C* algebras

A C* algebra is a space that has about all the structure of the complex numbers. The general rule is, if the complex numbers have a property that isn't specific to them, then that property is required of a C* algebra. For example, the complex numbers are a complex vector space, so a C* algebra must also be a complex vector space. Additionally, the complex numbers are a complex vector space *of dimension one*, but this property is too specific, so a C* algebra doesn't have to have that dimension. Here, precisely, are the properties required:

Suppose you have a complex vector space. Suppose this space also has a norm (a function $x \mapsto \|x\|$ into the positive real numbers, including zero). Suppose the norm is compatible with the vector space structure in the sense that it satisfies definiteness (only the zero vector has zero for a norm), the triangle inequality ($\|x+y\| \leq \|x\| + \|y\|$), and homogeneity ($\|cx\| = |c|\,\|x\|$). Then the complex vector space is a normed vector space.

Now suppose you have a normed vector space. Now, this normed vector space is a metric space, if the distance between x and y is $\|x-y\|$. So suppose this metric space is complete. Then the normed vector space is a Banach space.

Now suppose you have a Banach space. Suppose this Banach space has an associative binary operation (multiplication). Suppose multiplication is compatible with the vector space in the sense that it is bilinear, so that the vector space is an algebra. Suppose that multiplication is also compatible with the norm in the sense that $\|xy\| \leq \|x\|\,\|y\|$. Then the Banach space is a Banach algebra.

Now suppose you have a Banach algebra. Suppose this Banach algebra has an involution (a function $x \mapsto x^*$ which is its own inverse). Suppose the involution is compatible with the algebraic structure in the sense that $(x+y)^* = x^* + y^*$, $(cx)^* = \bar{c}x^*$, and $(xy)^* = y^*x^*$. Suppose the involution is also compatible with the norm in the sense that $\|x^*x\| = \|x\|^2$. Then the Banach algebra is a C* algebra.

There are some properties of the complex numbers that are missing. For example, multiplication of complex numbers is commutative, but multiplication of a C* algebra is not required to be commutative. This requirement is not too specific; there are several commutative C* algebras, and the study of them is an interesting field. But Irving Segal, who invented C* algebras in 1947, didn't think that was important enough to include in the definition. So that's an exception to the general rule I began this document with. There are other exceptions which you might be able to find.

You might also wonder about division. The complex numbers are a division algebra (and, since they're commutative, also a field). But that requirement *is* too specific; the only C* algebra which is also a division algebra is itself the complex field. The proof is an exercise for you. (Hint: all you need to know about the C* algebra is that it's a normed algebra: a complex algebra with a compatible norm.)

Notice that the space of bounded linear transformations of a Hilbert space is a C* algebra. In fact, any C* algebra is a subspace of the space of bounded linear transformations of some Hilbert space. Notice also that the space of continuous functions from a compact Hausdorff space to the complex field is a commutative C* algebra. In fact, any commutative C* algebra is the space of continuous functions from some compact Hausdorff space. These facts can be used to elegantly prove the spectral theorem.