  (the space of objects, or points) and a space  $\mathcal{X}^2$  (the space of morphisms, or arrows) together with maps  $\mathcal{X}^2 \xrightarrow{d_0} \mathcal{X}^1$  (the source map) and  $\mathcal{X}^2 \xrightarrow{d_1} \mathcal{X}^1$  (the target map). This creates a pullback diagram:

$$\begin{array}{ccc} & & \mathcal{X}^2 \\ & & \downarrow d_1 \\ \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1 \end{array}$$

To have a 2-space, this pullback diagram must have a pullback  $\mathcal{X}^3$  (the space of composable pairs of arrows), giving a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 \\ d_{12} \downarrow & & \downarrow d_1 \\ \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1 \end{array}$$

Finally, consider the pullback diagram

$$\begin{array}{ccc} & & \mathcal{X}^3 \\ & & \downarrow d_{12} \\ \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 \end{array}$$

A pullback  $\mathcal{X}^4$  (the space of composable triples) must exist:

$$\begin{array}{ccc} \mathcal{X}^4 & \xrightarrow{d_{012}} & \mathcal{X}^3 \\ d_{123} \downarrow & & \downarrow d_{12} \\ \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 \end{array}$$

A 2-space is additionally equipped with maps  $\mathcal{X}^1 \xrightarrow{d_{00}} \mathcal{X}^2$  (taking an object to its identity morphism, or a point to its null arrow) and  $\mathcal{X}^3 \xrightarrow{d_{02}} \mathcal{X}^2$  (composing a composable pair of arrows), such that each of these diagrams commutes:

$$\begin{array}{ccc} \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 \\ & \searrow d_{02} & \downarrow d_0 \\ & & \mathcal{X}^2 \xrightarrow{d_0} \mathcal{X}^1 \end{array} \quad \begin{array}{ccc} \mathcal{X}^3 & & \\ d_{12} \downarrow & \searrow d_{02} & \\ \mathcal{X}^2 & & \mathcal{X}^2 \\ & \searrow d_1 & \downarrow d_1 \\ & & \mathcal{X}^1 \end{array} \quad \begin{array}{ccc} \mathcal{X}^1 & \xrightarrow{d_{00}} & \mathcal{X}^2 \\ & \searrow \underline{\mathcal{X}^1} & \downarrow d_1 \\ & & \mathcal{X}^1 \end{array} \quad \begin{array}{ccc} \mathcal{X}^1 & & \\ d_{00} \downarrow & \searrow \underline{\mathcal{X}^1} & \\ \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1 \end{array}$$

(These just say that identities and compositions all have the proper sources and

targets.) Next, consider the commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{X}^4 & \xrightarrow{d_{012}} & \mathcal{X}^3 & & \\
 \downarrow d_{123} & & \downarrow d_{12} & \searrow d_{01} & \\
 \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 & & \mathcal{X}^2 \\
 & \searrow d_{02} & \searrow d_0 & \downarrow d_1 & \\
 & & \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1
 \end{array}$$

This is a pullback cone into the diagram defining  $\mathcal{X}^3$ , so the universal property of that pullback gives a map  $\mathcal{X}^4 \xrightarrow{d_{013}} \mathcal{X}^3$ . Similarly, the commutative diagram

$$\begin{array}{ccccc}
 \mathcal{X}^4 & \xrightarrow{d_{012}} & \mathcal{X}^3 & & \\
 \downarrow d_{123} & & \downarrow d_{12} & \searrow d_{02} & \\
 \mathcal{X}^3 & \xrightarrow{d_{01}} & \mathcal{X}^2 & & \mathcal{X}^2 \\
 & \searrow d_{12} & \searrow d_1 & \downarrow d_1 & \\
 & & \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1
 \end{array}$$

defines a map  $\mathcal{X}^4 \xrightarrow{d_{023}} \mathcal{X}^3$ . In a 2-space, this diagram must also commute:

$$\begin{array}{ccc}
 \mathcal{X}^4 & \xrightarrow{d_{013}} & \mathcal{X}^3 \\
 \downarrow d_{023} & & \downarrow d_{02} \\
 \mathcal{X}^3 & \xrightarrow{d_{02}} & \mathcal{X}^2
 \end{array}$$

(This is the *associative law* for an category, since the composable triples in  $\mathcal{X}^4$  are being composed in different ways to give the same result.) Finally, consider the commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1 & \xrightarrow{d_{00}} & \mathcal{X}^2 \\
 & \searrow \mathcal{X}^2 & & \searrow \mathcal{X}^1 & \downarrow d_1 \\
 & & \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1
 \end{array}$$

This too is a pullback cone into the pullback diagram defining  $\mathcal{X}^3$ , so it defines a map

$\mathcal{X}^2 \xrightarrow{d_{001}} \mathcal{X}^3$ . Similarly, the commutative diagram

$$\begin{array}{ccc}
 \mathcal{X}^2 & & \\
 \downarrow d_1 & \searrow \mathcal{X}^2 & \\
 \mathcal{X}^1 & & \mathcal{X}^2 \\
 \downarrow d_{00} & \searrow \mathcal{X}^1 & \downarrow d_1 \\
 \mathcal{X}^2 & \xrightarrow{d_0} & \mathcal{X}^1
 \end{array}$$

defines a map  $\mathcal{X}^2 \xrightarrow{d_{011}} \mathcal{X}^3$ . The last requirement of a 2-space is that both parts of this diagram commute:

$$\begin{array}{ccc}
 \mathcal{X}^2 & \xrightarrow{d_{001}} & \mathcal{X}^3 \\
 \searrow \mathcal{X}^2 & & \downarrow d_{02} \\
 & & \mathcal{X}^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{X}^2 & & \\
 \downarrow d_{011} & \searrow \mathcal{X}^2 & \\
 \mathcal{X}^3 & \xrightarrow{d_{02}} & \mathcal{X}^2
 \end{array}
 \tag{2.13}$$

(This expresses the *unit laws* of the internal category.)

This definition is a complicated mess of superscripts and subscripts, but a simple combinatorial pattern results. The space  $\mathcal{X}^n$  is the space of composable  $(n - 1)$ -tuples of arrows— which is to say that  $n$  points are involved when you include all of the endpoints. These points can be numbered (in order) from 0 to  $n - 1$ . Then the various  $d$  maps from  $\mathcal{X}^n$  to  $\mathcal{X}^{n'}$  are given by listing  $n'$  numbers from 0 to  $n - 1$  (in order, allowing repetitions and omissions), indicating which points are retained. Repeated numbers indicate an identity arrow in the composable  $(n' - 1)$ -tuple, while skipped numbers in the middle indicate that some of the original  $n - 1$  arrows have been composed. The axioms above are sufficient to yield the natural combinatorial behaviour.

For example, consider the final commutative triangle in the final axiom (2.13). Following that around the long way, you start with an arrow  $\mathfrak{r}_0 \longrightarrow \mathfrak{r}_1$  in  $\mathcal{X}^2$ , then repeat position 1 (introducing an identity arrow) to get a composable pair  $\mathfrak{r}_0 \longrightarrow \mathfrak{r}_1 \longrightarrow \mathfrak{r}_1$  in  $\mathcal{X}^3$ , and then skip the middle position (composing across it) to get an arrow  $\mathfrak{r}_0 \longrightarrow \mathfrak{r}_1$  in  $\mathcal{X}^2$  again. If the right unit law is to hold in this category, then this must be the very arrow that you started with, and that is precisely what the commutative triangle guarantees.

### 2.2.2 2-Maps

Just as maps go between spaces, so 2-maps go between 2-spaces. The usual theory of internal categories, as described in [HCA1, chapter 8], uses internal functors, and early versions of this paper did the same. However, these are not sufficient, and I will need instead a notion of internal *anafunctors*; anafunctors (in the case of ordinary categories) were first described in [Mak]. (In the category  $\mathbf{C}$  of smooth manifolds and smooth functions, these 2-maps will be *smooth* anafunctors.)

Given 2-spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , a **2-map**  $\mathfrak{r}$  from  $\mathcal{X}$  to  $\mathcal{Y}$  consists in part of a cover  $|\mathfrak{r}|^1 \xrightarrow{j^1} \mathcal{X}^1$ . Since  $|\mathfrak{r}|^1$  is a cover, so is  $|\mathfrak{r}|^1 \times |\mathfrak{r}|^1$ , so this pullback exists:

$$\begin{array}{ccc} |\mathfrak{r}|^2 & \xrightarrow{j^2} & \mathcal{X}^1 \\ \langle d_0, d_1 \rangle \downarrow & & \downarrow \langle d_0, d_1 \rangle \\ |\mathfrak{r}|^1 \times |\mathfrak{r}|^1 & \xrightarrow{j^1 \times j^1} & \mathcal{X}^0 \times \mathcal{X}^0 \end{array}$$

Spaces like  $|\mathfrak{r}|^3$  and maps like  $|\mathfrak{r}|^3 \xrightarrow{d_{02}} |\mathfrak{r}|^2$  can all be defined similarly.

The 2-map  $\mathfrak{r}$  additionally consists of a map  $|\mathfrak{x}|^1 \xrightarrow{\mathfrak{r}^1} \mathcal{Y}^1$  and a map  $|\mathfrak{x}|^2 \xrightarrow{\mathfrak{r}^2} \mathcal{Y}^2$ .

The following diagrams must commute:

$$\begin{array}{ccc}
 |\mathfrak{x}|^2 & \xrightarrow{d_0} & |\mathfrak{x}|^1 \\
 \searrow \mathfrak{r}^2 & & \searrow \mathfrak{r}^1 \\
 & & \mathcal{Y}^2 \xrightarrow{d_0} \mathcal{Y}^1
 \end{array}
 \qquad
 \begin{array}{ccc}
 |\mathfrak{x}|^2 & & \\
 \downarrow d_1 & \searrow \mathfrak{r}^2 & \\
 |\mathfrak{x}|^1 & & \mathcal{Y}^2 \\
 \searrow \mathfrak{r}^1 & & \downarrow d_1 \\
 & & \mathcal{Y}^1
 \end{array}
 \tag{2.14}$$

(These indicate the proper source and target for an arrow in the range of the 2-map.)

Now consider this commutative diagram:

$$\begin{array}{ccccc}
 |\mathfrak{x}|^3 & \xrightarrow{d_{01}} & |\mathfrak{x}|^2 & & \\
 \downarrow d_{12} & & \downarrow d_1 & \searrow \mathfrak{r}^2 & \\
 |\mathfrak{x}|^2 & \xrightarrow{d_0} & |\mathfrak{x}|^1 & & \mathcal{Y}^2 \\
 \searrow \mathfrak{r}^2 & & \searrow \mathfrak{r}^1 & & \downarrow d_1 \\
 & & \mathcal{Y}^2 & \xrightarrow{d_0} & \mathcal{Y}^1
 \end{array}$$

This is a pullback cone into the pullback diagram defining  $\mathcal{Y}^3$ , so it defines a map  $|\mathfrak{x}|^3 \xrightarrow{\mathfrak{r}^3} \mathcal{Y}^3$ . The final requirement for a 2-map is that the following *functoriality* diagrams commute:

$$\begin{array}{ccc}
 |\mathfrak{x}|^1 & \xrightarrow{\mathfrak{r}^1} & \mathcal{Y}^1 \\
 \downarrow d_{00} & & \downarrow d_{00} \\
 |\mathfrak{x}|^2 & \xrightarrow{\mathfrak{r}^2} & \mathcal{Y}^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 |\mathfrak{x}|^3 & \xrightarrow{\mathfrak{r}^3} & \mathcal{Y}^3 \\
 \downarrow d_{02} & & \downarrow d_{02} \\
 |\mathfrak{x}|^2 & \xrightarrow{\mathfrak{r}^2} & \mathcal{Y}^2
 \end{array}
 \tag{2.15}$$

These axioms are enough to ensure that you can construct arbitrary  $\mathfrak{r}^n$  and that  $\mathfrak{r}$  always commutes with  $d$ .

### 2.2.3 Natural transformations

There is a new twist with 2-spaces. Not only are there 2-maps between 2-spaces, but there is also a kind of mapping between 2-maps. Since these have no name in ordinary (uncategorified) geometry, I borrow the name ‘natural transformation’ from pure category theory. But most properly, these are *smooth* natural transformations.

So, given 2-spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , and given 2-maps  $\mathfrak{x}$  and  $\mathfrak{y}$ , both from  $\mathcal{X}$  to  $\mathcal{Y}$ , a **natural transformation**  $\chi$  from  $\mathfrak{x}$  to  $\mathfrak{y}$  consists of a single map  $|\mathfrak{x}|^1 \cap |\mathfrak{y}|^1 \xrightarrow{|\chi|} \mathcal{Y}^2$ . Of course, there are conditions. First, these diagrams must commute, as usual specifying the proper source and target:

$$\begin{array}{ccc}
 |\mathfrak{x}|^1 \cap |\mathfrak{y}|^1 & \xrightarrow{j_{[0]}^1} & |\mathfrak{x}|^1 \\
 \downarrow |\chi| & & \downarrow \mathfrak{x}^1 \\
 \mathcal{Y}^2 & \xrightarrow{d_0} & \mathcal{Y}^1
 \end{array}
 \qquad
 \begin{array}{ccc}
 |\mathfrak{x}|^1 \cap |\mathfrak{y}|^1 & \xrightarrow{|\chi|} & \mathcal{Y}^2 \\
 \downarrow j_{[1]}^1 & & \downarrow d_1 \\
 |\mathfrak{y}|^1 & \xrightarrow{\mathfrak{y}^1} & \mathcal{Y}^1
 \end{array}
 \qquad (2.16)$$

Now the commutative diagram

$$\begin{array}{ccccc}
 |\mathfrak{x}|^2 \cap |\mathfrak{y}|^2 & \xrightarrow{d_0} & |\mathfrak{x}|^1 \cap |\mathfrak{y}|^1 & & \\
 \downarrow j_{[1]}^2 & & \downarrow j_{[1]}^1 & \searrow |\chi| & \\
 |\mathfrak{y}|^2 & \xrightarrow{d_0} & |\mathfrak{y}|^1 & & \mathcal{Y}^2 \\
 \searrow \mathfrak{y}^2 & & \searrow \mathfrak{y}^1 & & \downarrow d_1 \\
 & & \mathcal{Y}^2 & \xrightarrow{d_0} & \mathcal{Y}^1
 \end{array}$$

defines a map  $|\mathfrak{x}|^2 \cap |\mathfrak{h}|^2 \xrightarrow{|\chi|_{[0]1}} \mathcal{Y}^3$ , and the commutative diagram

$$\begin{array}{ccccc}
 |\mathfrak{x}|^2 \cap |\mathfrak{h}|^2 & \xrightarrow{j_{[0]}^2} & |\mathfrak{x}|^2 & & \\
 \downarrow d_1 & & \downarrow d_1 & \searrow r^2 & \\
 |\mathfrak{x}|^1 \cap |\mathfrak{h}|^1 & \xrightarrow{j_{[0]}^1} & |\mathfrak{x}|^1 & & \mathcal{Y}^2 \\
 \searrow |\chi| & & \searrow r^1 & & \downarrow d_1 \\
 & & \mathcal{Y}^2 & \xrightarrow{d_0} & \mathcal{Y}^2
 \end{array}$$

defines a map  $|\mathfrak{x}|^2 \cap |\mathfrak{h}|^2 \xrightarrow{|\chi|_{[0]1}} \mathcal{Y}^3$ . Then the final requirement for a natural transformation is that the following diagram commute:

$$\begin{array}{ccc}
 |\mathfrak{x}|^2 \cap |\mathfrak{h}|^2 & \xrightarrow{|\chi|_{[0]1}} & \mathcal{Y}^3 \\
 \downarrow |\chi|_{[0]1} & & \downarrow d_{02} \\
 \mathcal{Y}^3 & \xrightarrow{d_{02}} & \mathcal{Y}^2
 \end{array} \tag{2.17}$$

This condition is the *naturality square*; although because of the internalised presentation, it's not the same shape as the familiar square with this name.

**Proposition 12** *Using suitable operations, 2-spaces, 2-maps, and natural transformations now form a bicategory (a weak 2-category)  $\mathfrak{C}$ .*

**Proof:** The argument is a combination of the argument in [Ehr] that internal categories, (strict) internal functors, and internal natural transformations form a (strict) 2-category and the argument in [Mak] that (small) categories, (small) anafunctors, and natural transformations form a bicategory.

Let me start by proving that, given 2-spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , the 2-maps from  $\mathcal{X}$  to  $\mathcal{Y}$  and the natural transformations between them form a category. First, given a 2-map



$\mathcal{X} \xrightarrow{\underline{\mathfrak{r}}} \mathcal{Y}$ , I want to define the identity natural transformation  $\underline{\mathfrak{r}}$  of  $\mathfrak{r}$ . Consider the space  $|\mathfrak{r}|^{1[2]}$  (that is the kernel pair of  $|\mathfrak{r}|^1 \xrightarrow{j^1} \mathcal{X}^1$ , which should be the source of  $|\underline{\mathfrak{r}}|$ ).

This diagram commutes:

$$\begin{array}{ccccc} |\mathfrak{r}|^{1[2]} & \xrightarrow{j^{1[2]}} & \mathcal{X}^1 & \xrightarrow{d_{00}} & \mathcal{X}^2 \\ \langle j_{[0]}^1, j_{[1]}^2 \rangle \downarrow & & & & \downarrow \langle d_0, d_1 \rangle \\ |\mathfrak{r}|^1 \times |\mathfrak{r}|^1 & \xrightarrow{j^1 \times j^1} & \mathcal{X}^1 & & \mathcal{X}^1 \end{array}$$

Since  $|\mathfrak{r}|^2$  is given as a pullback, this diagram defines a map  $|\mathfrak{r}|^{1[2]} \longrightarrow |\mathfrak{r}|^2$ ; let  $|\mathfrak{r}|^{1[2]} \xrightarrow{|\underline{\mathfrak{r}}|} \mathcal{Y}^2$  be the composite of this map with  $\mathfrak{r}^2$ . Then the diagrams (2.16, 2.17) easily commute.

Now given 2-maps  $\mathfrak{r}$ ,  $\mathfrak{h}$ , and  $\mathfrak{z}$ , all from  $\mathcal{X}$  to  $\mathcal{Y}$ , let  $\chi$  be a natural transformation from  $\mathfrak{r}$  to  $\mathfrak{h}$ , and let  $\psi$  be a natural transformation from  $\mathfrak{h}$  to  $\mathfrak{z}$ . To define the composite of  $\chi$  and  $\psi$ , the key is this commutative diagram:

$$\begin{array}{ccccc} |\mathfrak{r}|^1 \cap |\mathfrak{h}|^{1[2]} \cap |\mathfrak{z}|^1 & \xrightarrow{j_{[013]}^1} & |\mathfrak{r}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{z}|^1 & & \\ \downarrow j_{[023]}^1 & & \downarrow \langle j_{[01]}^1, j_{[12]}^1 \rangle & & \\ & & |\mathfrak{r}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{z}|^1 & & \\ & & \downarrow |\chi| \cap |\psi| & & \\ & & \mathcal{Y}^{[3]} & & \\ & & \downarrow j_{[02]}^1 & & \\ |\mathfrak{r}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{z}|^1 & \xrightarrow{\langle j_{[01]}^1, j_{[12]}^1 \rangle} & |\mathfrak{r}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{z}|^1 & \xrightarrow{|\chi| \cap |\psi|} & \mathcal{Y}^{[3]} \xrightarrow{j_{[02]}^1} \mathcal{Y}^{[2]} \end{array}$$

This diagram is an example of (1.4) for the cover  $|\mathfrak{r}|^1 \cap |\mathfrak{h}|^1 \cap |\mathfrak{z}|^1 \xrightarrow{j_{[02]}^1} |\mathfrak{r}|^1 \cap |\mathfrak{z}|^1$ , so (1.5) defines the map  $|\mathfrak{r}|^1 \cap |\mathfrak{z}|^1 \longrightarrow \mathcal{Y}^2$  that defines the composite natural transformation.

It is straightforward (but tedious) to check that this composition is associative and that the identity natural transformations are indeed identities.

For the identity 2-map  $\underline{\mathcal{X}}$  on the 2-space  $\mathcal{X}$ , take  $|\underline{\mathcal{X}}|^1 := \mathcal{X}^1$ ,  $|\underline{\mathcal{X}}|^2 := \mathcal{X}^2$ ,  $\underline{\mathcal{X}}^1 := \underline{\mathcal{X}}^1$ , and  $\underline{\mathcal{X}}^2 := \underline{\mathcal{X}}^2$ . All of the conditions to be met are trivial.

If  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{Y}$  and  $\mathcal{Y} \xrightarrow{\mathfrak{y}} \mathcal{Z}$  are 2-maps, then I need to define a 2-map  $\mathcal{X} \xrightarrow{\mathfrak{z}} \mathcal{Z}$ . Let  $|\mathfrak{z}|^1$  be given by the pullback of  $|\mathfrak{x}|^1 \xrightarrow{\mathfrak{x}^1} \mathcal{Y}^1$  along the cover  $|\mathfrak{y}|^1 \xrightarrow{j_{\mathfrak{y}}^1} \mathcal{Y}^1$ . This pullback comes with a map  $|\mathfrak{z}|^1 \longrightarrow |\mathfrak{x}|^1$ , which composes with  $|\mathfrak{x}|^1 \xrightarrow{j_{\mathfrak{x}}^1} \mathcal{X}^1$  to form the cover  $|\mathfrak{z}|^1 \xrightarrow{j_{\mathfrak{z}}^1} \mathcal{X}^1$ ; and it comes with a map  $|\mathfrak{z}|^1 \longrightarrow |\mathfrak{y}|^1$ , which composes with  $|\mathfrak{y}|^1 \xrightarrow{\mathfrak{y}^1} \mathcal{Z}^1$  to form  $|\mathfrak{z}|^1 \xrightarrow{\mathfrak{z}^1} \mathcal{Z}^1$ . Form the pullback  $|\mathfrak{z}|^2$ ; the cover  $|\mathfrak{z}|^2 \xrightarrow{j_{\mathfrak{z}}^2} \mathcal{X}^1$  factors through  $|\mathfrak{x}|^2$  (using the latter's universal property) to define a map  $|\mathfrak{z}|^2 \longrightarrow |\mathfrak{x}|^2$ , which composes with  $|\mathfrak{x}|^2 \xrightarrow{\mathfrak{x}^2} \mathcal{Y}^2$  to form a map  $|\mathfrak{z}|^2 \longrightarrow \mathcal{Y}^2$ . Now the universal property of  $|\mathfrak{y}|^2$  defines a map  $|\mathfrak{z}|^2 \longrightarrow |\mathfrak{y}|^2$ , which composes with  $|\mathfrak{y}|^2 \xrightarrow{j_{\mathfrak{y}}^2} \mathcal{Y}^2$  to form  $|\mathfrak{z}|^2 \xrightarrow{\mathfrak{z}^2} \mathcal{Z}^2$ .

The various coherence conditions in a (weak) 2-category are now tedious but straightforward to check. ■

An internal **natural isomorphism** is simply an internal natural transformation equipped with an inverse in the sense of this 2-category  $\mathfrak{C}$ .

## 2.2.4 Spaces as 2-spaces

The category  $\mathbf{C}$  of spaces may be embedded into the 2-category  $\mathfrak{C}$  of 2-spaces. That is, each space may be interpreted as a 2-space, and maps between spaces will be certain 2-maps between the corresponding 2-spaces. Specifically, if  $X$  is a space, then

let  $\mathcal{X}^1$  and  $\mathcal{X}^2$  each be  $X$ , and let all the  $d$  maps in the definition of 2-space be the identity on  $X$ ; this defines a 2-space  $\mathcal{X}$ . To define a 2-map out of this 2-space  $\mathcal{X}$ , you need a cover  $|\mathfrak{x}|^1 \xrightarrow{j^1} X$ ; but I will call this simply  $|\mathfrak{x}| \xrightarrow{j} X$  now, because in fact  $|\mathfrak{x}|^n$  is simply  $|\mathfrak{x}|^{[n]}$  in this case. So if  $\mathcal{Y}$  is a 2-space, then a 2-map  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{Y}$  consists of a cover  $|\mathfrak{x}| \xrightarrow{j} X$ , a map  $|\mathfrak{x}| \xrightarrow{\mathfrak{r}^1} \mathcal{Y}^1$  and a map  $|\mathfrak{x}|^{[2]} \xrightarrow{\mathfrak{r}^2} \mathcal{Y}^2$ , such that these diagrams commute:

$$\begin{array}{ccccc}
|\mathfrak{x}|^{[2]} & \xrightarrow{j_{[0]}} & |\mathfrak{x}| & & \\
& \searrow \mathfrak{r}^2 & \searrow \mathfrak{r}^1 & & \\
& & \mathcal{Y}^2 & \xrightarrow{d_0} & \mathcal{Y}^1
\end{array}
\qquad
\begin{array}{ccc}
|\mathfrak{x}|^{[2]} & & \\
j_{[1]} \downarrow & \searrow \mathfrak{r}^2 & \\
|\mathfrak{x}| & & \mathcal{Y}^2 \\
& \searrow \mathfrak{r}^1 & \downarrow d_1 \\
& & \mathcal{Y}^1
\end{array}
\qquad
\begin{array}{ccc}
|\mathfrak{x}| & \xrightarrow{\mathfrak{r}^1} & \mathcal{Y}^1 \\
j_{[00]} \downarrow & & \downarrow d_{00} \\
|\mathfrak{x}|^{[2]} & \xrightarrow{\mathfrak{r}^2} & \mathcal{Y}^2
\end{array}
\qquad
\begin{array}{ccc}
|\mathfrak{x}|^{[3]} & \xrightarrow{\mathfrak{r}^3} & \mathcal{Y}^3 \\
j_{[02]} \downarrow & & \downarrow d_{02} \\
|\mathfrak{x}|^{[2]} & \xrightarrow{\mathfrak{r}^2} & \mathcal{Y}^2
\end{array}$$

Note that if  $\mathcal{Y}$  is derived from a space  $Y$ , then the map  $|\mathfrak{x}| \xrightarrow{\mathfrak{r}^1} Y$  defines a map  $X \xrightarrow{x} Y$  (because  $X$  is the quotient of  $|\mathfrak{x}|^{[2]}$ ); conversely, any such map defines a 2-map where  $|\mathfrak{x}| := X$ . Furthermore, if you start with  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{Y}$ , turn it into  $X \xrightarrow{x} Y$ , and turn that back into a new  $\mathcal{X}' \xrightarrow{\mathfrak{r}'} \mathcal{Y}$ , then  $\mathfrak{r}$  and  $\mathfrak{r}'$  are naturally isomorphic, with the natural isomorphism  $\chi$  given by  $|\chi| := \mathfrak{r}^2$ .

Finally, a group  $G$  defines a 2-space  $\mathcal{G}$  as follows. Let  $\mathcal{G}^1$  be the trivial space 1, and let  $\mathcal{G}^2$  be  $G$ ; then composition in  $\mathcal{G}$  is given by multiplication in  $G$ . (If  $G$  is abelian, then  $\mathcal{G}$  will also have the structure of a 2-group —see section 2.4.1—, but not in general.)

Comparing the previous paragraphs, I get this interesting result:

**Proposition 13** *Given a space  $B$  and a group  $G$ , interpret  $B$  as a 2-space  $\mathcal{B}$  where*

$\mathcal{B}^1, \mathcal{B}^2 := B$ , and interpret  $G$  as a 2-space  $\mathcal{G}$  where  $\mathcal{G}^1 := 1$  and  $\mathcal{G}^2 := G$ . Then the category of 2-maps from  $\mathcal{B}$  to  $\mathcal{G}$  is equivalent to the category of  $G$ -transitions on  $B$ .

**Proof:** The cover  $|\mathfrak{x}|$  of the 2-map is the cover  $U$  of the bundle, the map  $\mathfrak{x}^1$  is trivial, the map  $\mathfrak{x}^2$  is the transition map  $g$ , the laws (2.14) are trivial, and the functoriality diagrams (2.15) are (respectively) the laws  $\eta$  and  $\gamma$  described in the diagrams (1.9). Given a pair of 2-maps and the corresponding pair of transitions, the map  $|\chi|$  of a natural transformation is the map  $b$  of a transition morphism; checking that composition is well behaved as well is tedious but straightforward.

■

Thus the theory of (uncategorified) bundles is intimately tied up in the general notion of (categorified) 2-space; this is just an interesting aside now, but it will be important in part 3, where I link the to the notions of gerbe and bundle gerbe.

### 2.2.5 2-Covers in the 2-category of 2-spaces

If  $\mathcal{B}$  is a 2-space, then let a **2-cover** of  $\mathcal{B}$  be a 2-space  $\mathcal{U}$  equipped with a 2-map  $\mathcal{U} \xrightarrow{j} \mathcal{B}$  such that the maps  $|\mathfrak{j}|^1 \xrightarrow{j^1} \mathcal{B}^1$  and  $|\mathfrak{j}|^2 \xrightarrow{j^2} \mathcal{B}^2$  are both covers in the category  $\mathbf{C}$ .

Just as pullbacks along covers always exist, so 2-pullbacks along 2-covers always exist. This is because you can pullback along the covers  $j^1$  and  $j^2$  to define the spaces in the 2-pullback, and the universal property of these pullbacks will give the coherence laws of the 2-pullback as a 2-space.

In the case of 2-quotients, the map  $j^1$  in the 2-cover itself becomes the cover for a 2-map out of  $B$ , so in this way 2-covers have 2-quotients as well.

In the case that the 2-spaces involved are spaces (2-spaces that are discrete in the category-theoretic sense), then their 2-products, 2-pullbacks, and 2-quotients in  $\mathfrak{C}$  are the same as in  $\mathbf{C}$  (when they exist).

## 2.3 2-Bundles

The most general notion of 2-bundle is quite simple; if  $\mathcal{B}$  is a 2-space, then a **2-bundle** over  $\mathcal{B}$  is simply a 2-space  $\mathcal{E}$  together with a 2-map  $\mathcal{E} \xrightarrow{p} \mathcal{B}$ . Before long, of course, I will restrict attention to 2-bundles with more structure than this— eventually, to locally trivial 2-bundles with a structure 2-group.

In physical applications where  $\mathcal{B}$  is spacetime, the 2-space  $\mathcal{B}$  will be discrete in the category-theoretic sense; that is, it comes from a space. This would simplify some of the following presentation.

### 2.3.1 The 2-category of 2-bundles

For a proper notion of equivalence of 2-bundles, I should define the 2-category  $\mathfrak{C}/\mathcal{B}$  of 2-bundles over  $\mathcal{B}$ .

Given 2-bundles  $\mathcal{E}$  and  $\mathcal{E}'$  over  $\mathcal{B}$ , a **2-bundle morphism** from  $\mathcal{E}$  to  $\mathcal{E}'$  is a 2-map

$\mathcal{E} \xrightarrow{f} \mathcal{E}'$  equipped with a natural isomorphism  $\pi$ :

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{f} & \mathcal{E}' \\
 \downarrow p & \Downarrow \pi & \downarrow p' \\
 & & \mathcal{B}
 \end{array}$$

Given 2-bundle morphisms  $f$  from  $\mathcal{E}$  to  $\mathcal{E}'$  and  $f'$  from  $\mathcal{E}'$  to  $\mathcal{E}''$ , the composite 2-bundle morphism is the composite 2-map  $\mathcal{E} \xrightarrow{f} \mathcal{E}' \xrightarrow{f'} \mathcal{E}''$  equipped with this natural transformation:

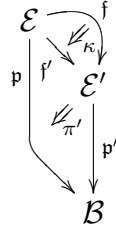
$$\begin{array}{ccccc}
 \mathcal{E} & \xrightarrow{f} & \mathcal{E}' & \xrightarrow{f'} & \mathcal{E}'' \\
 \downarrow p & \Downarrow \pi & \downarrow p' & \Downarrow \pi' & \downarrow p'' \\
 & & \mathcal{B} & & 
 \end{array}$$

Also, the identity 2-bundle morphism on  $\mathcal{E}$  is simply the identity 2-map on  $\mathcal{E}$  equipped with its identity natural transformation.

So far, this section is quite analogous to section 1.3.1, with only a few natural isomorphisms added in. But now the categorified case has something distinctly new. Given 2-bundle morphisms  $f$  and  $f'$  both from  $\mathcal{E}$  to  $\mathcal{E}'$ , a **2-bundle 2-morphism** from  $f$  to  $f'$  is a natural transformation  $\kappa$  from  $f$  to  $f'$ :

$$\begin{array}{ccc}
 \mathcal{E} & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \kappa \\ \xrightarrow{f'} \end{array} & \mathcal{E}'
 \end{array}$$

such that the following natural transformation:

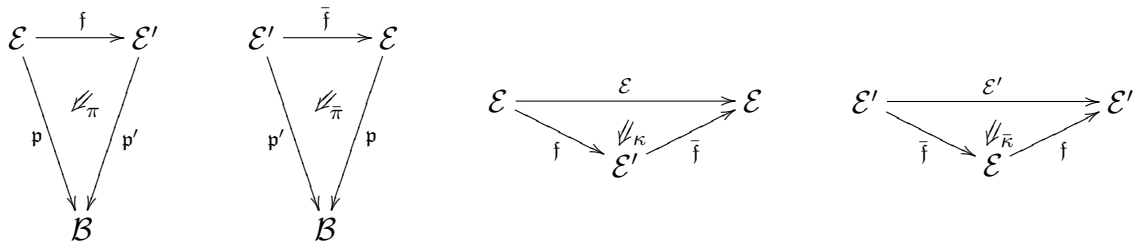


is equal to  $\pi$ .

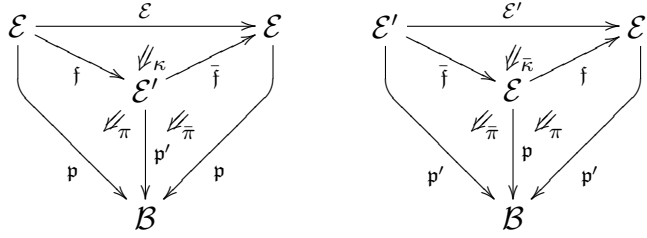
**Proposition 14** *Given a 2-space  $\mathcal{B}$ , 2-bundles over  $\mathcal{B}$ , 2-bundle morphisms, and 2-bundle 2-morphisms form a 2-category.*

**Proof:**  $\mathfrak{C}/\mathcal{B}$  is just a slice 2-category of the 2-category  $\mathfrak{C}$  of 2-spaces. ■

I now know what it means for 2-bundles  $\mathcal{E}$  and  $\mathcal{E}'$  to be **equivalent 2-bundles**: weakly isomorphic objects in the 2-category  $\mathfrak{C}/\mathcal{B}$ . That is, there must be a 2-bundle morphism  $f$  from  $\mathcal{E}$  to  $\mathcal{E}'$  equipped with a weak inverse, which is a 2-bundle morphism  $\bar{f}$  from  $\mathcal{E}'$  to  $\mathcal{E}$ , an invertible 2-bundle 2-morphism  $\kappa$  from the identity 2-bundle morphism on  $\mathcal{E}$  to the composite of  $f$  and  $\bar{f}$ , and an invertible 2-bundle 2-morphism  $\bar{\kappa}$  from the identity 2-bundle morphism on  $\mathcal{E}'$  to the composite of  $\bar{f}$  and  $f$ . What this amounts to, then, are 2-maps  $\mathcal{E} \xrightarrow{f} \mathcal{E}'$  and  $\mathcal{E}' \xrightarrow{\bar{f}} \mathcal{E}$  equipped with natural isomorphisms  $\pi$ ,  $\bar{\pi}$ ,  $\kappa$ , and  $\bar{\kappa}$ :



such that the following natural transformations



are identity natural transformations on (respectively)  $\mathcal{E} \xrightarrow{p} \mathcal{B}$  and  $\mathcal{E}' \xrightarrow{p'} \mathcal{B}$ . In particular,  $\mathcal{E}$  and  $\mathcal{E}'$  are equivalent as 2-spaces.

### 2.3.2 Trivial 2-bundles

If  $\mathcal{B}$  is a space and  $\mathcal{F}$  is a 2-space, then the Cartesian 2-product  $\mathcal{F} \times \mathcal{B}$  is automatically a 2-bundle over  $\mathcal{B}$ . Just let the 2-map be  $\mathcal{F} \times \mathcal{B} \xrightarrow{\hat{\mathcal{F}} \times \mathcal{B}} \mathcal{B}$ , the projection onto the right factor in the product.

Of this 2-bundle, it can rightly be said that  $\mathcal{F}$  is its fibre. I'll want to define a more general notion, however, of 2-bundle over  $\mathcal{B}$  with fibre  $\mathcal{F}$ . To begin, I'll generalise this example to the notion of a trivial 2-bundle. Specifically, a **trivial 2-bundle** over  $\mathcal{B}$  with fibre  $\mathcal{F}$  is simply a 2-bundle over  $\mathcal{B}$  equipped with a 2-bundle equivalence to it from  $\mathcal{F} \times \mathcal{B}$ .

In more detail, this is a 2-space  $\mathcal{E}$  equipped with 2-maps  $\mathcal{E} \xrightarrow{p} \mathcal{B}$ ,  $\mathcal{F} \times \mathcal{B} \xrightarrow{f} \mathcal{E}$ ,



and  $\mathcal{E} \xrightarrow{\bar{f}} \mathcal{F} \times \mathcal{B}$  equipped with natural isomorphisms  $\pi$ ,  $\bar{\pi}$ ,  $\kappa$ , and  $\bar{\kappa}$ :

$$\begin{array}{ccc}
 \begin{array}{c} \mathcal{F} \times \mathcal{B} \xrightarrow{f} \mathcal{E} \\ \hat{f} \times \mathcal{B} \searrow \swarrow \pi \\ \mathcal{B} \end{array} & 
 \begin{array}{c} \mathcal{E} \xrightarrow{\bar{f}} \mathcal{F} \times \mathcal{B} \\ \mathcal{E} \searrow \swarrow \bar{\pi} \\ \mathcal{B} \end{array} & 
 \begin{array}{ccc} \mathcal{F} \times \mathcal{B} & \xrightarrow{\mathcal{F} \times \mathcal{B}} & \mathcal{F} \times \mathcal{B} \\ \downarrow f & \Downarrow \kappa & \downarrow \bar{f} \\ & \mathcal{E} & \end{array} & 
 \begin{array}{ccc} \mathcal{E} & \xrightarrow{\mathcal{E}} & \mathcal{E} \\ \downarrow \bar{f} & \Downarrow \bar{\kappa} & \downarrow f \\ & \mathcal{F} \times \mathcal{B} & \end{array}
 \end{array}$$

such that the following natural transformations

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{B} & \xrightarrow{\mathcal{F} \times \mathcal{B}} & \mathcal{F} \times \mathcal{B} \\
 \downarrow f & \Downarrow \kappa & \downarrow \bar{f} \\
 \mathcal{E} & & \mathcal{E} \\
 \downarrow \hat{f} \times \mathcal{B} & \Downarrow \pi & \downarrow \hat{f} \times \mathcal{B} \\
 \mathcal{B} & & \mathcal{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\mathcal{E}} & \mathcal{E} \\
 \downarrow \bar{f} & \Downarrow \bar{\kappa} & \downarrow f \\
 \mathcal{F} \times \mathcal{B} & & \mathcal{F} \times \mathcal{B} \\
 \downarrow \mathcal{E} & \Downarrow \bar{\pi} & \downarrow \hat{f} \times \mathcal{B} \\
 \mathcal{B} & & \mathcal{B}
 \end{array}$$

are identity natural transformations on (respectively)  $\mathcal{F} \times \mathcal{B} \xrightarrow{\hat{f} \times \mathcal{B}} \mathcal{B}$  and  $\mathcal{E} \xrightarrow{p} \mathcal{B}$ .

### 2.3.3 Restrictions of 2-bundles

Ultimately, I'll want to deal with *locally* trivial 2-bundles, so I need a notion of restriction to a 2-cover. Analogously to section 1.3.3, I'll initially model the 2-cover by any 2-map  $\mathcal{U} \xrightarrow{j} \mathcal{B}$ . I call such a map an (unqualified) **2-subspace** of  $\mathcal{B}$ . (Thus logically, there is no difference between a 2-subspace of  $\mathcal{B}$  and a 2-bundle over  $\mathcal{B}$ ; but one does different things with them, and they will be refined in different ways.)

Given a 2-bundle  $\mathcal{E}$  and a 2-subspace  $\mathcal{U}$  (equipped with the 2-map  $j$  as above), I get a 2-pullback diagram:

$$\begin{array}{ccc}
 & \mathcal{E} & \\
 & \downarrow p & \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{B}
 \end{array}$$

Then the **restriction**  $\mathcal{E}|_{\mathcal{U}}$  of  $\mathcal{E}$  to  $\mathcal{U}$  is *any* 2-pullback of this diagram, if any exists. I name the associated 2-maps and natural isomorphism as in this commutative diagram:

$$\begin{array}{ccc} \mathcal{E}|_{\mathcal{U}} & \xrightarrow{\tilde{j}} & \mathcal{E} \\ \text{p}|_{\mathcal{U}} \downarrow & \cong & \downarrow \text{p} \\ \mathcal{U} & \xrightarrow{j} & \mathcal{B} \end{array}$$

Notice that  $\mathcal{E}|_{\mathcal{U}}$  becomes both a 2-subspace of  $\mathcal{E}$  and a 2-bundle over  $\mathcal{U}$ .

By the unicity of 2-pullbacks, the restriction  $\mathcal{E}|_{\mathcal{U}}$ , if it exists, is well defined up to 2-diffeomorphism (equivalence in  $\mathfrak{C}$ ); but is it well defined up to equivalence of 2-bundles over  $\mathcal{U}$ ? The answer is yes, because the 2-pullback cone morphisms in diagram (2.2) become 2-bundle morphisms when applied to this situation.

### 2.3.4 Locally trivial 2-bundles

I can now combine the preceding ideas to define a locally trivial 2-bundle with a given fibre, which is the general notion of fibre 2-bundle without a fixed structure 2-group.

Given a 2-space  $\mathcal{B}$  and a 2-space  $\mathcal{F}$ , suppose that  $\mathcal{B}$  has been supplied with a 2-cover  $\mathcal{U} \xrightarrow{j} \mathcal{B}$ ; this 2-cover is a 2-subspace. Then a **locally trivial 2-bundle** over  $\mathcal{B}$  with fibre  $\mathcal{F}$  subordinate to  $\mathcal{U}$  is a 2-bundle  $\mathcal{E}$  over  $\mathcal{B}$  such that the trivial 2-bundle  $\mathcal{F} \times \mathcal{U}$  over  $\mathcal{U}$  is a restriction  $\mathcal{E}|_{\mathcal{U}}$  (once an appropriate  $\tilde{j}$  has been specified).

So to sum up, this consists of the following items:

- a **base 2-space**  $\mathcal{B}$ ;
- a **cover 2-space**  $\mathcal{U}$ ;

- a **fibre 2-space**  $\mathcal{F}$ ;
- a **total 2-space**  $\mathcal{E}$ ;
- a **cover 2-map**  $\mathcal{U} \xrightarrow{j} \mathcal{B}$ ;
- a **projection 2-map**  $\mathcal{E} \xrightarrow{p} \mathcal{B}$ ; and
- a **pulled-back 2-map**  $\mathcal{F} \times \mathcal{U} \xrightarrow{\tilde{j}} \mathcal{E}$ ;

equipped with a **2-pullback** natural isomorphism  $\tilde{\omega}$ :

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\tilde{j}} & \mathcal{E} \\
 \hat{\mathcal{F}} \times \mathcal{U} \downarrow & \Downarrow_{\tilde{\omega}} & \downarrow p \\
 \mathcal{U} & \xrightarrow{j} & \mathcal{B}
 \end{array} \tag{2.18}$$

Even though  $\omega$  could be ignored in section 2.1.5, the natural transformation  $\tilde{\omega}$  incorporates the equivalence of  $\mathcal{F} \times \mathcal{U}$  and  $\mathcal{E}|_{\mathcal{U}}$ , so it is an essential ingredient. (However, since most of the action takes place back in  $\mathcal{U}$ , it doesn't come up much.)

In studying fibre 2-bundles, I regard  $\mathcal{B}$  as a fixed structure on which the 2-bundle is defined. The fibre  $\mathcal{F}$  (like the 2-group  $\mathcal{G}$  in the next section) indicates the type of problem that one is considering; while the 2-cover  $\mathcal{U}$  (together with  $j$ ) is subsidiary structure that is not preserved by 2-bundle morphisms. Accordingly,  $\mathcal{E}$  (together with  $p$ ) 'is' the 2-bundle; I may use superscripts or primes on  $\mathfrak{p}$  if I'm studying more than one 2-bundle.

## 2.4 $\mathcal{G}$ -2-spaces

Just as a group can act on a space, so a 2-group can act on a 2-space.

### 2.4.1 2-Groups

I define here what [HDA5] calls a ‘coherent’ 2-group. For me, they are the default type of 2-group. My notation differs slightly from that of [HDA5] in order to maintain the conventions indicated earlier. But my definition is the same as [HDA5, Definition 19].

First, a **2-monoid** is a 2-space  $\mathcal{G}$  together with 2-maps  $1 \xrightarrow{\epsilon} \mathcal{G}$  (the identity, or unit, object) and  $\mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}$  (multiplication), as well as natural isomorphisms  $\alpha$ ,  $\lambda$ , and  $\rho$ :

$$\begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} & \mathcal{G} \xrightarrow{\epsilon \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} & \mathcal{G} \xrightarrow{\underline{\mathcal{G}} \times \epsilon} \mathcal{G} \times \mathcal{G} \\
 \underline{\mathcal{G}} \times m \downarrow \quad \swarrow \alpha \quad \downarrow m & \searrow \underline{\mathcal{G}} \quad \swarrow \lambda \quad \downarrow m & \searrow \underline{\mathcal{G}} \quad \swarrow \rho \quad \downarrow m \\
 \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G} & \mathcal{G} & \mathcal{G}
 \end{array} \quad (2.19)$$

These isomorphisms are (respectively) the *associator* and the *left and right unitors*. Additionally, in each of the following pairs of diagrams, the diagram on the left must define the same composite natural transformation as the diagram on the right:

$$\begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times \underline{\mathcal{G}} \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} \times \mathcal{G} & & \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times \underline{\mathcal{G}} \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} \times \mathcal{G} \\
 \underline{\mathcal{G}} \times \underline{\mathcal{G}} \times m \downarrow \quad \searrow \underline{\mathcal{G}} \times m \times \underline{\mathcal{G}} \quad \swarrow \alpha \times \underline{\mathcal{G}} \quad \searrow m \times \underline{\mathcal{G}} & & \underline{\mathcal{G}} \times \underline{\mathcal{G}} \times m \downarrow \quad \parallel \quad \downarrow \underline{\mathcal{G}} \times m \quad \searrow m \times \underline{\mathcal{G}} \\
 \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{\underline{\mathcal{G}} \times \alpha} \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} & = & \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times \underline{\mathcal{G}}} \mathcal{G} \times \mathcal{G} \xrightarrow{\underline{\mathcal{G}} \times \alpha} \mathcal{G} \times \mathcal{G} \\
 \underline{\mathcal{G}} \times m \downarrow \quad \swarrow \alpha \quad \downarrow m & & \underline{\mathcal{G}} \times m \downarrow \quad \swarrow \alpha \quad \downarrow m \\
 \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G} & & \mathcal{G} \xrightarrow{m} \mathcal{G}
 \end{array}$$

and:

$$\begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} & \xrightarrow{\underline{\mathcal{G}} \times \epsilon \times \underline{\mathcal{G}}} & \mathcal{G} \times \mathcal{G} \times \mathcal{G} \\
 & \searrow_{\underline{\mathcal{G}} \times \underline{\mathcal{G}}} & \downarrow_{\underline{\mathcal{G}} \times \lambda} \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \swarrow_{\alpha} \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \downarrow_{\mathfrak{m}} \\
 & & \mathcal{G}
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} & \xrightarrow{\underline{\mathcal{G}} \times \epsilon \times \underline{\mathcal{G}}} & \mathcal{G} \times \mathcal{G} \times \mathcal{G} \\
 & \searrow_{\underline{\mathcal{G}} \times \underline{\mathcal{G}}} & \downarrow_{\mathfrak{m} \times \underline{\mathcal{G}}} \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \xrightarrow{\underline{\mathcal{G}} \times \underline{\mathcal{G}}} \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \downarrow_{\mathfrak{m}} \\
 & & \mathcal{G}
 \end{array}
 \quad (2.21)$$

These coherence requirements are known (respectively) as the *pentagon identity* and the *triangle identity*; these names ultimately derive from the number of natural isomorphisms that appear in each.

In terms of generalised elements, if  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{G}$  and  $\mathcal{X} \xrightarrow{\eta} \mathcal{G}$  are elements of  $\mathcal{G}$ , then I will (again) write  $(\mathfrak{r}\eta)$  for the composite  $\mathcal{X} \xrightarrow{\langle \mathfrak{r}, \eta \rangle} \mathcal{G} \times \mathcal{G} \xrightarrow{\mathfrak{m}} \mathcal{G}$ ; and I'll write  $1$  for  $\mathcal{X} \xrightarrow{\mathfrak{r}} 1 \xrightarrow{\epsilon} \mathcal{G}$  (suppressing  $\mathcal{X}$ ). Then if  $\mathfrak{r}$ ,  $\eta$ , and  $\mathfrak{z}$  are elements of  $\mathcal{G}$ , then the associator defines an arrow  $((\mathfrak{r}\eta)\mathfrak{z}) \xrightarrow{\alpha(\mathfrak{r}\eta\mathfrak{z})} (\mathfrak{r}(\eta\mathfrak{z}))$ , and the left and right unitors define (respectively) arrows  $(1\mathfrak{r}) \xrightarrow{\lambda(\mathfrak{r})} \mathfrak{r}$  and  $(\mathfrak{r}1) \xrightarrow{\rho(\mathfrak{r})} \mathfrak{r}$ . The pentagon identity then says that  $((((\mathfrak{w}\mathfrak{r})\eta)\mathfrak{z})) \xrightarrow{\alpha((\mathfrak{w}\mathfrak{r})\eta\mathfrak{z})} ((\mathfrak{w}(\mathfrak{r}\eta))\mathfrak{z}) \xrightarrow{\alpha(\mathfrak{w}(\mathfrak{r}\eta)\mathfrak{z})} (\mathfrak{w}((\mathfrak{r}\eta)\mathfrak{z})) \xrightarrow{(\mathfrak{w}\alpha(\mathfrak{r}\eta\mathfrak{z}))} (\mathfrak{w}(\mathfrak{r}(\eta\mathfrak{z})))$  is equal to  $((((\mathfrak{w}\mathfrak{r})\eta)\mathfrak{z})) \xrightarrow{\alpha((\mathfrak{w}\mathfrak{r})\eta\mathfrak{z})} ((\mathfrak{w}\mathfrak{r})(\eta\mathfrak{z})) \xrightarrow{\alpha(\mathfrak{w}\mathfrak{r}(\eta\mathfrak{z}))} (\mathfrak{w}(\mathfrak{r}(\eta\mathfrak{z})))$ ; and the triangle identity says that  $((\mathfrak{r}1)\eta) \xrightarrow{\alpha(\mathfrak{r}1\eta)} (\mathfrak{r}(1\eta)) \xrightarrow{(\mathfrak{r}\lambda(\eta))} (\mathfrak{r}\eta)$  is equal to  $(\rho(\mathfrak{r})\eta)$ .

2-Monoids have been studied extensively as ‘weak monoidal categories’; see [CWM, chapter 11] for a summary of the noninternalised case. But by thinking of them as categorifications of monoids, [HDA5] motivates the following definition to categorify groups: A **2-group** is a 2-monoid  $\mathcal{G}$  together with a 2-map  $\mathcal{G} \xrightarrow{i} \mathcal{G}$  (the inverse

operator), as well as natural isomorphisms  $\epsilon$  and  $\iota$ :

$$\begin{array}{ccc}
 \mathcal{G} & \xrightarrow{\langle i, \underline{\mathcal{G}} \rangle} & \mathcal{G} \times \mathcal{G} \\
 \hat{\mathcal{G}} \downarrow & \swarrow \epsilon & \downarrow m \\
 1 & \xrightarrow{\epsilon} & \mathcal{G}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{G} & \xrightarrow{\hat{\mathcal{G}}} & 1 \\
 \langle \underline{\mathcal{G}}, i \rangle \downarrow & \swarrow \iota & \downarrow e \\
 \mathcal{G} \times \mathcal{G} & \xrightarrow{m} & \mathcal{G}
 \end{array}
 \quad (2.22)$$

These are (respectively) the *left and right invertors*, but they're also called (respectively) the *counit* and *unit*, and this is not merely an analogy— they in fact form an (internal) adjunction; compare [HDA5, Definition 7].

These must also satisfy coherence laws:

and:

These laws are known as the *zig-zag identities*, because of their representation in string diagrams; see section 2.4.2.

In terms of generalised elements, if  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{G}$  is an element of  $\mathcal{G}$ , then I will (again) write  $\mathfrak{x}^{-1}$  for the composite  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{G} \xrightarrow{i} \mathcal{G}$ . Then the left and right invertors define (respectively) arrows  $(\mathfrak{x}^{-1}\mathfrak{x}) \xrightarrow{\epsilon(\mathfrak{x})} 1$  and  $(\mathfrak{x}\mathfrak{x}^{-1}) \xrightarrow{\iota(\mathfrak{x})} 1$ . The zig-zag identities state, respectively, that  $((\mathfrak{x}\mathfrak{x}^{-1})\mathfrak{x}) \xrightarrow{\alpha(\mathfrak{x}\mathfrak{x}^{-1}\mathfrak{x})} (\mathfrak{x}(\mathfrak{x}^{-1}\mathfrak{x})) \xrightarrow{\beta(\mathfrak{x})} (\mathfrak{x}1) \xrightarrow{\rho(\mathfrak{x})} \mathfrak{x}$  equals  $((\mathfrak{x}\mathfrak{x}^{-1})\mathfrak{x}) \xrightarrow{\iota(\mathfrak{x})} (1\mathfrak{x}) \xrightarrow{\lambda(\mathfrak{x})} \mathfrak{x}$ ; and that  $((\mathfrak{x}^{-1}\mathfrak{x})\mathfrak{x}^{-1}) \xrightarrow{\alpha(\mathfrak{x}^{-1}\mathfrak{x}\mathfrak{x}^{-1})} (\mathfrak{x}^{-1}(\mathfrak{x}\mathfrak{x}^{-1})) \xrightarrow{\beta(\mathfrak{x}^{-1})} (\mathfrak{x}^{-1}1) \xrightarrow{\rho(\mathfrak{x}^{-1})} \mathfrak{x}^{-1}$  equals  $((\mathfrak{x}^{-1}\mathfrak{x})\mathfrak{x}^{-1}) \xrightarrow{\epsilon(\mathfrak{x}^{-1})} (1\mathfrak{x}^{-1}) \xrightarrow{\lambda(\mathfrak{x}^{-1})} \mathfrak{x}^{-1}$ .

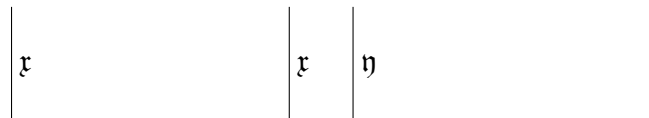
Note that since maps in the category  $\mathbf{C}$  are smooth functions, so that 2-maps in the 2-category  $\mathfrak{C}$  are smooth functors, a 2-group in that 2-category is automatically a *Lie 2-group*. At this point, I could define (internal) homomorphisms and 2-homomorphisms to get the 2-category  $\mathfrak{C}^{\text{Grp}}$  of internal 2-groups; this is done in [HDA5]. But in the following, I'm only interested in a single fixed 2-group. For the rest of part 2, let  $\mathcal{G}$  be this fixed 2-group.

## 2.4.2 String diagrams for 2-groups

The formulas in the previous section were complicated, and they'll only get worse further on. To simplify these, I'll make use of string diagrams, which are introduced in [Str] and used heavily in [HDA5]. The validity of these diagrams rests once again on the Mac Lane Coherence Theorem, which states, roughly speaking, that when manipulating element expressions in any 2-monoid, one may safely assume that  $\alpha$ ,

$\lambda$ , and  $\rho$  are all identities, which is the case in ‘strict’ 2-monoids. (More precisely, there is a unique coherent natural isomorphism between any two functors interpreting an expression from the free monoidal category on the variable names.) Thus I can prove coherence laws for general 2-groups by proving them for these ‘semistrict’ 2-groups. (However, the invertors  $\epsilon$  and  $\iota$  are unavoidable! Although one may prove —see [HDA5, Proposition 45]— that every 2-group is equivalent to a strict 2-group where  $\epsilon$  and  $\iota$  are also identities, this depends essentially on the axiom of choice and does not apply internally to the 2-category  $\mathfrak{C}$ .)

If  $\mathcal{X} \xrightarrow{x} \mathcal{G}$  is a generalised element of the 2-group  $\mathcal{G}$ , then this (or rather, its identity natural isomorphism) is drawn as in the diagram on the left below. Multiplication is shown by juxtaposition;  $(\mathfrak{x}\eta)$  is in the middle below. The identity element 1 is generally invisible, but it may be shown as on the right below to stress its presence.



Since string diagrams use the Mac Lane Coherence Theorem to pretend that all 2-monoids are strict, there is no distinction between  $((\mathfrak{x}\eta)\mathfrak{z})$  and  $(\mathfrak{x}(\eta\mathfrak{z}))$ , both of which are shown on the left below. Similarly, the picture on the right below may be interpreted as  $\mathfrak{x}$ ,  $(1\mathfrak{x})$ , or  $(\mathfrak{x}1)$ ; 1 is properly invisible here. You can also consider these maps to be pictures of the associator and unitors; these are invisible to string



diagrams.

$$\left| \begin{array}{c} \mathfrak{r} \\ \mathfrak{h} \\ \mathfrak{z} \end{array} \right| \quad \left| \begin{array}{c} \mathfrak{r} \end{array} \right|$$

String diagrams are boring for 2-monoids (arguably, that's the point), but the inverse operation is more interesting. I will denote  $\mathfrak{r}^{-1}$  simply by the diagram at the left, but the invertors are now interesting:

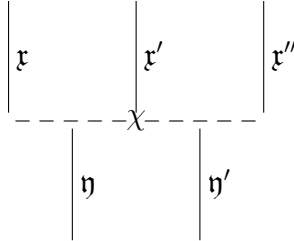
$$\left| \begin{array}{c} \mathfrak{r}^{-1} \end{array} \right| \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \left| \begin{array}{c} \mathfrak{r} \\ \mathfrak{r}^{-1} \end{array} \right| \end{array} \quad \begin{array}{c} \left| \begin{array}{c} \mathfrak{r}^{-1} \\ \mathfrak{r} \end{array} \right| \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}$$

The zig-zag identities are far from invisible, and you see where they get their name:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \left| \begin{array}{c} \mathfrak{r} \\ \mathfrak{r}^{-1} \\ \mathfrak{r} \end{array} \right| \end{array} = \left| \begin{array}{c} \mathfrak{r} \end{array} \right| \quad (2.23)$$



$\chi$  is an arrow from  $((\mathfrak{r}\mathfrak{r}')\mathfrak{r}'')$  (or  $(\mathfrak{r}(\mathfrak{r}'\mathfrak{r}''))$ ) to  $(\eta\eta')$ :



As another point, any equation between string diagrams (giving an equation between natural transformations) may be substituted into another string diagram; this follows from the naturality of the arrows involved.

### 2.4.3 Action on a 2-space

I want  $\mathcal{G}$  to act on the fibre of a 2-bundle.  $\mathfrak{i}$ ,  $\epsilon$ , and  $\iota$  will play no role in this section, which could be applied when  $\mathcal{G}$  is just a 2-monoid (see [Moer, 6.1]).

A **right  $\mathcal{G}$ -2-space** is a 2-space  $\mathcal{F}$  equipped with a 2-map  $\mathcal{F} \times \mathcal{G} \xrightarrow{\tau} \mathcal{F}$  and natural isomorphisms  $\mu$  and  $v$ :

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathfrak{r} \times \underline{\mathcal{G}}} & \mathcal{F} \times \mathcal{G} \\
 \mathfrak{F} \times \mathfrak{m} \downarrow & \swarrow \mu & \downarrow \tau \\
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\tau} & \mathcal{F}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\mathfrak{F} \times \epsilon} & \mathcal{F} \times \mathcal{G} \\
 \mathfrak{F} \searrow & \swarrow v & \downarrow \tau \\
 & & \mathcal{F}
 \end{array}$$

such that the following pairs of diagrams each define equal natural transformations:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathfrak{r} \times \mathcal{G} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \times \mathcal{G} \\
 \downarrow \mathfrak{F} \times \mathcal{G} \times \mathfrak{m} & \searrow \mathfrak{F} \times \mathfrak{m} \times \mathcal{G} & \searrow \mathfrak{r} \times \mathcal{G} \\
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathfrak{F} \times \mathfrak{m} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \times \mathcal{G} \\
 & \searrow \mathfrak{F} \times \mathfrak{m} & \searrow \mathfrak{r} \times \mathcal{G} \\
 & \mathcal{F} \times \mathcal{G} & \mathcal{F} \times \mathcal{G} \\
 & \downarrow \mathfrak{F} \times \mathfrak{m} & \downarrow \mathfrak{r} \\
 & \mathcal{F} \times \mathcal{G} & \mathcal{F} \\
 & \downarrow \mathfrak{F} \times \mathfrak{m} & \downarrow \mathfrak{r} \\
 & \mathcal{F} & \mathcal{F}
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathfrak{r} \times \mathcal{G} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \times \mathcal{G} \\
 \downarrow \mathfrak{F} \times \mathcal{G} \times \mathfrak{m} & \searrow \mathfrak{F} \times \mathfrak{m} & \searrow \mathfrak{r} \times \mathcal{G} \\
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathfrak{r} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \\
 & \searrow \mathfrak{F} \times \mathfrak{m} & \searrow \mathfrak{r} \\
 & \mathcal{F} \times \mathcal{G} & \mathcal{F} \\
 & \downarrow \mathfrak{F} \times \mathfrak{m} & \downarrow \mathfrak{r} \\
 & \mathcal{F} & \mathcal{F}
 \end{array}$$

and:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathfrak{F} \times \mathfrak{e} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \times \mathcal{G} \\
 \searrow \mathfrak{F} \times \mathcal{G} & \downarrow \mathfrak{F} \times \lambda & \downarrow \mathfrak{F} \times \mathfrak{m} \\
 & \mathcal{F} \times \mathcal{G} & \mathcal{F} \times \mathcal{G} \\
 & \downarrow \mathfrak{F} \times \mathcal{G} & \downarrow \mathfrak{r} \\
 & \mathcal{F} & \mathcal{F}
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathfrak{F} \times \mathfrak{e} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \times \mathcal{G} \\
 \searrow \mathfrak{F} \times \mathcal{G} & \downarrow \mathfrak{F} \times \mathcal{G} & \downarrow \mathfrak{r} \\
 & \mathcal{F} \times \mathcal{G} & \mathcal{F} \times \mathcal{G} \\
 & \downarrow \mathfrak{F} \times \mathcal{G} & \downarrow \mathfrak{r} \\
 & \mathcal{F} & \mathcal{F}
 \end{array}
 \quad (2.25)$$

If more than one  $\mathcal{G}$ -2-space is around at a time, then I'll use subscripts or primes on  $\mathfrak{r}$  to keep things straight.

In terms of generalised elements, if  $\mathcal{X} \xrightarrow{\mathfrak{w}} \mathcal{F}$  is a generalised element of  $\mathcal{F}$  and  $\mathcal{X} \xrightarrow{\mathfrak{r}} \mathcal{G}$  is a generalised element of  $\mathcal{G}$ , then I will again write  $(\mathfrak{w}\mathfrak{r})$  for the composite  $\mathcal{X} \xrightarrow{(\mathfrak{w},\mathfrak{r})} \mathcal{F} \times \mathcal{G} \xrightarrow{\mathfrak{r}} \mathcal{F}$ . Then if  $\mathfrak{w}$  is an element of  $\mathcal{F}$  and  $\mathfrak{r}$  and  $\mathfrak{\eta}$  are elements of  $\mathcal{G}$ , then  $\mu$  defines an arrow  $((\mathfrak{w}\mathfrak{r})\mathfrak{\eta}) \xrightarrow{\mu((\mathfrak{w}\mathfrak{r})\mathfrak{\eta})} (\mathfrak{w}(\mathfrak{r}\mathfrak{\eta}))$ , and  $v$  defines an arrow  $(\mathfrak{w}1) \xrightarrow{v(\mathfrak{w})} \mathfrak{w}$ . Then the law (2.24) says that  $((((\mathfrak{w}\mathfrak{r})\mathfrak{\eta})\mathfrak{z})) \xrightarrow{(\mu((\mathfrak{w}\mathfrak{r})\mathfrak{\eta})\mathfrak{z})} ((\mathfrak{w}(\mathfrak{r}\mathfrak{\eta}))\mathfrak{z}) \xrightarrow{\mu((\mathfrak{w}(\mathfrak{r}\mathfrak{\eta}))\mathfrak{z})} (\mathfrak{w}((\mathfrak{r}\mathfrak{\eta})\mathfrak{z})) \xrightarrow{(\mathfrak{w}\alpha(\mathfrak{r}\mathfrak{\eta})\mathfrak{z})} (\mathfrak{w}(\mathfrak{r}(\mathfrak{\eta}\mathfrak{z})))$  is equal to  $((((\mathfrak{w}\mathfrak{r})\mathfrak{\eta})\mathfrak{z})) \xrightarrow{\mu((\mathfrak{w}\mathfrak{r})\mathfrak{\eta}\mathfrak{z})} ((\mathfrak{w}\mathfrak{r})(\mathfrak{\eta}\mathfrak{z})) \xrightarrow{\mu((\mathfrak{w}\mathfrak{r})(\mathfrak{\eta}\mathfrak{z}))} (\mathfrak{w}(\mathfrak{r}(\mathfrak{\eta}\mathfrak{z})))$ , for an element  $\mathfrak{w}$  of  $\mathcal{F}$  and elements  $\mathfrak{r}$ ,  $\mathfrak{\eta}$ , and  $\mathfrak{z}$  of  $\mathcal{G}$ ; while the law (2.25) says that  $((\mathfrak{w}1)\mathfrak{r}) \xrightarrow{\mu(\mathfrak{w}1)\mathfrak{r}} (\mathfrak{w}1\mathfrak{r}) \xrightarrow{(\mathfrak{w}\lambda(\mathfrak{r}))} (\mathfrak{w}\mathfrak{r})$  is equal to  $(v(\mathfrak{w})\mathfrak{r})$ .

The natural isomorphisms  $\mu$  and  $\nu$  are (respectively) the associator and right unitor of the right action, while the coherence laws (2.24) and (2.25) are (respectively) the pentagon and triangle identities for the action, analogous to (2.20, 2.21). This fits in with the laws for the 2-group itself:

**Proposition 15** *The 2-group  $\mathcal{G}$  is itself a right  $\mathcal{G}$ -2-space, acting on itself by right multiplication.*

**Proof:** Set  $\mathfrak{r}_{\mathcal{G}} := \mathfrak{m}$ ,  $\mu_{\mathcal{G}} := \alpha$ , and  $\nu_{\mathcal{G}} := \rho$ . The requirements above then reduce to (some of) the requirements for a 2-group. ■

I'll use this to define  $\mathcal{G}$ -2-torsors and principal  $\mathcal{G}$ -2-bundles.

String diagrams also work with 2-group actions. For a right action, where elements of  $\mathcal{F}$  are written on the left, I write the leftmost string with a double line to indicate a boundary; there is nothing more further to the left. For example, this string diagram describes both  $((\mathfrak{w}\mathfrak{x})\mathfrak{y})$  and  $(\mathfrak{w}(\mathfrak{x}\mathfrak{y}))$ :

$$\mathfrak{w} \parallel \quad \left| \mathfrak{x} \right. \quad \left. \left| \mathfrak{y} \right. \right.$$

You can also think of this as a picture of  $\mu(\mathfrak{w}\mathfrak{x}\mathfrak{y})$ ;  $\mu$  and  $\nu$  (like  $\alpha$ ,  $\lambda$ , and  $\rho$ ) are invisible to string diagrams. (This again relies on the Mac Lane Coherence Theorem.)

#### 2.4.4 The 2-category of $\mathcal{G}$ -2-spaces

Just as 2-spaces form a 2-category  $\mathfrak{C}$ , so  $\mathcal{G}$ -2-spaces form a 2-category  $\mathfrak{C}^{\mathcal{G}}$ .

Given right  $\mathcal{G}$ -2-spaces  $\mathcal{F}$  and  $\mathcal{F}'$ , a  $\mathcal{G}$ -2-**map** from  $\mathcal{F}$  to  $\mathcal{F}'$  is a 2-map  $\mathcal{F} \xrightarrow{t} \mathcal{F}'$  together with a natural isomorphism  $\phi$ :

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{G} & \xrightarrow{t \times \underline{\mathcal{G}}} & \mathcal{F}' \times \mathcal{G} \\ \tau \downarrow & \swarrow \phi & \downarrow \tau' \\ \mathcal{F} & \xrightarrow{t} & \mathcal{F}' \end{array}$$

Note that a composition of  $\mathcal{G}$ -2-maps is a  $\mathcal{G}$ -2-map, using the following composite natural transformation:

$$\begin{array}{ccccc} \mathcal{F} \times \mathcal{G} & \xrightarrow{t \times \underline{\mathcal{G}}} & \mathcal{F}' \times \mathcal{G} & \xrightarrow{t' \times \underline{\mathcal{G}}} & \mathcal{F}'' \times \mathcal{G} \\ \tau \downarrow & \swarrow \phi & \tau' \downarrow & \swarrow \phi' & \downarrow \tau'' \\ \mathcal{F} & \xrightarrow{t} & \mathcal{F}' & \xrightarrow{t'} & \mathcal{F}'' \end{array}$$

The identity map on  $\mathcal{F}$  is also a  $\mathcal{G}$ -2-map. Indeed,  $\mathcal{G}$ -2-spaces and  $\mathcal{G}$ -2-maps form a category  $\mathfrak{C}^{\mathcal{G}}$ .

Given right  $\mathcal{G}$ -2-spaces  $\mathcal{F}$  and  $\mathcal{F}'$  and  $\mathcal{G}$ -2-maps  $t$  and  $t'$  from  $\mathcal{F}$  to  $\mathcal{F}'$ , a  $\mathcal{G}$ -**natural transformation** from  $t$  to  $t'$  is a natural transformation  $\tau$ :

$$\mathcal{F} \begin{array}{c} \xrightarrow{t} \\ \Downarrow \tau \\ \xrightarrow{t'} \end{array} \mathcal{F}'$$

such that the following pair of diagrams defines equal natural transformations:

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{G} & \xrightarrow{t \times \underline{\mathcal{G}}} & \mathcal{F}' \times \mathcal{G} \\ \tau \downarrow & \swarrow \phi & \downarrow \tau' \\ \mathcal{F} & \xrightarrow{t} & \mathcal{F}' \\ & \Downarrow \tau & \\ & \xrightarrow{t'} & \end{array} = \begin{array}{ccc} & \xrightarrow{t \times \underline{\mathcal{G}}} & \\ \mathcal{F} \times \mathcal{G} & \xrightarrow{t' \times \underline{\mathcal{G}}} & \mathcal{F}' \times \mathcal{G} \\ \tau \downarrow & \swarrow \phi & \downarrow \tau' \\ \mathcal{F} & \xrightarrow{t'} & \mathcal{F}' \end{array}$$

Now  $\mathfrak{C}^{\mathcal{G}}$  is in fact a 2-category.

Now I know what it means for  $\mathcal{G}$ -2-spaces  $\mathcal{F}$  and  $\mathcal{F}'$  to be **equivalent**: There must be maps  $\mathcal{F} \xrightarrow{t} \mathcal{F}'$  and  $\mathcal{F}' \xrightarrow{\bar{t}} \mathcal{F}$  equipped with natural isomorphisms:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{t \times \mathcal{G}} & \mathcal{F}' \times \mathcal{G} \\
 \downarrow \tau & \Downarrow \phi & \downarrow \tau' \\
 \mathcal{F} & \xrightarrow{t} & \mathcal{F}' \\
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \downarrow t & \Downarrow \tau & \downarrow \bar{t} \\
 & \mathcal{F}' & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{F}' \times \mathcal{G} & \xrightarrow{\mathcal{G} \times \bar{t}} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau' & \Downarrow \phi & \downarrow \tau \\
 \mathcal{F}' & \xrightarrow{\bar{t}} & \mathcal{F} \\
 \mathcal{F}' & \xrightarrow{\mathcal{F}'} & \mathcal{F}' \\
 \downarrow \bar{t} & \Downarrow \bar{\tau} & \downarrow t \\
 & \mathcal{F} & 
 \end{array}$$

such that each pair of diagrams below defines equal natural transformations:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathcal{F} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau & \Downarrow & \downarrow \tau \\
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \downarrow t & \Downarrow \tau & \downarrow \bar{t} \\
 & \mathcal{F}' & 
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathcal{F} \times \mathcal{G}} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau & \begin{array}{ccc} \searrow t \times \mathcal{G} & \Downarrow \mathcal{G} \times \bar{t} & \nearrow \mathcal{G} \times \bar{t} \end{array} & \downarrow \tau \\
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \downarrow t & \Downarrow \tau & \downarrow \bar{t} \\
 & \mathcal{F}' & 
 \end{array}$$

and:

$$\begin{array}{ccc}
 \mathcal{F}' \times \mathcal{G} & \xrightarrow{\mathcal{F}' \times \mathcal{G}} & \mathcal{F}' \times \mathcal{G} \\
 \downarrow \tau' & \Downarrow & \downarrow \tau' \\
 \mathcal{F}' & \xrightarrow{\mathcal{F}'} & \mathcal{F}' \\
 \downarrow \bar{t} & \Downarrow \bar{\tau} & \downarrow t \\
 & \mathcal{F} & 
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{F}' \times \mathcal{G} & \xrightarrow{\mathcal{F}' \times \mathcal{G}} & \mathcal{F}' \times \mathcal{G} \\
 \downarrow \tau' & \begin{array}{ccc} \searrow \mathcal{G} \times \bar{t} & \Downarrow \mathcal{G} \times \bar{t} & \nearrow t \times \mathcal{G} \end{array} & \downarrow \tau' \\
 \mathcal{F}' & \xrightarrow{\mathcal{F}'} & \mathcal{F}' \\
 \downarrow \bar{t} & \Downarrow \bar{\tau} & \downarrow t \\
 & \mathcal{F} & 
 \end{array}$$

### 2.4.5 $\mathcal{G}$ -2-torsors

When I define principal 2-bundles, I will want the fibre of the 2-bundle to be  $\mathcal{G}$ . However, it should be good enough if the fibre is only *equivalent* to  $\mathcal{G}$ . So, let a **right  $\mathcal{G}$ -2-torsor** be any right  $\mathcal{G}$ -2-space that is equivalent, as a  $\mathcal{G}$ -2-space, to  $\mathcal{G}$  itself.

In more detail, this is a 2-space  $\mathcal{F}$  equipped with 2-maps  $\mathcal{F} \times \mathcal{G} \xrightarrow{\tau} \mathcal{F}$ ,  $\mathcal{F} \xrightarrow{t} \mathcal{G}$ ,

and  $\mathcal{G} \xrightarrow{\bar{t}} \mathcal{F}$ , and natural isomorphisms:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} \times \mathcal{G} & \xrightarrow{\mathcal{F} \times m} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau \times \underline{g} & \swarrow \mu & \downarrow \tau \\
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\tau} & \mathcal{F}
 \end{array} & 
 \begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\mathcal{F} \times \epsilon} & \mathcal{F} \times \mathcal{G} \\
 \searrow \mathcal{F} & \swarrow \psi_\nu & \downarrow \tau \\
 & & \mathcal{F}
 \end{array} & 
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{t \times \underline{g}} & \mathcal{G} \times \mathcal{G} \\
 \downarrow \tau & \swarrow \phi & \downarrow m \\
 \mathcal{F} & \xrightarrow{t} & \mathcal{G}
 \end{array} \\
 \\ 
 \begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} & \xrightarrow{g \times \bar{t}} & \mathcal{F} \times \mathcal{G} \\
 \downarrow m & \swarrow \phi & \downarrow \tau \\
 \mathcal{G} & \xrightarrow{\bar{t}} & \mathcal{F}
 \end{array} & 
 \begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \searrow t & \swarrow \psi_\tau & \downarrow \bar{t} \\
 & & \mathcal{G}
 \end{array} & 
 \begin{array}{ccc}
 \mathcal{G} & \xrightarrow{g} & \mathcal{G} \\
 \searrow \bar{t} & \swarrow \psi_{\bar{t}} & \downarrow t \\
 & & \mathcal{F}
 \end{array}
 \end{array}$$

such that the coherence laws (2.24, 2.25),

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathcal{F} \times g} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau & \parallel & \downarrow \tau \\
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \searrow t & \swarrow \psi_\tau & \downarrow \bar{t} \\
 & & \mathcal{G}
 \end{array} & = & 
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{G} & \xrightarrow{\mathcal{F} \times g} & \mathcal{F} \times \mathcal{G} \\
 \downarrow \tau & \swarrow t \times \underline{g} & \downarrow \tau \\
 \mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
 \searrow t & \swarrow \psi_\phi & \downarrow m \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \swarrow g \times \bar{t} & \searrow \psi_\phi \\
 & & & \downarrow \tau \\
 & & & \mathcal{F}
 \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} & \xrightarrow{g \times g} & \mathcal{G} \times \mathcal{G} \\
 \downarrow m & \parallel & \downarrow m \\
 \mathcal{G} & \xrightarrow{g} & \mathcal{G} \\
 \searrow \bar{t} & \swarrow \psi_{\bar{t}} & \downarrow t \\
 & & \mathcal{F}
 \end{array} & = & 
 \begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} & \xrightarrow{g \times g} & \mathcal{G} \times \mathcal{G} \\
 \downarrow m & \swarrow g \times \bar{t} & \downarrow m \\
 \mathcal{G} & \xrightarrow{g} & \mathcal{G} \\
 \searrow \bar{t} & \swarrow \psi_\phi & \downarrow \tau \\
 & & \mathcal{F} \times \mathcal{G} \\
 & & \swarrow t \times \underline{g} & \searrow \psi_\phi \\
 & & & \downarrow m \\
 & & & \mathcal{G}
 \end{array}
 \end{array}$$

are all satisfied.

## 2.4.6 Crossed modules

As I remarked in the Introduction, previous work with gerbes [Bre90] and bundle gerbes [Jur] has been generalised to the case of crossed modules, which are equivalent to strict 2-groups, but not to general 2-groups. In this extra section accordingly, I



will explain how crossed modules work as 2-groups, including how to apply string diagrams to them.

A **strict 2-group** is a 2-group whose structural 2-maps ( $\mathfrak{m}$ ,  $\epsilon$ , and  $\mathfrak{i}$ ) are all strict and whose structural isomorphisms ( $\alpha$ ,  $\lambda$ ,  $\rho$ ,  $\epsilon$ , and  $\iota$ ) are all identity isomorphisms. (This is an unnatural condition in that it forces certain diagrams of morphisms in a 2-category —specifically, the backbones in (2.19, 2.22)— to commute ‘on the nose’, rather than merely up to coherent isomorphism.)

A **left crossed module** consists of the following data and conditions:

- a *base group*  $H$ ;
- an *automorphism group*  $D$ ;
- a group homomorphism  $d$  from  $H$  to  $D$ ; and
- a (left) group action  $l$  of  $D$  on  $H$ ; such that
- the homomorphism  $d$  is equivariant (an intertwiner) between the action  $l$  of  $D$  on  $H$  and the action by (left) conjugation of  $D$  on itself; and
- the action of  $H$  on itself (given by applying  $d$  and then  $l$ ) is equal to (left) conjugation.

(The final item is the *Peiffer identity*.) Expanding on this, a crossed module consists of groups (as in section 1.4.1)  $D$  and  $H$  and maps  $H \xrightarrow{d} D$  and  $D \times H \xrightarrow{l} H$  such that the following diagrams commute:

$$\begin{array}{ccc}
 H \times H & \xrightarrow{d \times d} & D \times D \\
 m_H \downarrow & & \downarrow m_D \\
 H & \xrightarrow{d} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 1 & & \\
 e_H \downarrow & \searrow e_D & \\
 H & \xrightarrow{d} & D
 \end{array}$$



with the structure maps  $d_{01}$  and  $d_{12}$  being (respectively)  $H \times H \times D \xrightarrow{H \times d \times D} H \times D \times D \xrightarrow{H \times m_D} H \times D$  and  $H \times H \times D \xrightarrow{\hat{H} \times H \times D} H \times D$ . Finally,  $d_{00}$  and  $d_{02}$  are (respectively)  $D \xrightarrow{e_H \times D} H \times D$  and  $H \times H \times D \xrightarrow{m_H \times D} H \times D$ ; it's easy to check that this makes  $\mathcal{G}$  into a 2-space.

To define the 2-maps  $\mathbf{m}$ ,  $\mathbf{e}$ , and  $\mathbf{i}$  of the 2-group  $\mathcal{G}$ , first let their sources  $|\mathbf{m}|$ ,  $|\mathbf{e}|$ , and  $|\mathbf{i}|$  be (respectively)  $D \times D \xrightarrow{D \times D} D \times D$ ,  $1 \xrightarrow{1} 1$ , and  $D \xrightarrow{D} D$ ; that is, they are all strict 2-maps. Since  $\mathcal{G}^1$  is already a group  $D$ , let the object part of these 2-maps be the corresponding maps for the group  $D$ ; that is,  $\mathbf{m}^1$  is  $D \times D \xrightarrow{m_D} D$ ,  $\mathbf{e}^1$  is  $1 \xrightarrow{e_D} D$ , and  $\mathbf{i}^1$  is  $D \xrightarrow{i_D} D$ . Finally, let  $\mathbf{m}^2$  be  $H \times D \times H \times D \xrightarrow{H \times \langle l, D \times \hat{H} \rangle \times D} H \times H \times D \times D \xrightarrow{m_H \times m_D} H \times D$ , let  $\mathbf{e}^2$  be  $1 \xrightarrow{e_H \times e_D} H \times D$ , and let  $\mathbf{i}^2$  be  $H \times D \xrightarrow{i_H \times i_D} H \times D \xrightarrow{\langle l, \hat{H} \times D \rangle} H \times D$ . Then it is straightforward to check, first that  $\mathbf{m}$ ,  $\mathbf{e}$ , and  $\mathbf{i}$  are all 2-maps, then that they make  $\mathcal{G}$  into a (strict) 2-group. ■

For the complete equivalence between crossed modules and strict 2-groups, refer to [F-B]; for now, I will consider how to translate between statements about crossed modules and string diagrams about 2-groups. An element  $X \xrightarrow{y} D$  of  $D$  defines to an element  $\mathcal{X} \xrightarrow{\eta} \mathcal{G}$  of  $\mathcal{G}$ , where  $\mathcal{X}$  and  $\eta$  are built out of  $X$  and  $y$  as in section 2.2.4. Given elements  $X \xrightarrow{y} D$  and  $X \xrightarrow{y'} D$ , an element  $X \xrightarrow{x} H$  may correspond to an arrow  $\eta \xrightarrow{\chi} \eta'$  in  $\mathcal{G}$ , where the map  $|\chi|$  is given by  $X \xrightarrow{\langle x, y' \rangle} H \times D$ ; but of course this only really works if  $X \xrightarrow{\langle x, y' \rangle} H \times D \xrightarrow{d \times D} D \times D \xrightarrow{m_D} D$  is equal to  $y$ . Often in the literature on crossed modules (and certainly in [Bre90], which I will make contact

with in part 3), to indicate (in effect) that  $x$  should be interpreted as an arrow from  $\eta$  to  $\eta'$ , one reads that  $X \xrightarrow{x} H \xrightarrow{d} D$  is equal to  $(yy'^{-1})$ .

Given an equation about elements of  $H$ , this can be turned into an equation about arrows in  $\mathcal{G}$  if the sources and targets can be matched properly, but they don't have to match completely. For example, suppose that  $y, y'$ , and  $y''$  are elements of  $D$ , that  $x, x'$ , and  $x''$  are elements of  $H$ , and that these may be interpreted (because  $d$  takes the appropriate values) as  $\eta \xrightarrow{x} \eta'$ ,  $\eta' \xrightarrow{x'} \eta''$ , and  $\eta \xrightarrow{x''} \eta''$ . Then the equation  $(xx') = x''$  (in  $H$ ) may be interpreted using these string diagrams:

$$\begin{array}{c}
 | \\
 \eta \\
 | \\
 x \\
 | \\
 \eta' \\
 | \\
 x' \\
 | \\
 \eta''
 \end{array}
 =
 \begin{array}{c}
 | \\
 \eta \\
 | \\
 x'' \\
 | \\
 \eta''
 \end{array}$$

But more interesting equations are possible. Suppose instead that  $x$  may still be interpreted as  $\eta \xrightarrow{x} \eta'$  but now  $x'$  may be interpreted as  $(\eta\eta'') \xrightarrow{x'} (\eta'\eta'')$ . Then notice that  $x$  can also be interpreted as  $(\eta\eta'') \implies (\eta'\eta'')$ , because  $(yy'^{-1}) = ((yy'')(y'y'')^{-1})$  in the group  $D$ . Thus, the equation  $x = x'$  in  $H$  may be interpreted using these string

diagrams:

$$\begin{array}{c} | \\ \eta \\ \chi \\ | \\ \eta' \end{array} \quad \begin{array}{c} | \\ \eta'' \end{array} = \begin{array}{c} | \\ \eta \\ - \chi' - \\ | \\ \eta' \end{array} \quad \begin{array}{c} | \\ \eta'' \\ | \\ \eta'' \end{array}$$

The lesson is that  $\eta''$  can appear to the right of  $\chi$ , without messing things up.

On the other hand, suppose that now  $x'$  may be interpreted as  $(\eta''\eta) \xrightarrow{x'} (\eta''\eta')$ .

Now  $x$  may *not* be interpreted similarly, because  $(yy'^{-1}) = ((y''y)(y''y')^{-1})$  is not valid in a group; instead, the left side must be conjugated by  $y''$  to produce the right side. However, the action  $l$  of  $D$  on  $H$  now comes into play; if  $y''$  is applied to  $x$  to produce an element  $x''$ , then, using (2.26),  $x''$  may be interpreted as  $(\eta''\eta) \implies (\eta''\eta')$ .

Thus, the following string diagrams, rather than stating  $x = x'$ , instead say that  $X \xrightarrow{\langle y'', x \rangle} D \times H \xrightarrow{l} H$  is equal to  $x'$ .

$$\begin{array}{c} | \\ \eta'' \end{array} \quad \begin{array}{c} | \\ \eta \\ \chi \\ | \\ \eta' \end{array} = \begin{array}{c} | \\ \eta \\ - \chi' - \\ | \\ \eta' \end{array} \quad \begin{array}{c} | \\ \eta'' \\ | \\ \eta'' \end{array}$$

I will apply these principles in part 3.

## 2.5 $\mathcal{G}$ -2-bundles

I can now put the above ideas together to get the concept of  $\mathcal{G}$ -2-bundle.

### 2.5.1 Definition of $\mathcal{G}$ -2-bundle

Suppose that I have a 2-space  $\mathcal{B}$  with a 2-cover  $\mathcal{U} \xrightarrow{j} \mathcal{B}$ , as well as a 2-group  $\mathcal{G}$ . In these circumstances, a  **$\mathcal{G}$ -2-transition** of the 2-cover  $\mathcal{U}$  is a 2-map  $\mathcal{U}^{[2]} \xrightarrow{\mathfrak{g}} \mathcal{G}$  together with natural isomorphisms  $\gamma$  and  $\eta$ :

$$\begin{array}{ccc}
 \mathcal{U}^{[3]} & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & \mathcal{U}^{[2]} \times \mathcal{U}^{[2]} \xrightarrow{\mathfrak{g} \times \mathfrak{g}} \mathcal{G} \times \mathcal{G} \\
 & \searrow j_{[02]} & \swarrow \gamma \quad \downarrow m \\
 & & \mathcal{U}^{[2]} \xrightarrow{\mathfrak{g}} \mathcal{G}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\hat{u}} & 1 \\
 j_{[00]} \downarrow & \swarrow \eta & \downarrow \epsilon \\
 \mathcal{U}^{[2]} & \xrightarrow{\mathfrak{g}} & \mathcal{G}
 \end{array}
 \tag{2.27}$$

that satisfy three coherence laws.

To write these coherence laws in a manner that fits on the page while remaining legible, I will use string diagrams. As in section 1.5, if  $\langle \mathfrak{x}, \mathfrak{y} \rangle$  defines an element of  $\mathcal{U}^{[2]}$ , then let  $\mathfrak{g}_{\mathfrak{x}\mathfrak{y}}$  be the composite  $\mathcal{X} \xrightarrow{\langle \mathfrak{x}, \mathfrak{y} \rangle} \mathcal{U}^{[2]} \xrightarrow{\mathfrak{g}} \mathcal{G}$ . Then  $\gamma$  defines an arrow  $(\mathfrak{g}_{\mathfrak{x}\mathfrak{y}}\mathfrak{g}_{\mathfrak{y}\mathfrak{z}}) \xrightarrow{\gamma_{\mathfrak{x}\mathfrak{y}\mathfrak{z}}} \mathfrak{g}_{\mathfrak{x}\mathfrak{z}}$  (for an element  $\langle \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \rangle$  of  $\mathcal{U}^{[3]}$ ), while  $\eta$  defines an arrow  $1 \xrightarrow{\eta_{\mathfrak{x}}} \mathfrak{g}_{\mathfrak{x}\mathfrak{x}}$  (for an element  $\mathcal{X} \xrightarrow{\mathfrak{x}} \mathcal{U}$  of  $\mathcal{U}$ ). Then the string diagrams for (the identity on)  $\mathfrak{g}_{\mathfrak{x}\mathfrak{y}}$ , for  $\gamma_{\mathfrak{x}\mathfrak{y}\mathfrak{z}}$ , and for  $\eta_{\mathfrak{x}}$  are (respectively):

$$\begin{array}{c}
 \mathfrak{x} \\
 | \\
 \mathfrak{g} \quad \mathfrak{y} \\
 |
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & | & & | & \\
 & \mathfrak{g} & \mathfrak{y} & \mathfrak{g} & \\
 & \text{---} & \gamma & \text{---} & \\
 \mathfrak{x} & & | & & \mathfrak{z} \\
 & & \mathfrak{g} & & 
 \end{array}
 \qquad
 \begin{array}{c}
 \mathfrak{x} \\
 \vdots \\
 \eta \\
 | \\
 \mathfrak{g}
 \end{array}$$

(Here I've drawn in the invisible identity string diagram to clarify the height of the diagram for  $\eta$ .) As in section 1.5, only some diagrams can be drawn; there is no diagram for, say,  $(\mathfrak{g}_{\mathfrak{w}\mathfrak{x}}\mathfrak{g}_{\mathfrak{y}\mathfrak{z}})$  —nor will there be any need for such a diagram.

In these terms, the coherence laws for a  $\mathcal{G}$ -2-transition are:

$$\begin{array}{c}
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \quad \mathfrak{r} \quad \mathfrak{g} \\
 | \quad | \quad | \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \quad \quad \quad \gamma \\
 \quad \quad \quad | \\
 \quad \quad \quad \mathfrak{g} \\
 \quad \quad \quad \quad \quad \eta \\
 \quad \quad \quad \quad \quad | \\
 \quad \quad \quad \quad \quad \mathfrak{g} \\
 \quad \quad \quad \quad \quad \quad \quad \delta \\
 \mathfrak{w}
 \end{array}
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \quad \eta \quad \mathfrak{g} \\
 | \quad | \quad | \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \quad \quad \quad \gamma \\
 \quad \quad \quad | \\
 \quad \quad \quad \mathfrak{g} \\
 \quad \quad \quad \quad \quad \mathfrak{r} \\
 \quad \quad \quad \quad \quad | \\
 \quad \quad \quad \quad \quad \mathfrak{g} \\
 \quad \quad \quad \quad \quad \quad \quad \mathfrak{z} \\
 \mathfrak{w}
 \end{array}
 \end{array}
 \end{array}
 \tag{2.28}$$

and:

$$\begin{array}{c}
 \begin{array}{c}
 \mathfrak{r} \\
 \quad \quad \quad | \\
 \eta \quad \quad \quad \mathfrak{g} \\
 | \quad \quad \quad | \\
 \text{---} \quad \quad \quad \text{---} \\
 \quad \quad \quad \gamma \\
 \quad \quad \quad | \\
 \quad \quad \quad \mathfrak{g} \\
 \quad \quad \quad \quad \quad \eta
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{r} \quad \mathfrak{g} \quad \eta \\
 | \quad | \quad | \\
 \text{---} \quad \text{---} \quad \text{---} \\
 \quad \quad \quad \gamma \\
 \quad \quad \quad | \\
 \quad \quad \quad \mathfrak{g}
 \end{array}
 =
 \begin{array}{c}
 \eta \\
 \quad \quad \quad | \\
 \mathfrak{g} \quad \quad \quad \eta \\
 | \quad \quad \quad | \\
 \text{---} \quad \quad \quad \text{---} \\
 \quad \quad \quad \gamma \\
 \quad \quad \quad | \\
 \quad \quad \quad \mathfrak{r} \\
 \quad \quad \quad \quad \quad \mathfrak{g}
 \end{array}
 \end{array}
 \tag{2.29}$$

The analogy between a  $G$ -transition and a group is stronger now as an analogy between a  $\mathcal{G}$ -2-transition and a group. Again,  $\mathfrak{g}$  corresponds to the underlying set of the group,  $\gamma$  corresponds to the operation of multiplication, and  $\eta$  corresponds to the identity element. But now the coherence law (2.28) corresponds to the associative law, while the laws (2.29) correspond (respectively) to the left and right unit laws. Law (2.28) is essentially the tetrahedron in the definition of 2-dimensional descent datum in [Dusk, page 258].

As in part 1,  $\eta$  may be derived from  $\gamma$ . But in fact, I can replace any  $\mathcal{G}$ -2-transition  $\mathfrak{g}$  with an equivalent  $\mathcal{G}$ -2-transition for which  $\eta$  takes a particularly simple form; so I defer this to section 2.5.5 after the notion of equivalence between 2-transitions has been defined.

If  $\mathcal{G}$  acts on some (right)  $\mathcal{G}$ -2-space  $\mathcal{F}$ , and if  $\mathcal{E}$  is a locally trivial 2-bundle over  $\mathcal{B}$  with fibre  $\mathcal{F}$ , then I can consider a natural isomorphism  $\theta$ :

$$\begin{array}{ccccc}
 \mathcal{F} \times \mathcal{U}^{[2]} & \xrightarrow{\mathcal{F} \times \langle \mathfrak{g}, \mathfrak{j}_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U} & \xrightarrow{\mathfrak{r} \times \mathcal{U}} & \mathcal{F} \times \mathcal{U} \\
 & \searrow_{\mathcal{F} \times \mathfrak{j}_{[0]}} & & & \downarrow \tilde{\mathfrak{j}} \\
 & & \mathcal{F} \times \mathcal{U} & \xrightarrow{\tilde{\mathfrak{j}}} & \mathcal{E}
 \end{array} \quad (2.30)$$

In terms of elements, if  $\mathcal{X} \xrightarrow{\mathfrak{w}} \mathcal{F}$  is an element of  $\mathcal{F}$ , and  $\langle \mathfrak{x}, \mathfrak{y} \rangle$  defines an element of  $\mathcal{U}^{[2]}$ , then  $\theta$  defines an arrow  $(\mathfrak{w}\mathfrak{g}_{\mathfrak{xy}})/_{\mathfrak{y}} \xrightarrow{\theta\mathfrak{w}_{\mathfrak{x}\mathfrak{y}}} \mathfrak{w}/_{\mathfrak{x}}$ . I will draw the string diagram for  $\theta\mathfrak{w}_{\mathfrak{xy}}$  the same way as in section 1.5:

$$\begin{array}{ccc}
 \parallel & & \parallel \\
 \mathfrak{w} & & \mathfrak{g} \quad \mathfrak{y} \\
 & & \text{---} \text{---} \text{---} \theta \\
 & \mathfrak{x} & \parallel
 \end{array}$$

As in part 1, the bottom half of this diagram represents  $\mathfrak{w}/_{\mathfrak{x}}$ , a general notation for  $\tilde{\mathfrak{j}}$ ; while the top half represents  $(\mathfrak{w}\mathfrak{g}_{\mathfrak{xy}})/_{\mathfrak{y}}$ , which cannot be generalised to all valid expressions (but does include all appropriate expressions that I need).

By definition, the natural isomorphism  $\theta$  defines a  $\mathcal{G}$ -2-**bundle** if the following



coherence laws are satisfied:

$$\begin{array}{c}
 \begin{array}{c}
 \parallel \\
 \parallel \\
 \mathfrak{w}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \quad \eta \quad \mathfrak{g} \\
 \hline
 \gamma \\
 \mathfrak{g} \quad \delta \\
 \hline
 \theta
 \end{array} \\
 \mathfrak{r}
 \end{array}
 \quad
 \parallel \\
 \parallel
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 \parallel \\
 \parallel \\
 \mathfrak{w}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \quad \delta \\
 \hline
 \theta \\
 \mathfrak{g} \quad \eta \\
 \hline
 \theta
 \end{array} \\
 \mathfrak{r}
 \end{array}
 \quad
 \parallel \\
 \parallel
 \end{array}
 \quad (2.31)
 \end{array}$$

and:

$$\begin{array}{c}
 \begin{array}{c}
 \parallel \\
 \parallel \\
 \mathfrak{w}
 \end{array}
 \quad
 \begin{array}{c}
 \mathfrak{r} \\
 \eta \\
 \mathfrak{g} \\
 \hline
 \theta
 \end{array}
 \quad
 \parallel \\
 \parallel
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 \parallel \\
 \parallel \\
 \mathfrak{w}
 \end{array}
 \quad
 \begin{array}{c}
 \mathfrak{r} \\
 \parallel \\
 \parallel
 \end{array}
 \quad (2.32)
 \end{array}$$

If you think of  $\theta$  as analogous to a (left) group action, then law (2.31) is analogous to the associative law, while (2.32) is the corresponding (left only) unit law. Certainly the string diagrams are similar to the associative and left unit laws in (2.28, 2.29)!

So in summary, a  $\mathcal{G}$ -2-bundle consists of the data in bullet points in section 2.3.4, and the data  $(\mathfrak{g}, \gamma, \eta, \theta)$  involved in diagrams (2.27, 2.30), such that the coherence laws (2.24, 2.25, 2.28, 2.29, 2.31, 2.32) are all satisfied. A **principal  $\mathcal{G}$ -2-bundle** is simply a  $\mathcal{G}$ -2-bundle whose fibre  $\mathcal{F}$  is  $\mathcal{G}$  itself.

## 2.5.2 The 2-category of $\mathcal{G}$ -2-transitions

To define a morphism of  $\mathcal{G}$ -2-bundles properly, I first need the notion of morphism of  $\mathcal{G}$ -2-transitions.

Given 2-covers  $\mathcal{U}$  and  $\mathcal{U}'$  of  $\mathcal{B}$ , a  $\mathcal{G}$ -2-transition  $\mathfrak{g}$  on  $\mathcal{U}$ , and a  $\mathcal{G}$ -2-transition  $\mathfrak{g}'$  on  $\mathcal{U}'$ , a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}$  to  $\mathfrak{g}'$  is a 2-map  $\mathcal{U} \cap \mathcal{U}' \xrightarrow{\mathfrak{b}} \mathcal{G}$  together with natural isomorphisms  $\sigma$ :

$$\begin{array}{ccc} \mathcal{U}^{[2]} \cap \mathcal{U}' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} \mathcal{U}^{[2]} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathfrak{g} \times \mathfrak{b}} \mathcal{G} \times \mathcal{G} \\ & \searrow j_{[02]} & \swarrow \sigma \quad \downarrow m \\ & \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathfrak{b}} \mathcal{G} \end{array}$$

and  $\delta$ :

$$\begin{array}{ccc} \mathcal{U} \cap \mathcal{U}'^{[2]} & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} \mathcal{U} \cap \mathcal{U}' \times \mathcal{U}'^{[2]} & \xrightarrow{\mathfrak{b} \times \mathfrak{g}'} \mathcal{G} \times \mathcal{G} \\ & \searrow j_{[02]} & \swarrow \delta \quad \downarrow m \\ & \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathfrak{b}} \mathcal{G} \end{array}$$

satisfying five coherence laws.

If  $\langle \mathfrak{x}, \mathfrak{x}' \rangle$  defines an element of  $\mathcal{U} \cap \mathcal{U}'$ , then let  $\mathfrak{b}_{\mathfrak{x}\mathfrak{x}'}$  be  $\mathcal{X} \xrightarrow{\langle \mathfrak{x}, \mathfrak{x}' \rangle} \mathcal{U} \cap \mathcal{U}' \xrightarrow{\mathfrak{b}} \mathcal{G}$ . Now given an element  $\langle \mathfrak{x}, \mathfrak{\eta}, \mathfrak{x}' \rangle$  of  $\mathcal{U}^{[2]} \cap \mathcal{U}'$ ,  $\sigma$  defines an arrow  $(\mathfrak{g}_{\mathfrak{x}\mathfrak{\eta}} \mathfrak{b}_{\mathfrak{\eta}\mathfrak{x}'}) \xrightarrow{\sigma_{\mathfrak{x}\mathfrak{\eta}\mathfrak{x}'}} \mathfrak{b}_{\mathfrak{x}\mathfrak{x}'}$ ; and given an element  $\langle \mathfrak{x}, \mathfrak{x}', \mathfrak{\eta}' \rangle$  of  $\mathcal{U} \cap \mathcal{U}'^{[2]}$ ,  $\delta$  defines an arrow  $(\mathfrak{b}_{\mathfrak{x}\mathfrak{x}'} \mathfrak{g}'_{\mathfrak{x}'\mathfrak{\eta}'}) \xrightarrow{\delta_{\mathfrak{x}\mathfrak{x}'\mathfrak{\eta}'}} \mathfrak{b}_{\mathfrak{x}\mathfrak{\eta}'}$ . As in section 1.5.2, I draw  $\mathfrak{b}_{\mathfrak{x}\mathfrak{x}'}$ ,  $\sigma_{\mathfrak{x}\mathfrak{\eta}\mathfrak{x}'}$ , and  $\delta_{\mathfrak{x}\mathfrak{x}'\mathfrak{\eta}'}$  as these string diagrams:

$$\begin{array}{c} \mathfrak{x} \quad \left| \quad \mathfrak{b} \quad \mathfrak{x}' \right. \\ \hline \mathfrak{x} \quad \left| \quad \mathfrak{g} \quad \mathfrak{\eta} \quad \mathfrak{b} \right. \\ \hline \mathfrak{x} \quad \left| \quad \mathfrak{b} \quad \mathfrak{\eta}' \right. \end{array} \quad \begin{array}{c} \left| \quad \mathfrak{b} \quad \mathfrak{x}' \quad \mathfrak{g}' \right. \\ \hline \left| \quad \mathfrak{b} \quad \mathfrak{x}' \quad \mathfrak{\eta}' \right. \\ \hline \left| \quad \mathfrak{b} \quad \mathfrak{\eta}' \right. \end{array} \quad (2.33)$$

Then these are the coherence laws required of a  $\mathcal{G}$ -2-transition morphism:

$$\begin{array}{c}
 \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \eta \quad \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \\
 \hline \text{---} \gamma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \mathfrak{z} \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \\
 \hline \text{---} \sigma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{z}' \\
 \mathfrak{x}
 \end{array} = \begin{array}{c}
 \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{z} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \\
 \hline \text{---} \sigma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \eta \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \\
 \hline \text{---} \sigma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{z}' \\
 \mathfrak{x}
 \end{array} \tag{2.34}$$

and:

$$\begin{array}{c}
 \mathfrak{x} \\
 \eta \\
 \begin{array}{|c|} \hline \mathfrak{g} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \\
 \hline \text{---} \sigma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{x}'
 \end{array} = \begin{array}{c}
 \begin{array}{|c|} \hline \mathfrak{x} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{x}' \\ \hline \end{array} \\
 \hline \text{---} \sigma \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array}
 \end{array} \tag{2.35}$$

and then:

$$\begin{array}{c}
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{x}' \quad \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \\
 \hline \text{---} \delta \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \eta' \quad \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \\
 \hline \text{---} \delta \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{z}' \\
 \mathfrak{x}
 \end{array} = \begin{array}{c}
 \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \quad \eta' \quad \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \\
 \hline \text{---} \gamma' \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{x}' \quad \begin{array}{|c|} \hline \mathfrak{g}' \\ \hline \end{array} \\
 \hline \text{---} \delta \text{---} \\
 \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} \quad \mathfrak{z}' \\
 \mathfrak{x}
 \end{array} \tag{2.36}$$

and next:

$$\begin{array}{c}
 \begin{array}{c}
 \mathfrak{x} \\
 | \\
 \mathfrak{b} \\
 | \\
 \mathfrak{x}'
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{x}' \\
 | \\
 \mathfrak{b} \\
 | \\
 \mathfrak{x}
 \end{array}
 \quad
 \begin{array}{c}
 \eta' \\
 | \\
 \mathfrak{g}' \\
 | \\
 \delta
 \end{array}
 \end{array}
 \end{array}
 \quad (2.37)$$

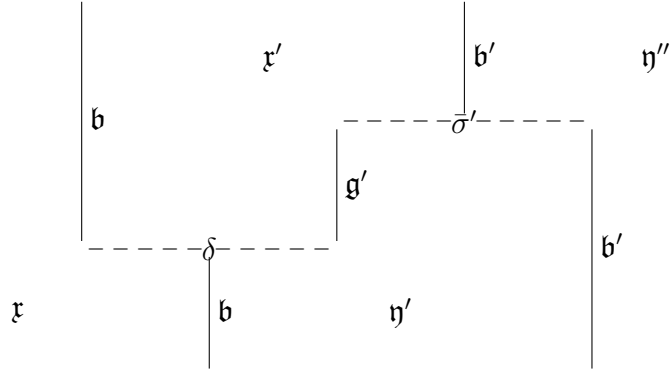
and finally:

$$\begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \quad \eta \quad \mathfrak{b} \\
 | \quad | \quad | \\
 \text{---} \sigma \text{---} \\
 | \\
 \mathfrak{b} \\
 | \\
 \delta \\
 | \\
 \mathfrak{b}
 \end{array}
 \quad
 \begin{array}{c}
 \mathfrak{g}' \\
 | \\
 \eta' \\
 | \\
 \delta \\
 | \\
 \mathfrak{b} \\
 | \\
 \mathfrak{z}'
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 \mathfrak{g} \\
 | \\
 \eta \\
 | \\
 \sigma \\
 | \\
 \mathfrak{b} \\
 | \\
 \mathfrak{x}
 \end{array}
 \quad
 \begin{array}{c}
 \mathfrak{b} \quad \eta' \quad \mathfrak{g}' \\
 | \quad | \quad | \\
 \text{---} \delta \text{---} \\
 | \\
 \mathfrak{b} \\
 | \\
 \sigma \\
 | \\
 \mathfrak{b}
 \end{array}
 \quad
 \begin{array}{c}
 \mathfrak{z}'
 \end{array}
 \end{array}
 \quad (2.38)$$

In the analogy where the  $\mathcal{G}$ -2-transitions  $\mathfrak{g}, \mathfrak{g}'$  are analogous to groups, the 2-transition morphism  $\mathfrak{b}$  is again analogous to a bihomogeneous space acted on by  $\mathfrak{g}$  and  $\mathfrak{g}'$ ;  $\sigma$  and  $\delta$  are again analogous (respectively) to the left and right actions. But now the coherence laws (2.34, 2.36) are analogous to the associative laws for the actions, while (2.35, 2.37) are analogous to the unit laws. Finally, (2.38) is analogous to the associative law relating the two actions: the law that it doesn't matter whether you can multiply first on the left or the right (so here it doesn't matter whether you apply  $\sigma$  or  $\delta$  first).

Again, a  $\mathcal{G}$ -2-transition  $\mathfrak{g}$  serves as its own identity  $\mathcal{G}$ -2-transition morphism, with  $\sigma$  and  $\delta$  both set to  $\gamma$ .

To see how to compose  $\mathcal{G}$ -2-transition morphisms, let  $\langle \mathfrak{x}, \mathfrak{x}', \eta', \eta'' \rangle$  be an element of  $\mathcal{U} \cap \mathcal{U}'^{[2]} \cap \mathcal{U}''$ . Using that  $\sigma'$  is invertible (with inverse  $\bar{\sigma}'$ , say), I form this string diagram:



(Here  $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$  are all  $\mathcal{G}$ -2-transitions,  $\mathfrak{b}$  is a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}$  to  $\mathfrak{g}'$ , and  $\mathfrak{b}'$  is a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}'$  to  $\mathfrak{g}''$ .) This diagram describes a natural isomorphism of this form:

$$\begin{array}{ccc}
 \mathcal{U} \cap \mathcal{U}'^{[2]} \cap \mathcal{U}'' & \xrightarrow{j_{[013]}} & \mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}'' \\
 \downarrow j_{[023]} & & \downarrow \langle j_{[01]}, j_{[12]} \rangle \\
 & & \mathcal{U} \cap \mathcal{U}' \times \mathcal{U}' \cap \mathcal{U}'' \\
 & \cong & \downarrow \mathfrak{b} \times \mathfrak{b}' \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \downarrow m \\
 \mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}'' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & \mathcal{U} \cap \mathcal{U}' \times \mathcal{U}' \cap \mathcal{U}'' \xrightarrow{\mathfrak{b} \times \mathfrak{b}'} \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}
 \end{array}$$

This natural isomorphism is an example of (2.10) for the cover  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}'' \xrightarrow{j_{[02]}} \mathcal{U} \cap \mathcal{U}''$ ,

so (2.11) defines a 2-map  $\mathcal{U} \cap \mathcal{U}'' \xrightarrow{\mathbf{b}; \mathbf{b}' } \mathcal{G}$  together with a natural isomorphism  $\beta$ :

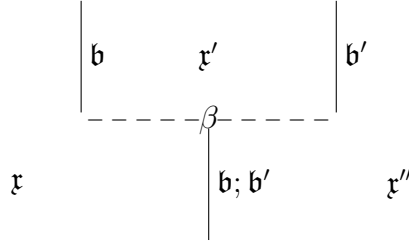
$$\begin{array}{ccc}
 \mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}'' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & \mathcal{U} \cap \mathcal{U}' \times \mathcal{U}' \cap \mathcal{U}'' \xrightarrow{\mathbf{b}; \mathbf{b}' } \mathcal{G} \times \mathcal{G} \\
 & \searrow j_{[02]} & \downarrow m \\
 & & \mathcal{U} \cap \mathcal{U}'' \xrightarrow{\mathbf{b}; \mathbf{b}' } \mathcal{G}
 \end{array}$$

$\beta$

Given an element  $\langle \mathfrak{x}, \mathfrak{x}'' \rangle$  of  $\mathcal{U} \cap \mathcal{U}''$ , I will denote its composite with  $\mathbf{b}; \mathbf{b}'$  as  $(\mathbf{b}; \mathbf{b}')_{\mathfrak{x}''}$ .

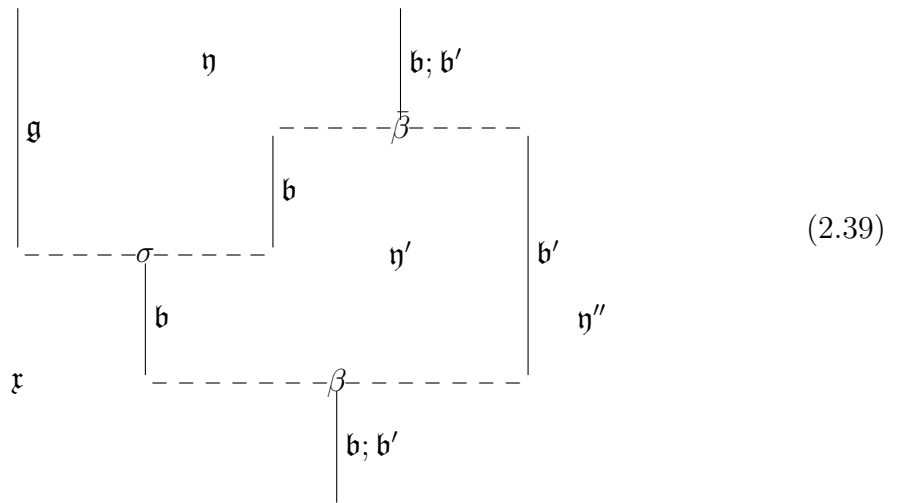
Then given an element  $\langle \mathfrak{x}, \mathfrak{x}', \mathfrak{x}'' \rangle$  of  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}''$ , the transformation  $\beta$  above defines

an arrow  $(\mathbf{b}_{\mathfrak{x}'} \mathbf{b}'_{\mathfrak{x}'}) \xrightarrow{\beta_{\mathfrak{x}' \mathfrak{x}''}} (\mathbf{b}; \mathbf{b}')_{\mathfrak{x}''}$ . As a string diagram:



**Proposition 17** *This 2-map  $\mathbf{b}; \mathbf{b}'$  is a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}$  to  $\mathfrak{g}''$ .*

**Proof:** Given an element  $\langle \mathfrak{x}, \eta, \eta', \eta'' \rangle$  of  $\mathcal{U}^{[2]} \cap \mathcal{U}' \cap \mathcal{U}''$ , I construct this string diagram:



(Here  $\bar{\beta}$  is the inverse of  $\beta$ .) This describes a natural isomorphism:

$$\begin{array}{ccc}
 \mathcal{U}^{[2]} \cap \mathcal{U}' \cap \mathcal{U}'' & \xrightarrow{j_{[013]}} & \mathcal{U}^{[2]} \cap \mathcal{U}'' \\
 \downarrow j_{[013]} & \swarrow \cong & \downarrow \langle j_{[01]}, j_{[12]} \rangle \\
 & & \mathcal{U}^{[2]} \times \mathcal{U} \cap \mathcal{U}'' \\
 & & \downarrow \mathfrak{g} \times \mathfrak{b}; \mathfrak{b}' \\
 & & \mathcal{G} \times \mathcal{G} \\
 & & \downarrow \mathfrak{m} \\
 \mathcal{U}^{[2]} \cap \mathcal{U}'' & \xrightarrow{j_{[02]}} \mathcal{U} \cap \mathcal{U}'' & \xrightarrow{\mathfrak{b}; \mathfrak{b}'} \mathcal{G}
 \end{array}$$

Because the 2-cover  $\mathcal{U}^{[2]} \cap \mathcal{U}' \cap \mathcal{U}'' \xrightarrow{j_{[013]}} \mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}''$  is an epimorphism, this natural isomorphism follows:

$$\begin{array}{ccccc}
 \mathcal{U}^{[2]} \cap \mathcal{U}'' & \xrightarrow{\langle j_{[01]}, j_{[12]} \rangle} & \mathcal{U}^{[2]} \times \mathcal{U} \cap \mathcal{U}'' & \xrightarrow{\mathfrak{g}' \times \mathfrak{b}; \mathfrak{b}'} & \mathcal{G} \times \mathcal{G} \\
 \searrow j_{[02]} & & \swarrow \cong & & \downarrow \mathfrak{m} \\
 & & \mathcal{U} \cap \mathcal{U}'' & \xrightarrow{\mathfrak{b}; \mathfrak{b}'} & \mathcal{G}
 \end{array}$$

This is precisely the natural isomorphism  $\sigma$  for  $\mathfrak{b}; \mathfrak{b}'$ . A natural transformation  $\mathfrak{b}; \delta'$  follows by an analogous argument.

To reason with  $\sigma; \mathfrak{b}'$  with string diagrams, simply expand the middle diagram in (2.33) —but for  $\mathfrak{b}; \mathfrak{b}'$  instead of  $\mathfrak{b}$ , of course— into (2.39). For example, to prove the coherence law (2.35), let  $\langle \mathfrak{x}, \mathfrak{x}', \mathfrak{x}'' \rangle$  be an element of  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}''$ . Then these string

diagrams are all equal:

$$\begin{array}{c}
 \begin{array}{c}
 \mathfrak{x} \\
 \eta \left| \begin{array}{c} \mathfrak{g} \\ \hline \sigma; \mathfrak{b}' \end{array} \right. \begin{array}{c} \mathfrak{b}; \mathfrak{b}' \\ \hline \mathfrak{x}'' \end{array}
 \end{array} \\
 = \\
 \begin{array}{c}
 \mathfrak{x} \\
 \eta \left| \begin{array}{c} \mathfrak{g} \\ \hline \sigma \\ \hline \mathfrak{b} \end{array} \right. \begin{array}{c} \mathfrak{b} \\ \hline \mathfrak{x}' \\ \hline \beta \\ \hline \mathfrak{b}; \mathfrak{b}' \end{array} \\
 \hline \begin{array}{c} \mathfrak{b}; \mathfrak{b}' \\ \hline \bar{\beta} \\ \hline \mathfrak{b}' \\ \hline \mathfrak{x}'' \end{array}
 \end{array} \\
 = \\
 \begin{array}{c}
 \mathfrak{x} \\
 \left| \begin{array}{c} \mathfrak{b}; \mathfrak{b}' \\ \hline \bar{\beta} \\ \hline \mathfrak{b} \quad \mathfrak{x}' \quad \mathfrak{b}' \\ \hline \beta \\ \hline \mathfrak{b}; \mathfrak{b}' \end{array} \right. \mathfrak{x}'' \\
 = \\
 \begin{array}{c}
 \mathfrak{x} \\
 \left| \begin{array}{c} \mathfrak{b}; \mathfrak{b}' \\ \hline \mathfrak{x}'' \end{array} \right.
 \end{array}
 \end{array}
 \end{array}$$

Here I use, in turn, the formula for  $\sigma; \mathfrak{b}'$ , the law (2.35) for  $\mathfrak{b}$  (the same law that I'm trying to prove for  $\mathfrak{b}; \mathfrak{b}'$ ), and that  $\bar{\beta}$  is the inverse of  $\beta$ .

Note that this does not *quite* prove what I want; this proves the equality of two natural transformations whose 2-space domain is  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}''$ , not  $\mathcal{U} \cap \mathcal{U}''$  as desired. However, since  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}'' \xrightarrow{j_{[02]}} \mathcal{U} \cap \mathcal{U}''$  is 2-epic, the equation that I want indeed follows.

The other coherence laws may all be derived similarly. Therefore,  $\mathfrak{b}; \mathfrak{b}'$  is a  $\mathcal{G}$ -2-



transition morphism, as desired.

■

This  $\mathfrak{b}; \mathfrak{b}'$  is the composite of  $\mathfrak{b}$  and  $\mathfrak{b}'$  in the 2-category  $\mathcal{G}^{\mathcal{B}}$  of  $\mathcal{G}$ -2-transitions in  $\mathcal{B}$ .

There is also a notion of 2-morphism between  $\mathcal{G}$ -2-transition morphisms. Specifically, if  $\mathfrak{b}$  and  $\mathfrak{b}'$  are both  $\mathcal{G}$ -2-transition morphisms from  $\mathfrak{g}$  to  $\mathfrak{g}'$ , then a  $\mathcal{G}$ -2-transition 2-morphism from  $\mathfrak{b}$  to  $\mathfrak{b}'$  is a natural transformation  $\xi$ :

$$\mathcal{U}^{[2]} \begin{array}{c} \xrightarrow{\mathfrak{b}} \\ \Downarrow \xi \\ \xrightarrow{\mathfrak{b}'} \end{array} \mathcal{F}'$$

where I draw  $\mathfrak{b}_{\mathfrak{r}\mathfrak{r}'} \xrightarrow{\xi_{\mathfrak{r}\mathfrak{r}'}} \mathfrak{b}'_{\mathfrak{r}\mathfrak{r}'}$  in string diagrams as follows:

$$\begin{array}{c} | \\ \mathfrak{b} \\ | \\ \mathfrak{r} \quad \xi \quad \mathfrak{r}' \\ | \\ \mathfrak{b}' \\ | \end{array}$$

such that these two coherence laws are satisfied:

$$\begin{array}{c} \left| \begin{array}{ccc} \mathfrak{g} & \eta & \mathfrak{b} \\ \hline \sigma & & \end{array} \right| \\ \left| \begin{array}{ccc} \mathfrak{r} & \xi & \eta' \\ \hline \mathfrak{b} & & \end{array} \right| \end{array} = \begin{array}{c} \left| \begin{array}{ccc} \mathfrak{g} & \eta & \mathfrak{b} \\ \hline \sigma' & & \end{array} \right| \\ \left| \begin{array}{ccc} \mathfrak{r} & \xi & \eta' \\ \hline \mathfrak{b}' & & \end{array} \right| \end{array} \quad (2.40)$$

and:

$$\begin{array}{ccc}
 \begin{array}{c}
 \left| \begin{array}{ccc}
 \mathfrak{b} & & \mathfrak{g}' \\
 \hline
 & \delta & \\
 \hline
 \mathfrak{x} & \xi & \eta' \\
 \hline
 & \mathfrak{b}' & \\
 \hline
 \end{array} \right. \\
 \end{array} & = & \begin{array}{c}
 \left| \begin{array}{ccc}
 \mathfrak{b} & & \mathfrak{g}' \\
 \xi & & \\
 \mathfrak{b}' & & \\
 \hline
 & \delta' & \\
 \hline
 \mathfrak{x} & & \eta' \\
 \hline
 & \mathfrak{b}' & \\
 \hline
 \end{array} \right. \\
 \end{array}
 \end{array} \tag{2.41}$$

A  $\mathcal{G}$ -2-transition 2-morphism like  $\xi$  is analogous to a homomorphism of bimodules; these coherence laws are analogous to preserving left and right multiplication.

**Proposition 18** *Given  $\mathcal{G}$ -2-transitions  $\mathfrak{g}$  and  $\mathfrak{g}'$ , the  $\mathcal{G}$ -2-transition morphisms between them and the  $\mathcal{G}$ -2-transition 2-morphisms between those form a category.*

**Proof:** Composition of  $\mathcal{G}$ -2-transition 2-morphisms is simply given by composition of their underlying natural transformations. All I must prove is that these composites (and the identity) are indeed  $\mathcal{G}$ -2-transition 2-morphisms. In the case of the identity, there is nothing to prove; in the case of a composite, just apply each coherence law for  $\xi$  and  $\xi'$  separately to prove that coherence law for their composite. ■

In fact, I have:

**Proposition 19** *Given a 2-group  $\mathcal{G}$  and a 2-space  $\mathcal{B}$ , the  $\mathcal{G}$ -2-transitions, with their morphisms and 2-morphisms, form a 2-category  $\mathcal{G}^{\mathcal{B}}$ .*

The various coherence laws of a 2-category follow directly from the properties of 2-quotients.

I now know what it means that two  $\mathcal{G}$ -2-transitions are equivalent: that they are equivalent in this 2-category.

When  $\mathcal{G}$  is a strict 2-group, it's possible to assume that the isomorphism  $\eta$  in a  $\mathcal{G}$ -2-transition is an identity; that is, every  $\mathcal{G}$ -2-transition is equivalent to such a 'semistrict'  $\mathcal{G}$ -2-transition. This will be necessary in part 3, and I will discuss it in section 2.5.5.

### 2.5.3 The 2-category of $\mathcal{G}$ -2-bundles

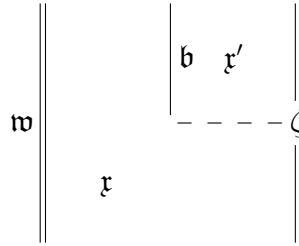
To classify  $\mathcal{G}$ -2-bundles, I need a proper notion of equivalence of  $\mathcal{G}$ -2-bundles. For this, I should define the 2-category  $\mathbf{Bun}_{\mathfrak{e}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$  of  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$  with fibre  $\mathcal{F}$ .

Assume  $\mathcal{G}$ -2-bundles  $\mathcal{E}$  and  $\mathcal{E}'$  over  $\mathcal{B}$ , both with the given fibre  $\mathcal{F}$ , and associated with the  $\mathcal{G}$ -2-transitions  $\mathfrak{g}$  and  $\mathfrak{g}'$  (respectively). Then a  $\mathcal{G}$ -2-bundle **morphism** from  $\mathcal{E}$  to  $\mathcal{E}'$  is a 2-bundle morphism  $\mathfrak{f}$  from  $\mathcal{E}$  to  $\mathcal{E}'$  together with a  $\mathcal{G}$ -2-transition morphism  $\mathfrak{b}$  from  $\mathfrak{g}$  to  $\mathfrak{g}'$  and a natural isomorphism  $\zeta$ :

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathcal{F} \times \langle \mathfrak{b}, \mathfrak{j}_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \\
 \downarrow \mathcal{F} \times \mathfrak{j}_{[0]} & & \downarrow \mathfrak{r} \times \mathcal{U}' \\
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\zeta} & \mathcal{F} \times \mathcal{U}' \\
 \searrow \tilde{\mathfrak{j}} & & \downarrow \tilde{\mathfrak{j}} \\
 \mathcal{E} & \xrightarrow{\mathfrak{f}} & \mathcal{E}'
 \end{array} \tag{2.42}$$

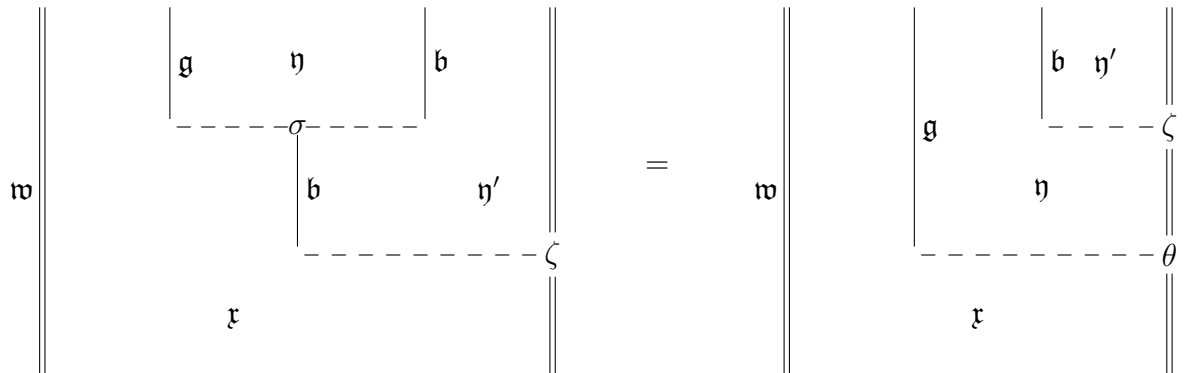
satisfying the two coherence laws given below. I think of  $\mathfrak{f}$  as *being* the  $\mathcal{G}$ -2-bundle morphism and say that  $\mathfrak{f}$  is **associated** with the 2-transition morphism  $\mathfrak{b}$ .

In terms of generalised elements, if  $\mathfrak{w}$  is an element of  $\mathcal{F}$  and  $\langle \mathfrak{x}, \mathfrak{x}' \rangle$  defines an element of  $\mathcal{U} \cap \mathcal{U}'$ , then  $\zeta \mathfrak{w}_{\mathfrak{x}'}$  is an arrow from  $(\mathfrak{w} \mathfrak{b}_{\mathfrak{x}'})/\mathfrak{x}'$  to the composite of  $\mathfrak{w}/\mathfrak{x}$  with  $\mathfrak{f}$ . I can draw  $\zeta$  with this string diagram:



where it is understood that  $\mathfrak{f}$  is applied to the portion in  $\mathcal{E}$  so that the whole diagram may be interpreted in  $\mathcal{E}'$ .

In these terms, the coherence laws that  $\zeta$  must satisfy are:



and:

$$\begin{array}{ccc}
 \begin{array}{c} \parallel \\ \parallel \\ \mathfrak{w} \end{array} & \begin{array}{c} \begin{array}{ccc} \mathfrak{b} & \mathfrak{r}' & \mathfrak{g}' \\ \hline \text{---} \delta \text{---} \\ \mathfrak{b} & & \eta' \\ \hline \mathfrak{x} & & \zeta \end{array} \\ \parallel \\ \parallel \end{array} & = & \begin{array}{c} \begin{array}{ccc} & \mathfrak{g}' & \eta' \\ \hline \mathfrak{b} & & \theta' \\ \hline \mathfrak{x} & & \zeta \end{array} \\ \parallel \\ \parallel \end{array}
 \end{array}$$

If  $\mathcal{E}$  and  $\mathcal{E}'$  are the same  $\mathcal{G}$ -2-bundle (so associated with the same  $\mathcal{G}$ -2-transition) and  $f$  is the identity 2-map  $\underline{\mathcal{E}}$ , then  $\zeta$  is a special case of  $\theta$  (with  $\mathfrak{b}$  taken to be the identity  $\mathcal{G}$ -2-transition morphism on  $\mathfrak{g}$ , which is  $\mathfrak{g}$  itself); both coherence laws for  $\zeta$  reduce to the associativity law for  $\theta$ . In this way, every  $\mathcal{G}$ -2-bundle has an identity  $\mathcal{G}$ -2-bundle automorphism.

Given  $\mathcal{G}$ -2-bundle morphisms  $f$  from  $\mathcal{E}$  to  $\mathcal{E}'$  and  $f'$  from  $\mathcal{E}'$  to  $\mathcal{E}''$ , the composite  $\mathcal{G}$ -2-bundle morphism  $f; f'$  is simply the composite 2-bundle morphism  $\mathcal{E} \xrightarrow{f} \mathcal{E}' \xrightarrow{f'} \mathcal{E}''$  together with the composite  $\mathcal{G}$ -2-transition  $\mathfrak{b}; \mathfrak{b}'$  as described in section 2.5.2.

**Proposition 20** *This  $f; f'$  really is a  $\mathcal{G}$ -2-bundle morphism.*

**Proof:** Let  $\mathfrak{w}$  be an element of  $\mathcal{F}$ , and let  $\langle \mathfrak{x}, \mathfrak{x}', \mathfrak{x}'' \rangle$  define an element of  $\mathcal{U} \cap \mathcal{U}' \cap \mathcal{U}''$ .



to  $\mathcal{E}'$ , a  $\mathcal{G}$ -2-bundle 2-morphism from  $\mathfrak{f}$  to  $\mathfrak{f}'$  consists of a bundle 2-morphism  $\kappa$  and a  $\mathcal{G}$ -2-transition 2-morphism  $\xi$  (between the underlying  $\mathcal{G}$ -2-transition morphisms  $\mathfrak{b}$  and  $\mathfrak{b}'$ ) satisfying this coherence law:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathcal{E} \times \langle \mathfrak{b}, j_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \\
 \downarrow \mathcal{E} \times j_{[0]} & & \downarrow \tau \times \mathcal{U}' \\
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\zeta} & \mathcal{F} \times \mathcal{U}' \\
 \searrow \tilde{j} & & \downarrow \tilde{j} \\
 \mathcal{E} & \xrightarrow{\mathfrak{f}} & \mathcal{E}' \\
 & \downarrow \kappa & \\
 & \mathfrak{f}' & 
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\mathcal{E} \times \langle \mathfrak{b}, j_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \\
 \downarrow \mathcal{E} \times j_{[0]} & \begin{array}{c} \xrightarrow{\mathcal{E} \times \langle \xi, j_{[1]} \rangle} \\ \xrightarrow{\mathcal{E} \times \langle \mathfrak{b}', j_{[1]} \rangle} \end{array} & \downarrow \tau \times \mathcal{U}' \\
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\zeta'} & \mathcal{F} \times \mathcal{U}' \\
 \searrow \tilde{j} & & \downarrow \tilde{j} \\
 \mathcal{E} & \xrightarrow{\mathfrak{f}'} & \mathcal{E}'
 \end{array}
 \quad (2.44)$$

**Proposition 21** *Given a space  $\mathcal{B}$ ,  $\mathcal{G}$ -bundles over  $\mathcal{B}$  and their  $\mathcal{G}$ -bundle morphisms and  $\mathcal{G}$ -bundle 2-morphisms form a 2-category  $\mathbf{Bun}_{\mathfrak{C}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$ .*

**Proof:** 2-Bundles, 2-bundle morphisms, and 2-bundle 2-morphisms form a 2-category  $\mathfrak{C}/\mathcal{B}$ ;  $\mathcal{G}$ -2-transitions,  $\mathcal{G}$ -2-transition morphisms, and  $\mathcal{G}$ -2-transition 2-morphisms form a 2-category  $\mathcal{G}^{\mathcal{B}}$ .  $\mathbf{Bun}_{\mathfrak{C}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$  is a straightforward combination.  $\blacksquare$

I now know what it means for  $\mathcal{G}$ -2-bundles  $\mathcal{E}$  and  $\mathcal{E}'$  to be **equivalent  $\mathcal{G}$ -2-bundles**: equivalent objects in the 2-category  $\mathbf{Bun}_{\mathfrak{C}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$ . In particular,  $\mathcal{E}$  and  $\mathcal{E}'$  are equivalent as 2-bundles.

## 2.5.4 Associated $\mathcal{G}$ -2-bundles

Just as a  $\mathcal{G}$ -2-bundle may be reconstructed from its  $\mathcal{G}$ -transition, so a  $\mathcal{G}$ -2-bundle may be reconstructed from its  $\mathcal{G}$ -2-transition.





The 2-kernel pair of  $\underline{\mathcal{F}} \times j_{[0]}$  is  $\mathcal{F} \times \mathcal{U}^{[3]}$ :

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{U}^{[3]} & \xrightarrow{\underline{\mathcal{F}} \times j_{[02]}} & \mathcal{F} \times \mathcal{U}^{[2]} \\ \underline{\mathcal{F}} \times j_{[01]} \downarrow & & \downarrow \underline{\mathcal{F}} \times j_{[0]} \\ \mathcal{F} \times \mathcal{U}^{[2]} & \xrightarrow{\underline{\mathcal{F}} \times j_{[0]}} & \mathcal{F} \times \mathcal{U} \end{array}$$

Let the Euclideanness 2-map be  $\mathcal{F} \times \mathcal{U}^{[3]} \xrightarrow{\underline{\mathcal{F}} \times \langle j_{[01]}, j_{[12]} \rangle} \mathcal{F} \times \mathcal{U}^{[2]} \times \mathcal{U}^{[2]} \xrightarrow{\underline{\mathcal{F}} \times \mathfrak{g} \times \mathcal{U}^{[2]}} \mathcal{F} \times \mathcal{G} \times \mathcal{U}^{[2]} \xrightarrow{\tau \times \mathcal{U}^{[2]}} \mathcal{F} \times \mathcal{U}^{[2]}$ . In (2.6), the natural isomorphism  $\omega_{[01]}$  is trivial, while  $\omega_{[11]}$  is given by this string diagram:

Therefore, I really do have an equivalence 2-relation, so some 2-quotient  $\mathcal{F} \times \mathcal{U} \xrightarrow{\tilde{j}} \mathcal{E}$  must exist. The isomorphism (2.7) becomes  $\theta$ ; while the coherence laws (2.8, 2.9) become (respectively) the coherence laws (2.32, 2.31).

To define the bundle 2-map  $\mathcal{E} \xrightarrow{p} \mathcal{B}$ , first note this natural isomorphism:

$$\begin{array}{ccccc} \mathcal{F} \times \mathcal{U}^{[2]} & \xrightarrow{\underline{\mathcal{F}} \times \langle \mathfrak{g}, j_{[11]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U} & \xrightarrow{\tau \times \mathcal{U}} & \mathcal{F} \times \mathcal{U} \\ \downarrow \underline{\mathcal{F}} \times j_{[0]} & & & & \downarrow \hat{\mathcal{F}} \times \mathcal{U} \\ & & & \Leftarrow & \mathcal{U} \\ & & & & \downarrow j \\ \mathcal{F} \times \mathcal{U} & \xrightarrow{\hat{\mathcal{F}} \times \mathcal{U}} & \mathcal{U} & \xrightarrow{j} & \mathcal{B} \end{array}$$

which is simply one of the structural isomorphisms of the 2-cover  $\mathcal{U}$ . Since  $\mathcal{E}$  is a 2-quotient, this defines a 2-map  $\mathcal{E} \xrightarrow{p} \mathcal{B}$  and a natural isomorphism as in (2.18).

Therefore,  $\mathcal{E}$  is a 2-bundle over  $\mathcal{B}$  with fibre  $\mathcal{F}$  associated with the 2-transition  $\mathfrak{g}$ . ■

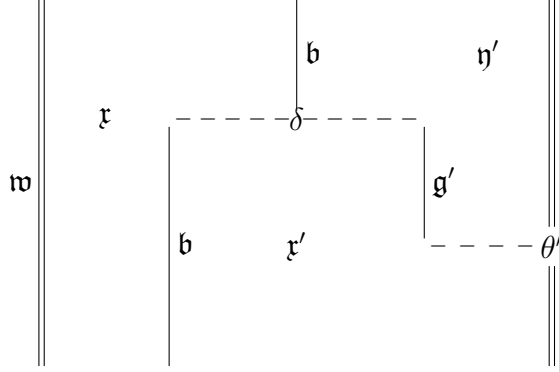
If I'm to conclude, as in part 1, that this  $\mathcal{E}$  is unique up to an essentially unique equivalence of  $\mathcal{G}$ -2-bundles, then the next step is to consider  $\mathcal{G}$ -2-bundle morphisms.

**Proposition 23** *Given  $\mathcal{G}$ -2-bundles  $\mathcal{E}$  and  $\mathcal{E}'$  (both over  $\mathcal{B}$  and with fibre  $\mathcal{F}$ ) associated with  $\mathcal{G}$ -2-transition morphisms  $\mathfrak{g}$  and  $\mathfrak{g}'$  (respectively), and given a  $\mathcal{G}$ -2-transition morphism  $\mathfrak{b}$  from  $\mathfrak{g}$  to  $\mathfrak{g}'$ , there is a  $\mathcal{G}$ -2-bundle morphism  $\mathfrak{f}$  from  $\mathcal{E}$  to  $\mathcal{E}'$  associated with  $\mathfrak{b}$ .*

**Proof:** First consider this natural isomorphism:

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}'^{[2]} & \xrightarrow{\underline{\mathcal{E}} \times j_{[02]}} & \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' \\
 \downarrow \underline{\mathcal{E}} \times j_{[01]} & & \downarrow \underline{\mathcal{E}} \times \langle \mathfrak{b}, j_{[1]} \rangle \\
 & \cong & \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \\
 & & \downarrow \tau \times \underline{\mathcal{U}} \\
 & & \mathcal{F} \times \mathcal{U} \\
 & & \downarrow \tilde{j}' \\
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\underline{\mathcal{E}} \times \langle \mathfrak{b}, j_{[1]} \rangle} \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \xrightarrow{\tau \times \underline{\mathcal{U}}'} \mathcal{F} \times \mathcal{U}' \xrightarrow{\tilde{j}'} & \mathcal{E}'
 \end{array}$$

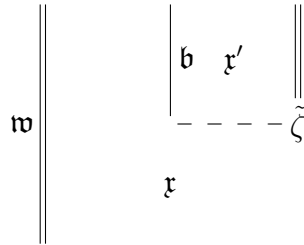
given by this string diagram:



Since  $\underline{\mathcal{F}} \times j_{[0]}$ , being a 2-cover, is a 2-quotient of its kernel pair, I can construct from this a 2-map  $\mathcal{F} \times \mathcal{U} \xrightarrow{\tilde{f}} \mathcal{E}$  and a natural isomorphism  $\tilde{\zeta}$ :

$$\begin{array}{ccc}
 \mathcal{F} \times \mathcal{U} \cap \mathcal{U}' & \xrightarrow{\underline{\mathcal{F}} \times \langle \mathfrak{b}, j_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U}' \\
 \searrow \underline{\mathcal{E}} \times j_{[0]} & & \downarrow \mathfrak{r} \times \mathcal{U}' \\
 & \tilde{\zeta} & \mathcal{F} \times \mathcal{U} \\
 & & \downarrow \tilde{f}' \\
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\tilde{f}} & \mathcal{E}'
 \end{array}$$

In terms of elements  $\mathfrak{w}$  of  $\mathcal{F}$  and  $\langle \mathfrak{r}, \mathfrak{r}' \rangle$  of  $\mathcal{U} \cap \mathcal{U}'$ , this gives an arrow  $\tilde{\zeta} \mathfrak{w}_{\mathfrak{r}'}$  from  $(\mathfrak{w} \mathfrak{b}_{\mathfrak{r}'})/\mathfrak{r}'$  to the composite of  $\langle \mathfrak{w}, \mathfrak{r} \rangle$  with  $\tilde{f}$ , as in this string diagram:

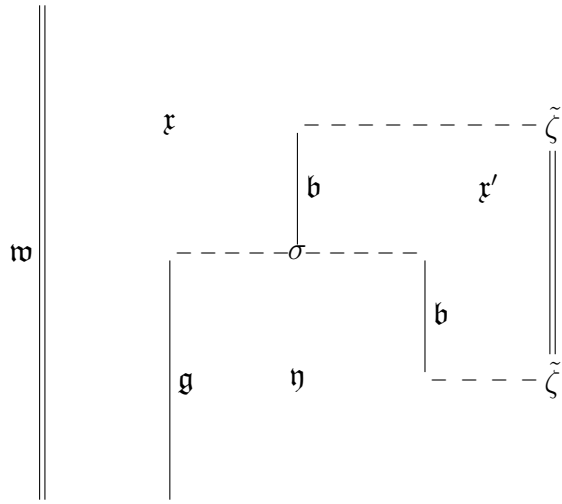


where  $\tilde{f}$  is applied to the portion of the diagram left open on the right.

Next, notice this natural isomorphism:

$$\begin{array}{ccccc}
 \mathcal{F} \times \mathcal{U}^{[2]} & \xrightarrow{\mathcal{F} \times \langle \mathfrak{g}, \mathfrak{j}_{[1]} \rangle} & \mathcal{F} \times \mathcal{G} \times \mathcal{U} & \xrightarrow{\mathfrak{r} \times \mathcal{U}} & \mathcal{F} \times \mathcal{U} \\
 \mathcal{F} \times \mathfrak{j}_{[0]} \downarrow & & \llcorner & & \downarrow \tilde{\mathfrak{f}} \\
 \mathcal{F} \times \mathcal{U} & \xrightarrow{\tilde{\mathfrak{f}}} & & & \mathcal{E}'
 \end{array}$$

given by this string diagram:



(Notice that I can drop  $\mathfrak{x}'$  because  $\mathcal{F} \times \mathcal{U}^{[2]} \cap \mathcal{U}' \xrightarrow{\mathfrak{j}_{[01]}} \mathcal{F} \times \mathcal{U}^{[2]}$  is 2-epic.) Since  $\tilde{\mathfrak{j}}$ , being the pullback of a 2-cover, is a 2-cover and hence the 2-quotient of its 2-kernel pair, I can construct from this a 2-map  $\mathcal{E} \xrightarrow{\tilde{\mathfrak{f}}} \mathcal{E}'$  and a natural isomorphism  $\zeta$  as in (2.42). As with  $\theta$  in the previous proposition, the coherence laws for  $\zeta$  come from the coherence laws for a 2-quotient.

In short, I've constructed a  $\mathcal{G}$ -2-bundle morphism  $\mathcal{E} \xrightarrow{\tilde{\mathfrak{f}}} \mathcal{E}'$  associated with  $\mathfrak{b}$ . ■

**Proposition 24** *Given  $\mathcal{G}$ -2-bundle morphisms  $\mathfrak{f}$  and  $\mathfrak{f}'$  associated (respectively) with*

$\mathfrak{b}$  and  $\mathfrak{b}'$ , any  $\mathcal{G}$ -2-transition 2-morphism  $\xi$  from  $\mathfrak{b}$  to  $\mathfrak{b}'$  has associated with it a unique  $\mathcal{G}$ -2-bundle 2-morphism  $\kappa$  from  $\mathfrak{f}$  to  $\mathfrak{f}'$ .

**Proof:** Let  $\tilde{\mathfrak{f}}'$  be the composite  $\mathcal{F} \times \mathcal{U} \xrightarrow{\tilde{\mathfrak{j}}} \mathcal{E} \xrightarrow{\mathfrak{f}} \mathcal{E}'$ . Then  $\kappa$  is  $\tilde{\omega}_{\mathfrak{t}}$  in (2.11), while (2.12) gives (2.44). ■

In fact, there is a 2-functor from the 2-category  $\mathbf{Bun}_{\mathfrak{e}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$  of  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$  with fibre  $\mathcal{F}$  to the 2-category  $\mathcal{B}^{\mathcal{G}}$  of  $\mathcal{G}$ -2-transitions over  $\mathcal{B}$ , defined by simply forgetting the total space  $\mathcal{E}$ . The propositions above, taken altogether, prove that this 2-functor is an equivalence of 2-categories. That is:

**Theorem 2** *Given a 2-space  $\mathcal{B}$ , a 2-group  $\mathcal{G}$ , and a  $\mathcal{G}$ -2-space  $\mathcal{F}$ , the 2-category  $\mathbf{Bun}_{\mathfrak{e}}(\mathcal{G}, \mathcal{B}, \mathcal{F})$  of  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$  with fibre  $\mathcal{F}$  is equivalent to the 2-category  $\mathcal{B}^{\mathcal{G}}$  of  $\mathcal{G}$ -2-transitions on  $\mathcal{B}$ .*

So depending on the point of view desired, you can think locally of  $\mathcal{G}$ -2-transitions over  $\mathcal{B}$ , or globally of principal  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$  (or  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$  with some other fibre  $\mathcal{F}$ ).

### 2.5.5 Semistrictification

In this extra section, I show how to replace any  $\mathcal{G}$ -2-transition (and hence any  $\mathcal{G}$ -2-bundle) with a **semistrict**  $\mathcal{G}$ -2-transition (or  $\mathcal{G}$ -2-bundle), defined as one where  $\eta$  is trivial, more precisely where  $\eta_{\mathfrak{t}} = \iota(\mathfrak{g}_{\text{tr}})$ . Notice that if  $\mathcal{G}$  is a strict 2-group, then this  $\eta$  is an identity; but reducing  $\eta$  to a special case of  $\iota$  is the best that you can hope

for in general, since you can't have an identity morphism between things that aren't equal! Unlike most of this part, it's *not* sufficient that  $\mathcal{G}$  be merely a 2-monoid; I will really need both  $\iota$  and  $\epsilon$  (even in the strict case).

Given a  $\mathcal{G}$ -2-transition  $\mathfrak{g}$ , to define the **semistrictification**  $\mathfrak{g}'$  of  $\mathfrak{g}$ , let  $\mathfrak{g}'_{\mathfrak{r}\mathfrak{h}}$  be  $(\mathfrak{g}_{\mathfrak{r}\mathfrak{h}}\mathfrak{g}_{\mathfrak{h}\mathfrak{r}}^{-1})$ , as in this string diagram:

$$\mathfrak{r} \left| \begin{array}{c} \mathfrak{g}' \quad \mathfrak{h} \\ \hline \end{array} \right. := \mathfrak{r} \left| \begin{array}{c} \mathfrak{g} \quad \mathfrak{h} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g}^{-1} \mathfrak{h} \\ \hline \end{array} \right.$$

Let  $\gamma'_{\mathfrak{r}\mathfrak{h}\mathfrak{s}}$  and  $\eta'_{\mathfrak{r}}$  be as follows:

$$\begin{array}{c} \left| \begin{array}{c} \mathfrak{g}' \quad \mathfrak{h} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g}' \\ \hline \end{array} \right. \\ \text{---} \gamma' \text{---} \\ \mathfrak{r} \quad \left| \begin{array}{c} \mathfrak{g}' \\ \hline \end{array} \right. \quad \mathfrak{s} \end{array} := \begin{array}{c} \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{h} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \\ \left| \begin{array}{c} \mathfrak{g}^{-1} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{h} \\ \hline \end{array} \right. \\ \text{---} \epsilon \text{---} \quad \text{---} \gamma \text{---} \\ \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{h} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \\ \text{---} \gamma \text{---} \\ \mathfrak{r} \quad \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \quad \mathfrak{s} \quad \left| \begin{array}{c} \mathfrak{g}^{-1} \\ \hline \end{array} \right. \quad \mathfrak{s} \end{array}$$

and:

$$\mathfrak{r} \quad \left| \begin{array}{c} \eta' \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g}' \\ \hline \end{array} \right. := \left| \begin{array}{c} \mathfrak{g} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{r} \\ \hline \end{array} \right. \left| \begin{array}{c} \mathfrak{g}^{-1} \\ \hline \end{array} \right.$$

That is,  $\eta'_{\mathfrak{r}} = \iota(\mathfrak{g}_{\mathfrak{r}\mathfrak{r}})$  as promised.



To see that  $\mathfrak{g}$  and  $\mathfrak{g}'$  are equivalent, let  $\mathfrak{b}_{\mathfrak{r}\eta}$  be  $(\mathfrak{g}_{\mathfrak{r}\eta}\mathfrak{g}_{\eta\mathfrak{r}}^{-1})$  (the same as  $\mathfrak{g}'_{\mathfrak{r}\eta}$ ), and let  $\mathfrak{b}'_{\mathfrak{r}\eta}$  be  $\mathfrak{g}_{\mathfrak{r}\eta}$ .

**Proposition 26**  *$\mathfrak{b}$  is a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}$  to  $\mathfrak{g}'$ ,  $\mathfrak{b}'$  is a  $\mathcal{G}$ -2-transition morphism from  $\mathfrak{g}'$  to  $\mathfrak{g}$ , there is an invertible  $\mathcal{G}$ -2-transition 2-morphism  $\xi$  from  $\mathfrak{b}; \mathfrak{b}'$  to the identity on  $\mathfrak{g}$ , and there is an invertible  $\mathcal{G}$ -2-transition 2-morphism  $\xi'$  from  $\mathfrak{b}'; \mathfrak{b}$  to the identity on  $\mathfrak{g}'$ .*

In other words,  $\mathfrak{g}$  and  $\mathfrak{g}'$  are equivalent.

**Proof:**  $\sigma$ ,  $\delta'$ , and  $\xi'$  work just like  $\gamma$ ; while  $\delta$ ,  $\sigma'$  and  $\xi$  work like  $\gamma'$ . (Sometimes there's an extra  $\mathfrak{g}_{\mathfrak{z}\mathfrak{z}}^{-1}$  multiplied on the right, but it doesn't get involved.) The proofs of the coherence laws are just like the associative law for  $\gamma$  (laws (2.34, 2.38) for  $\mathfrak{b}$ , (2.36, 2.38) for  $\mathfrak{b}'$ , (2.40) for  $\xi$ , and (2.40, 2.41) for  $\xi'$ ); the left unit law for  $\gamma$  (law (2.35) for  $\mathfrak{b}$ ); the associative law for  $\gamma'$  (law (2.36) for  $\mathfrak{b}$ , (2.34) for  $\mathfrak{b}'$ , and (2.41) for  $\xi$ ); the right unit law for  $\gamma'$  (law (2.37) for  $\mathfrak{b}$ ); the left unit law for  $\gamma'$  (law (2.35) for  $\mathfrak{b}'$ ); and the right unit law for  $\gamma$  (law (2.37) for  $\mathfrak{b}'$ ). ■

Notice that the original  $\eta$  is never used in any of these constructions. So the result is not only that every  $\mathcal{G}$ -2-transition may be semistrictified, but also that it's enough to give  $\mathfrak{g}$  and  $\gamma$  (and to check associativity for this  $\gamma$ ) to define an (equivalent) semistrict  $\mathcal{G}$ -2-transition.



## Part 3

# Examples

A few simple examples will help to clarify the ideas; a more complicated example will link them to the previous notion of gerbe. In these examples, I will not hesitate to assume that the category  $\mathbf{C}$  is indeed the category of smooth manifolds and smooth functions.

These examples, although they live in the 2-category  $\mathfrak{C}$  of 2-spaces, are built of the category  $\mathbf{C}$  of spaces. Accordingly, I will use italic Roman letters for spaces and maps; then, as in section 2.2.4, I will use calligraphic or Fraktur letters for the 2-spaces and 2-maps built out of them.

### 3.1 Bundles as 2-bundles

Just as every space is a categorically discrete 2-space, so every group is a categorically discrete 2-group, and every bundle is a categorically discrete 2-bundle. Just apply

section 2.2.4 to every space involved; all of the natural transformations needed can just be identities. This is trivial and uninteresting.

## 3.2 Vector bundles

Let  $E$  be a vector bundle over the base space  $B$ . That is, the fibre  $F$  is a (real or complex) vector space, and the action of  $G$  on  $F$  is a linear representation of  $G$ . To define a 2-bundle over  $B$ , let the space of points of the total 2-space be  $\mathcal{E}^1 := B$  and let its space of arrows be  $\mathcal{E}^2 := E$ . Let the source and target maps  $\mathcal{E}^2 \rightarrow \mathcal{E}^1$  both be the original projection map  $E \xrightarrow{p} B$ . Let the identity arrow map  $\mathcal{E}^1 \xrightarrow{d_{00}} \mathcal{E}^2$  be the zero section that assigns each point to the zero vector over it. Similarly, let the composition map  $d_{02}$  be given by vector addition. (In the case where  $E$  is the tangent bundle  $TB$ , this 2-space was described in the course of [HDA5, Example 48], as the ‘tangent groupoid’ of  $B$ , so called because this 2-space, viewed as a category, is a groupoid. But I would call this example the **tangent 2-bundle**, because as you will see, it has the structure of a 2-bundle.) Returning to the general case, I have a total 2-space  $\mathcal{E}$ ; the projection 2-map  $\mathcal{E} \xrightarrow{p} B$  is given by the identity  $B \xrightarrow{B} B$  on points and by  $E \xrightarrow{p} B$  on arrows.

Now, in order to get a 2-bundle, I need to identify the appropriate 2-group. This was defined in [HDA5, Example 50] in terms of Lie crossed modules. In the terms of the present paper, the space of points of the 2-group is simply  $\mathcal{G}^1 := G$ , while the space of arrows is  $\mathcal{G}^2 := G \times F$ . The source and target maps are both simply the

projection  $G \times F \xrightarrow{G \times \hat{F}} G$ . The identity arrow map maps  $x \in G$  to  $(0, x) \in G \times F$ , while composition is again given by vector addition.

The multiplication  $\mathbf{m}$  is more interesting. Since the vector space  $F$  is an abelian group and the action of  $G$  on  $F$  is linear, the space  $\mathcal{G}^2$  can be interpreted as a semidirect product  $G \ltimes F$ . Thus inspired, on points,  $\mathcal{G}^1 \times \mathcal{G}^1 \xrightarrow{\mathbf{m}^1} \mathcal{G}^1$  is simply  $G \times G \xrightarrow{m} G$ . But on arrows, I let the product of  $(x, \vec{w})$  and  $(x', \vec{w}')$  be  $(xx', \vec{w}x' + \vec{w}')$ . (This formula for multiplication can be internalised with a diagram, thus generalising the situation to an arbitrary category  $\mathbf{C}$ , but I won't do that here. In fact, to do it perfectly properly, I would need to replace the vector space  $F$  with an internal abelian group —although this is not a bad thing, since this can be an interesting generalisation even when  $\mathbf{C}$  is the category of smooth manifolds.)

Next, I'll define the fibre 2-space  $\mathcal{F}$ . Let the point space  $\mathcal{F}^1$  be the singleton space  $1$ , and let the arrow space  $\mathcal{F}^2$  be  $F$ . The source and target maps are both the unique map  $F \xrightarrow{\hat{F}} 1$ . Again, the identity arrow map is given by the zero vector, and the composition 2-map is given by vector addition. If  $U \xrightarrow{j} B$  is a cover map for the original vector bundle (and in the case of a vector bundle, any open cover by Euclidean neighbourhoods will suffice), then the pulled-back 2-map  $\mathcal{F} \times \mathcal{U} \xrightarrow{\tilde{j}} \mathcal{E}$  is given by  $U \xrightarrow{j} B$  on points and  $F \times U \xrightarrow{\tilde{j}} E$  on arrows. Some tedious checking shows that this gives a 2-pullback:

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{U} & \xrightarrow{\tilde{j}} & \mathcal{E} \\ \hat{F} \times \mathcal{U} \downarrow & & \downarrow \mathbf{p} \\ \mathcal{U} & \xrightarrow{j} & \mathcal{B} \end{array}$$

The action of  $\mathcal{G}$  on  $\mathcal{F}$  is simple enough. Of course, the point map  $\mathcal{F}^1 \times \mathcal{G}^1 \xrightarrow{\tau^1} \mathcal{F}^1$  is necessarily  $1 \times G \xrightarrow{\hat{G}} 1$ . But the arrow map  $\mathcal{F}^2 \times \mathcal{G}^2 \xrightarrow{\tau^2} \mathcal{F}^2$  is more interesting, mapping  $(\vec{w}, (x, \vec{w}'))$  to  $\vec{w}x + \vec{w}'$ .

Next, the 2-transition  $\mathcal{U}^{[2]} \xrightarrow{g} \mathcal{G}$  is simply  $U^{[2]} \xrightarrow{\hat{U}^{[2]}} 1$  on points, and  $U^{[2]} \xrightarrow{g} G$  on arrows. Now all of the 2-maps are defined.

I still need to define the natural transformations and verify the coherence laws. But the natural transformations can all be taken to be identities, so the coherence laws are all automatically true. Therefore, I have a  $\mathcal{G}$ -2-bundle.

### 3.3 Gerbes

For gerbes, I follow Breen in [Bre94, chapter 2]. As noted in [Bre94, 2.13], this discussion applies most generally to a crossed module, denoted by Breen as  $G \xrightarrow{\delta} \Pi$ . Now, for Breen, this is not a single crossed module but rather a *sheaf* of crossed modules over a space  $X$ . I will (again) take the base 2-space  $\mathcal{B}$  to be the 2-space built trivially out of the space  $X$  as in section 2.2.4; but I do not reach the generality of an arbitrary sheaf of crossed modules. Instead, given a strict 2-group  $\mathcal{G}$ , interpreted as a crossed module as in section 2.4.6, let Breen's  $G$  be the sheaf of  $H$ -valued functions on  $X$ , and let his  $\Pi$  be the sheaf of  $D$ -valued functions on  $X$ . Then acting pointwise, these form a sheaf of crossed modules  $G \xrightarrow{\delta} \Pi$ . (Thus gerbes are more general than 2-bundles, since only sheaves of this form can appear, much as sheaves themselves are more general than bundles; although 2-bundles are still more general than gerbes

—at least as so far as anything has been published— in that only strict 2-groups have so far appeared with gerbes.)

In [Bre94, 2.4], Breen describes a gerbe in terms of local data as follows:

- an open cover  $(U_i : i \in I)$  of  $X$ ;
- an open cover  $(U_{ij}^\alpha : \alpha \in J_{ij})$  of each double intersection  $U_i \cap U_j$ ;
- a section  $\lambda_{ij}^\alpha$  of  $\Pi$  over each  $U_{ij}^\alpha$ ; and
- a section  $g_{ijk}^{\alpha\beta\gamma}$  of  $G$  over each triple intersection  $U_{ij}^\alpha \cap U_{jk}^\beta \cap U_{ik}^\gamma$ ; such that
- there is only one  $\alpha$  in each  $J_{ii}$ ;
- $(\lambda_{ij}^\alpha \lambda_{jk}^\beta)$  is equal to the value of  $g_{ijk}^{\alpha\beta\gamma}$  in  $\Pi$  multiplied by  $\lambda_{ik}^\gamma$  ([Bre94, (2.4.5)], or [BM, (2.1.5)]);
- $(g_{ijk}^{\alpha\beta\gamma} g_{ikl}^{\gamma\eta\delta})$  is equal to the action of  $\lambda_{ij}^\alpha$  on  $g_{jkl}^{\beta\eta\epsilon}$ , multiplied by  $g_{ijl}^{\alpha\epsilon\delta}$ , on each sextuple intersection  $U_{ij}^\alpha \cap U_{jk}^\beta \cap U_{ik}^\gamma \cap U_{kl}^\eta \cap U_{jl}^\epsilon \cap U_{il}^\delta$  ([Bre94, (2.4.8)], or [BM, (2.1.6)]);
- $\lambda_{ii}^\alpha$  is the identity in  $\Pi$  (for  $\alpha$  the unique element of  $J_{ii}$ );
- $g_{iik}^{\alpha\beta\beta}$  is the identity in  $G$  (for  $\alpha$  again the unique element of  $J_{ii}$ ); and
- $g_{ikk}^{\alpha\beta\alpha}$  is the identity in  $G$  (for  $\beta$  now the unique element of  $J_{kk}$ ).

Since Breen nowhere gives precisely this list, a few words are in order: Breen’s definition of a ‘labeled decomposition’ is given directly in terms of certain  $\phi_{ij}^\alpha$ , out of which the  $\lambda$  and  $g$  are constructed; but [Bre94, 2.6] shows conversely how to recover  $\phi$  from  $\lambda$  and  $g$ . Also, Breen never mentions  $\Pi$  at this point, only the automorphism group  $\underline{Aut}(G)$  of  $G$ ; he is only considering the case where  $\Pi$  is this automorphism group,

but he explains how to generalise this when he discusses crossed modules, and I have followed that here. Finally, Breen suppresses the unique element of each  $J_{ii}$ , but I've restored this since I find it confusing to suppress it.

Here is how these data corresponds to the local data of a  $\mathcal{G}$ -2-bundle, that is a  $\mathcal{G}$ -2-transition:

- The open cover  $(U_i)$  corresponds to a cover  $U \xrightarrow{j} X$ , which (as explained in section 2.2.5) is equivalent to a 2-cover  $\mathcal{U} \xrightarrow{j} \mathcal{B}$ .
- The open cover  $(U_{ij}^\alpha)$  corresponds to the cover  $|\mathfrak{g}| \longrightarrow U$  involved in the transition 2-map  $\mathcal{U} \xrightarrow{\mathfrak{g}} \mathcal{G}$ .
- The section  $\lambda_{ij}^\alpha$  is simply the map  $|\mathfrak{g}| \xrightarrow{\mathfrak{g}^1} D = \mathcal{G}^1$ .
- The (disjoint union of the) triple intersections  $U_{ij}^\alpha \cap U_{jk}^\beta \cap U_{ik}^\gamma$  may be taken as the domain of the map  $|\gamma|$  for the natural isomorphism  $\gamma$  of diagram (2.27). Then  $|\gamma|$  takes values in  $\mathcal{G}^2 = H \rtimes D$ , and Breen's  $g$  is the  $H$ -valued component of this map.
- The simplicity of the open cover of  $U_i \cap U_j$  should be ensured before the transition begins by passing to a refinement. (This doesn't work for the  $U_i \cap U_j$  in general, since you might not be able to coordinate them. Breen discusses this in [Bre94, 2.3].)
- The requirement on the value of  $g_{ijk}^{\alpha\beta\gamma}$  in  $\Pi$  fixes the  $D$ -valued component of  $|\gamma|$ ; that is, it indicates an appropriate source and target for  $\gamma$ .

- The requirement for the action of  $\lambda_{ij}^\alpha$  on  $g_{jkl}^{\beta\eta\epsilon}$ , following the principles laid out in section 2.4.6, is precisely the associativity coherence law (2.28).
- That  $\lambda_{ii}^\alpha$  is trivial is ensured by passing to the semistrictification; see section 2.5.5.
- This done, that  $g_{iik}^{\alpha\beta\beta}$  and  $g_{ikk}^{\alpha\beta\alpha}$  are trivial is the unit coherence law (2.29).

Thus, gerbes correspond to (semistrict) 2-bundles.

But to get a good comparison, this should respect equivalence of each. (Already I had to replace an arbitrary 2-bundle with its equivalent semistrictification.) Given two gerbes (which Breen denotes as primed and unprimed), here are the local data for an equivalence of gerbes:

- a common refinement of the open covers  $(U_i)$  and  $(U'_i)$  (renamed  $(U_i)$  by Breen);
- common refinements of each  $(U_{ij}^\alpha)$  on a given  $U_i \cap U_j$ ;
- a further refinement to allow  $\mu_i$  below to make sense;
- a section  $\mu_i$  of  $\Pi$  on each  $U_i$ ; and
- a section  $\delta_{ij}^\alpha$  of  $G$  on each  $U_{ij}^\alpha$ ; such that
- $\delta_{ii}^\alpha$  is the identity in  $\Pi$  (for  $\alpha$  the unique element of  $J_{ii}$ );
- $(\lambda_{ij}^\alpha \mu_j)$  is equal to the value of  $\delta_{ij}^\alpha$  in  $\Pi$  multiplied by  $(\mu_i \lambda_{ij}^\alpha)$  ([Bre94, 2.4.16], or [BM, 2.1.11]); and
- $(g_{ijk}'^{\alpha\beta\gamma} \delta_{ik}^\gamma)$  is equal to the action of  $\lambda_{ij}'^\alpha$  on  $\delta_{jk}^\beta$ , multiplied by  $\delta_{ij}^\alpha$ , multiplied by the action of  $\mu_i$  on  $g_{ijk}^{\alpha\beta\gamma}$  ([Bre94, (2.4.17)], or [BM, (2.1.12)]).

I use primes like Breen, but it turns out that the order here is *backwards*; so as you

compare the above to the data of a  $\mathcal{G}$ -2-transition morphism, be sure to swap the primes in the data for the  $\mathcal{G}$ -2-transitions. Here is the precise correspondence:

- The common refinement of the original open covers is taken automatically in forming the pullback  $\mathcal{U} \cap \mathcal{U}'$ , the source of  $\mathfrak{b}$ .
- The common refinement of the open covers' hypercovers gives the source  $|\mathfrak{b}|$  of  $\mathfrak{b}^1$ .
- The final refinement is only taken to ensure that the new  $J_{ii}$  are again singletons.
- $\mu_i$  corresponds to  $\mathfrak{b}_{\mathfrak{r}\mathfrak{x}}$ . In general,  $\mathfrak{b}_{\mathfrak{r}\eta}$  may be recovered up to isomorphism as  $(\mathfrak{g}_{\mathfrak{r}\eta} \mathfrak{b}_{\eta\eta})$ .
- Breen's  $\delta_{ij}^\alpha$  corresponds to my  $\sigma_{\mathfrak{r}\eta\eta}$  followed by  $\bar{\delta}_{\mathfrak{r}\mathfrak{r}\eta}$ .
- That  $\sigma_{\mathfrak{r}\mathfrak{r}\mathfrak{x}}$  followed by  $\delta_{\mathfrak{r}\mathfrak{r}\mathfrak{x}}$  be trivial fixes  $\sigma$  and  $\delta$  up to isomorphism.
- The value of  $\delta_{ij}^\alpha$  in  $\Pi$  holds because  $(\mathfrak{g}_{\mathfrak{r}\eta} \mathfrak{b}_{\eta\eta}) \xrightarrow{\sigma_{\mathfrak{r}\eta\eta}} \mathfrak{b}_{\mathfrak{r}\eta} \xrightarrow{\bar{\delta}_{\mathfrak{r}\mathfrak{r}\eta}} (\mathfrak{b}_{\mathfrak{r}\mathfrak{x}} \mathfrak{g}'_{\mathfrak{r}\eta})$ .
- The last item is a combination of the associativity coherence laws (2.34, 2.36, 2.38).

Specifically, the formulation of the last item in [BM], translated into my notation,



gives these string diagrams:

The image shows two string diagrams representing gerbe compositions, separated by an equals sign. The left diagram consists of three main parts connected vertically. The top part has a vertical line on the left labeled  $\mathfrak{g}$ , a vertical line on the right labeled  $\mathfrak{g}$ , and a vertical line on the far right labeled  $\mathfrak{b}$ . A dashed horizontal line connects the two  $\mathfrak{g}$  lines, with a vertical line labeled  $\eta$  in the middle. A dashed horizontal line labeled  $\gamma$  connects the two  $\mathfrak{g}$  lines. The middle part has a vertical line on the left labeled  $\mathfrak{g}$ , a vertical line on the right labeled  $\mathfrak{b}$ , and a vertical line on the far right labeled  $\mathfrak{z}$ . A dashed horizontal line labeled  $\sigma$  connects the two  $\mathfrak{g}$  lines. The bottom part has a vertical line on the left labeled  $\mathfrak{b}$ , a vertical line on the right labeled  $\mathfrak{r}$ , and a vertical line on the far right labeled  $\mathfrak{g}'$ . A dashed horizontal line labeled  $\delta$  connects the two  $\mathfrak{b}$  lines. The right diagram is similar but with different labels and connections. It has a vertical line on the left labeled  $\mathfrak{g}$ , a vertical line on the right labeled  $\mathfrak{b}$ , and a vertical line on the far right labeled  $\mathfrak{b}$ . A dashed horizontal line labeled  $\sigma$  connects the two  $\mathfrak{g}$  lines. A dashed horizontal line labeled  $\delta$  connects the two  $\mathfrak{b}$  lines. The middle part has a vertical line on the left labeled  $\mathfrak{b}$ , a vertical line on the right labeled  $\mathfrak{b}$ , and a vertical line on the far right labeled  $\mathfrak{z}$ . A dashed horizontal line labeled  $\sigma$  connects the two  $\mathfrak{b}$  lines. The bottom part has a vertical line on the left labeled  $\mathfrak{b}$ , a vertical line on the right labeled  $\mathfrak{r}$ , and a vertical line on the far right labeled  $\mathfrak{g}'$ . A dashed horizontal line labeled  $\delta$  connects the two  $\mathfrak{b}$  lines. A dashed horizontal line labeled  $\gamma$  connects the two  $\mathfrak{r}$  lines.

(Note that [Bre94] uses slightly more complicated phrasing, as [BM] points out in a footnote; this corresponds to introducing into the string diagram an extra zig-zag, which can be removed by (2.23), the first zig-zag identity). Thus, equivalent gerbes and equivalent 2-bundles correspond.

To make this correspondence even more precise, I should consider composition of 2-bundle morphisms as well as 2-bundle 2-morphisms. The analogues of these for gerbes are not discussed in [Bre94], but they are discussed by Breen & Messing in [BM], to which I now turn exclusively. I should note that Breen & Messing are less general in [BM] than Breen alone was in [Bre94], in that they assume that a single open cover may be chosen once and for all that obviates the need for the upper

Greek superscripts. (As they remark in [BM, 2.1], this assumption is valid for certain choices of  $\mathbf{C}$ , such as affine schemes or paracompact manifolds, but not for more general schemes or manifolds.) I will restore the missing indices. Also, they change the notation ‘ $\mu$ ’ to ‘ $m$ ’ and use superscripts on this instead of primes; I will continue to write ‘ $\mu$ ’ and to use primes to distinguish morphisms (although again they will be reverse from my primes, since composition in both [BM] and [Bre94] is in the opposite order from this paper).

This in mind, a 2-morphism between morphisms of gerbes is described locally by the following data:

- a common refinement of the open covers involved in the gerbe morphisms (suppressed in [BM]);
- a further refinement to allow  $\theta_i$  below to make sense (again suppressed in [BM], but analogous to the refinement for  $\mu$  last time); and
- sections  $\theta_i$  of  $G$  on each  $U_i$ ; such that
- $\mu'_i$  is equal to value of  $\theta_i$  in  $\Pi$ , multiplied by  $\mu_i$  ([BM, (2.3.7)]); and
- $(\delta'_{ij}\theta_i)$  is equal to the action of  $\lambda_{ij}^\alpha$  on  $\theta_j$ , multiplied by  $\delta_{ij}^\alpha$  ([BM, (2.3.8)]);

These correspond to a 2-transition 2-morphism as follows:

- The common refinement is unnecessary, since the same pullback  $\mathcal{U} \cap \mathcal{U}'$  applies to both morphisms.
- The other refinement again ensures that each new  $J_{ii}$  is a singleton.
- $\theta_i$  corresponds to  $\xi_{\mathbb{R}^n}$ .

- The value of  $\theta_i$  in  $\Pi$  shows that  $\xi_{\mathfrak{r}\mathfrak{r}}$  is an arrow from  $\mathfrak{b}_{\mathfrak{r}\mathfrak{r}}$  to  $\mathfrak{b}'_{\mathfrak{r}\mathfrak{r}}$ .
- One of the coherence laws (2.40, 2.41) may be taken to define  $\xi_{\mathfrak{r}\eta}$  in general; then the other gives the desired action of  $\lambda_{ij}^\alpha$  on  $\theta_j$ .

Here are the full string diagrams for the last item:

Both here and in [BM], composition of 2-morphisms is given by multiplication in  $G$ , so the correspondence between 2-morphisms is functorial. In [BM], whiskering a 2-morphism by a morphism on the left does nothing, while whiskering on the right applies  $\mu$  to  $\theta$ . This is precisely what should happen, since ‘left’ and ‘right’ are reversed, in the string diagram for a crossed module. Thus I have proved the following theorem:

**Theorem 3** *Given a space  $B$  and a strict 2-group  $\mathcal{G}$ , the 2-category  $\mathcal{B}^{\mathcal{G}}$  of  $\mathcal{G}$ -2-transitions over  $\mathcal{B}$  (or equivalently, the category of principal  $\mathcal{G}$ -2-bundles over  $\mathcal{B}$ ) is equivalent to the 2-category of  $\mathcal{G}$ -gerbes over  $B$  (where  $\mathcal{G}$  is identified with its corresponding crossed module).*

It's worth mentioning that the usual sort of nonabelian gerbe, where  $\mathcal{G}^1$  is a group of automorphisms, may not actually work in a given category  $\mathbf{C}$ . (See [HDA5, 8.1] for precise instructions on how to build the 'automorphism 2-group' of a group.) For example, if  $\mathbf{C}$  is the category of finite-dimensional manifolds, then the automorphism group of a Lie group has (in general) infinite dimensions, so it's not an object of  $\mathbf{C}$ . One can either form a space of only those automorphisms desired, passing to crossed modules, or else extend  $\mathbf{C}$  to include the necessary automorphism groups. This is just one reason why Baez & Schreiber [HGT] would replace the category of manifolds with a more general category of 'smooth spaces'.

Abelian gerbes, in contrast, are easy; they use a strict 2-group where  $\mathcal{G}^1$  is the trivial group. As remarked in section 2.2.4, this requires an abelian group.

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