COMBINATORICS OF POLYHEDRA FOR n-CATEGORIES

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In these notes, we give some combinatorial techniques for constructing various polyhedra which appear to be intimately connected with the geometry of weak n-categories. These include

1. Associahedra (and "monoidahedra"; see below);
2. Permutoassociahedra, and the cellular structure of Fulton-MacPherson compactifications of moduli spaces
3. "Functoriahedra" (related to the A_n maps of Stasheff)

Our basic techniques use derivations on operads and bar constructions. In part A, we introduce derivations, which should be regarded as "boundary operators" on (set-valued) operads satisfying a Leibniz rule. Such derivations can be used to construct poset-valued operads; taking nerves, one gets polyhedral operads. In this way we reconstruct the associahedra and the Fulton-MacPherson compactifications.

Part A. Derivations on operads

Sections A.1. and A.2. preface the main definitions which begin in section A.3: they establish notation and terminology and categorical formalizations of fairly trivial facts. The reader may wish to skip them or refer back to them as necessary.

A.1.

Initially, we work with set-valued non-permutative operads M, which may be viewed as models of a multisorted algebraic theory with operations $\mathsf{sub}_k : M_p \times M_q \to M_{p+q-1}$. (This is the appropriate presentation when working with *non-unital* operads.) The sorts in this case are parametrized by natural numbers p, so that we have for each operad M an underlying sequence of sets. Such a sequence may be regarded as a functor $N \to \mathsf{Sets}$ on the discrete category $\mathbb{N}$ of natural numbers.

When working with permutative operads, it is often convenient to work with functors on the category of finite sets and bijections, rather than with functors on the permutation category (which is a skeleton). (Logically it makes no difference, but some things are clearer and easier to describe in the former setting.) There are analogues of the sub_k operations, parametrized by pairs (S, E) where S is a finite set and E is a subset. Let S/E denote the (pointed) set in which all of the points of E are identified with a basepoint; if E is empty then S/E is the disjoint sum of S with a basepoint. Then there is a substitution map

$$\mathsf{sub}_{S,E} : M(S/E) \times M(E) \to M(S). \qquad (1)$$

Throughout these notes, we work in a setting which can be adapted to both non-permutative and permutative operads.

A.2.

The functor category $[\mathbb{N}, \mathsf{Sets}]$ is a Boolean topos. In this case, this means that the power object of a sequence is the sequence of power sets. We let $P : [\mathbb{N}, \mathsf{Sets}] \to [\mathbb{N}, \mathsf{Sets}]$ denote the

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covariant power object functor. Thus if $f : X \to Y$ is a morphism (= sequence of functions), then $Pf : PX \to PY$ sends a sequence of subsets S_n to the sequence of images $f_n(S_n)$.

The functor P carries a structure of monoidal monad. The monoidal structure map $PA \times PB \to P(A \times B)$ sends a pair of subsets to their product. The monad unit $u_X : X \to PX$ sends a sequence of elements to the corresponding sequence of singletons, and the multiplication $m_X : PPX \to PX$ sends a sequence of subset-collections to the corresponding sequence of unions. (It is easily checked that u and m are monoidal natural transformations, so we do get a monoidal monad.)

The algebras of P are sup-lattice objects, *i.e.*, sequences of sup-lattices. Using the monoidal monad structure on P, one can produce a closed symmetric monoidal structure on $P\text{-}\mathsf{Alg}$. There is an underlying functor from $P\text{-}\mathsf{Alg}$ into the category of (idempotent) commutative monoids in $[\mathbb{N}, \mathsf{Sets}]$.

The morphisms $A \to B$ in the Kleisli category of P are maps $A \to PB$ in $[\mathbb{N}, \mathsf{Sets}]$, i.e., relations between A and B. Thus a binary relation on A is a Kleisli morphism $A \to A$, and the hom-object $\mathsf{Kl}(A, A)$ of binary relations is a monoid $P\text{-}\mathsf{Alg}(PA, PA)$ in the category of sup-lattices. The reflexive transitive closure of a binary relation R on A may be defined as the geometric series $1 + R + R^2 + \ldots$ in this sup-lattice monoid. Of course, this is an incarnation of the free monoid construction, although in this case, $R^* = 1 + R + R^2 + \ldots$ gives an idempotent monad T. A poset may be defined as an object A together with a T-algebra in $\mathsf{Kl}(A, A)$. Here it just means a sequence of ordinary posets.

All of this section carries over to species, braided species, and more generally permutation representations of arbitrary groupoids, starting from the observation that each of these gives a Boolean topos. In fact, the Boolean assumption isn't even needed.

A.3.

On $[\mathbb{N}, \mathsf{Sets}]$, we have a substitution tensor product whose monoids are non-permutative operads. If X is a sequence of sets, we let OX denote the sequence underlying the free operad on X.

Using the monoidal structure on the covariant power set functor P, each operad M induces an operad structure on PM. Thus we have for example an operad POX. It is very easy to see that the elementhood relation $e_M \rightarrowtail M \times PM$ defines a suboperad.

Definition A.3.1. If A is a set-valued operad, and B an operad valued in commutative monoids, then a derivation $d : A \to B$ is a map in $[\mathbb{N}, \mathsf{Sets}]$ such that $d(\mathsf{sub}_k(a, b)) = \mathsf{sub}_k(da, b) + \mathsf{sub}_k(a, db)$.

For the remainder of these notes, we take B to be of the form PM, and the main examples of derivations take the form $M \to PM$. Such derivations extend uniquely to derivations $PM \to PM$ which preserve arbitrary sups.

Given a derivation $d : M \to PM$, it is suggestive to regard the elements m of M as "cells", and $d(m)$ as the collection of face-cells lying on the boundary of m. The idea is to use (M, d) as combinatorial data to describe a polyhedral operad. (It seems that Clemens Berger has a notion of "cellular operad", but I don't know his definition, or how it fits with the notions given here.)

Many examples occurring in nature are derivations of the form $OX \to POX$. Notice that derivations of this form correspond bijectively to maps $X \to POX$ (much as linear maps $V \to A$ from a vector space V into an algebra A correspond bijectively to derivations $TV \to A$ on the tensor algebra).

Example 1. Let X be empty in degrees 0 and 1, and terminal in degrees 2 and higher. Then the elements of OX correspond bijectively to (combinatorial) planar rooted trees, in which every vertex adjacent to fewer than two incoming edges must be a leaf. For $n > 1$, the unit map $X \to OX$ sends the element in degree n to the "n-sprout" (no nodes except leaves and root). Let $[n]$ denote the n-sprout. Define $X \to POX$ by sending the element in X_n to the set of all trees in $(OX)_n$ of the form $\mathsf{sub}_k([p], [q])$. Extend this uniquely to a derivation $d : OX \to POX$.

This example is connected with Stasheff's associahedra, as we explain in a moment. First, some generalities.

Each derivation $d : M \to PM$ may be regarded as a binary relation on M. Consider its reflexive transitive closure d^*, *i.e.*, the poset generated by this binary relation, viewed as a map $M \to PM$, or as a sup-lattice map $PM \to PM$.

Proposition A.3.1. *$d^* : PM \to PM$ is an operad map.*

Proof. We must show that $d^*(\mathsf{sub}_k(x, y)) = \mathsf{sub}_k(d^*(x), d^*(y))$. Being sup-lattice morphisms, the sub_k operations distribute over arbitrary sums. Then we calculate, as with ordinary derivations (except that we may drop binomial coefficients, on account of the idempotency of $+$ in PM),

$$d^n(\mathsf{sub}_k(x, y)) = \sum_{p+q=n} \mathsf{sub}_k(d^p(x), d^q(y)) \quad (2)$$

and since $d^* = 1 + d + d^2 + \dots$, the desired equation follows easily. □

Remark. There is a well-known theorem that if A is a Banach algebra and if $d : A \to A$ is a bounded linear derivation, then $\exp(d) : A \to A$ is a Banach algebra map. The proposition is analogous: algebra multiplication is replaced by operad substitution maps, the completeness property of the Banach algebra by the sup-lattice property of PM, and $\exp(d)$ is replaced by the geometric series $d^* = 1 + d + d^2 + \dots$

Since P is a monoidal monad, the unit $u_M : M \to PM$ is an operad morphism, and we may compose this with $d^* : PM \to PM$ to get an operad map $M \to PM$, again denoted by $d^* : M \to PM$. Now, if we pull back (take the inverse image of) the elementhood relation $e_M \to M \times PM$ along the map $1 \times d^* : M \times M \to M \times PM$, we get a suboperad

$$R \to M \times M, \quad (3)$$

since this is obtained as a pullback in the category of operads. In other words, for each derivation $d : M \to PM$, we get a poset (M, R) in the category of operads, which is the same as an operad in the category of posets.

Definition A.3.2. Given a derivation $d : M \to PM$, we denote by M_d the posetal operad whose characteristic map is $d^* : M \to PM$.

Example 1 (continued)**.** Given planar trees t, t', let us write $t' \to t$ if t' belongs to $d^*(t)$. Thus $\to$ is a poset relation. From before, we have that t' is contained in $d([n])$, where $[n]$ is the n-sprout, if $t' = \mathsf{sub}_k([p], [q])$, *i.e.*, if t' has exactly one internal edge, and $[n]$ is obtained from t' by contracting that internal edge. In general, $t' \to t$ if t is obtained by contracting a finite number of internal edges of t'. Since contraction of all internal edges results in an n-sprout, we see that the n-th component of the posetal operad $(OX)_d$ has $[n]$ as terminal object, so that its nerve is contractible.

With a little work, one can show that the realization of the nerve of each component $(OX)_d(n)$ is homeomorphic to an n-disk. Indeed, the nerve gives a natural triangulation of Stasheff's associahedron in dimension n.

The remainder of part A is devoted to the construction of a polyhedral operad which corresponds to a natural triangulation of the Fulton-MacPherson compactification of moduli spaces (for R^k). In this case, we need to work with permutative operads. We begin by defining a cell structure for the moduli space.

For finite sets S of cardinality greater than 1, we let $F_k[S]$ denote the spatial species of injective functions $S \to R^k$. For $S = \{1, \dots, n\}$, this is usually denoted $F_k[n]$: the space of configurations of n points in R^k. We partition $F_k[n]$ into convex cells by using a kind of lexicographic order (in which the j-th coordinate takes priority over the i-th coordinate if $j > i$ [not $j < i$]). For example, $(x, y) < (x', y')$ if $y < y'$ or $y = y'$ and $x < x'$.

The points $p_1, \dots, p_n$ of a configuration may be rearranged in ascending lexicographic order: $p_{f_1} < \dots < p_{f_n}$ for some permutation f. (Note: if the species F_k is defined on finite sets S, then we get an induced linear order on S, not a permutation.) Let us define the type of the configuration to be a tuple $(f; k_1, \ldots, k_{n-1})$ where f is the permutation and k_j is the index of the last coordinate where p_{fj} and $p_{f(j+1)}$ disagree. One often uses a "bar notation" for denoting such types, by placing k_j bars between $f(j)$ and $f(j+1)$. For example, working with configurations in the plane, $2||3|1$ is the type of a configuration of three points where points 1 and 3 lie on the same horizontal line lying above point 2, and point 3 is to the left of point 1. It is immediate that the region of configurations of common type form a convex cell. Letting $T(k, n)$ denote the set of types, we have a typing function $F_k[n] \to T(k, n)$ whose convex fibers partition the configuration space.

Let $M_k[n]$ ("moduli space") denote the orbit space of $F_k[n]$ under the action of translations and positive dilatations on R^k. Observe that the typing function factors through $M_k[n]$, so that we have an induced typing function $M_k[n] \to T(k, n)$.

Keeping k fixed, we let $T_n = T(k, n)$ for $n > 1$, and take T_0 and T_1 to be empty: this gives a species T. We will define a boundary $d : T \to POT$ whose induced derivation $d : OT \to POT$ provides data for a posetal operad, whose nerve components naturally triangularize the Fulton-MacPherson (or really "Axelrod-Singer" for the real case) compactifications. First, a few preliminaries.

First, it is convenient to view a type in terms of a collection of mutually disjoint transitive relations $|_1, |_2, \ldots, |_k$ on the set of indices $\{ 1, \ldots, n \}$, whose join (in the poset of transitive relations) is a linear order $f(1) < \cdots < f(n)$. Namely, define these $|_j$ to be the minimal relations such that

(1) $f(i)|_j f(i+1)$ if there are j bars between $f(i)$ and $f(i+1)$;
(2) $p|_i q$ and $q|_j r$ implies $p|_{\max(i,j)} r$.

(The reader should consider what this means geometrically.) Instances of $p|_i q$ shall be called "consequences" of the type. The maximum i where $p|_i q$ occurs is called the "degree" of the type.

By (2), and the fact that the transitive join of all the relations $|_i$ is a linear order on $\{ 1, \ldots, n \}$, we see that the transitive join of $|_i$ for $i = 1$ up to j must be a disjoint sum of linear orders. Each of these orders will be called a j-connected component, or j-component for short. We may also speak of the j-component of a given element in $\{ 1, \ldots, n \}$.

In what follows, we will need certain kinds of "shufflings". Given two disjoint j-components A and B in a type, a j-shuffle of them is a shuffle on the underlying linear orders, in such a way that a in A is interposed between b and b' in B only when $b|_j b'$, and b in B is interposed between a and a' in A only when $a|_j a'$. Together with all the $|_i$ relations on A and on B, one adjoins new relations $b|_j a$, $a|_j b'$ in the former case, and $a|_j b$, $b|_j a'$ in the latter. As a special case, we have the notion of "intercalating" one j-component within another, where one places the entire component B between two successive elements a, c of A. Or, if $a < b < c$ are successive elements in A, we get the same thing by replacing b by B, and we speak then of intercalating B at b.

Now let us give rules for describing the boundary operator $d : T \to POT$.

Rule (1) If t and t' are two types, let us say t' is in $d(t)$ if t' can be obtained from t as follows: if t has the linear order $f(1) < \cdots < f(n)$, then there exists i such that $f(i)|_j f(i+1)$ in t, and t' is obtained by removing this pair and the set of consequences derived using this pair, followed by $(j-1)$-shuffling the $(j-1)$-components of $f(i)$ and $f(i+1)$, and deriving a new set of consequences from the relations which result.

Rule (2) The remaining elements in $d(t)$ are of the form $\mathsf{sub}_E(t', t'')$, where E is a subset of $S = \{ 1, \ldots, n \}$, t belongs to $X[S] = X_n$, t' to $X[S/E]$, and t'' to $X[E]$ (see section A.1. for the sub_E operations). Such an element belongs to $d(t)$ if the following conditions hold. Let b be the basepoint in S/E, and suppose t' has degree d. Then the type t'' on E should be the same as that obtained by restriction of t-consequences on S to the subset E, and there should exist a list of sets $C_1, \ldots, C_d$, nested in ascending order, where each C_j is a j-component in t'' which, when intercalated at b in the j-component of b in t', yields only correct consequences of t.

The ideas which govern this description are as follows. First, each type t represents a convex cell in moduli space, and the boundary of t (in the moduli space) should be the set of types whose cells are codimension 0 parts of the boundary of the cell for t. If p represents a configuration moving through the closure of the t-cell, with the points of p ordered lexicographically, then p undergoes a type transition when the relation between two points $p(i)$ and $p(i+1)$ changes at the j-th coordinate, from $p(i)_j < p(i+1)_j$ to $p(i)_j = p(i+1)_j$. (We are assuming $p(i)$ and $p(i+1)$ agree in coordinates with indices greater than j.) After the transition, the largest index where $p(i)$ and $p(i+1)$ disagree is at most $j-1$. And in order for p after the transition to be interior to the codimension 0 stratum of the boundary, that largest index must be equal to $j-1$. Now we take into account the possible ways in which the $(j-1)$-th coordinates differ, while still maintaining the order of points in p, and we are led to the first part of the description of the boundary operator.

As for the second part, the idea is that we are peering into a first-order infinitesimal neighborhood of a singular configuration, blowing it up, and examining what types of moving non-singular configurations can approach the inverse image or blow-up of the singular configuration. Now a singular configuration, as a function from a finite set S into R^k, factors through a quotient $S/\sim$ which maps injectively into R^k; in order to get the highest dimensional stratum (of the boundary of a t-cell in the Fulton-MacPherson compactification), we restrict to equivalence relations on S which merely squash a single subset E down to a point. Then we have a non-singular configuration on S/E, so that the singularity is concentrated on E; after blowing up, the points of E are then distinguished according to tangent directions, thus giving non-singular configurations on E in the blow-up. The resulting pairing of a type on S/E with a type on E may be viewed as an instance of a formal sub_E operation in the operad freely generated from the species of types, and the description of how boundaries of types intersect these applications of sub_E merely reflect properties of configuration-types as some of their points become infinitely close.

As this example of a polyhedral operad is somewhat complicated, it may be well to calculate the explicit structure of a few cells. In what follows, $*$ denotes the basepoint E/E in a quotient S/E of S. Hence $\mathsf{sub}(*||3, 1|2)$ indicates the expression obtained by substituting $1|2$ for $*$ in $*||3$. Substituted expressions are indicated using a bracket notation, *e.g.*, $[1|2]||3$.

Example 2. The boundary of $1||2$ is $\{\, 1|2, 2|1 \,\}$, by rule (1), and $1|2$, $2|1$ each have empty boundaries. So the picture is

$$1|2 \overset{1||2}{\rule[0.5ex]{4em}{0.4pt}} 1|2 \tag{4}$$

and one should think of a braiding $c_{x,y} : xy \to yx$.

Example 3. The boundary of $1||[2|3] = \mathsf{sub}(1||*, 2|3)$ is calculated by the Leibniz rule. We get $d(1||[2|3]) = \mathsf{sub}(d(1||*), 2|3) + \mathsf{sub}(1||*, d(2|3))$ where the second summand is 0 since $2|3$ has empty boundary. From the preceding example, the first summand is $\mathsf{sub}(1|*, 2|3) + \mathsf{sub}(*|1, 2|3) = \{\, 1|[2|3], [2|3]|1 \,\}$ so the picture becomes

$$1|[2|3] \overset{1||[2|3]}{\rule[0.5ex]{4em}{0.4pt}} [2|3]|1 \tag{5}$$

which is reminiscent of $c_{x,yz} : x(yz) \to (yz)x$.

Example 4. To calculate the boundary of $1|2|3$, we use rule (2). The only consequences are $1|2$, $2|3$, and $1|3$; and the only ones which occur as intercalated relations are $1|2$ and $2|3$. Hence the boundary is $\{\, [1|2]|3, 1|[2|3] \,\}$. The picture is

$$[1|2]|3 \overset{1|2|3}{\rule[0.5ex]{4em}{0.4pt}} 1|[2|3] \tag{6}$$

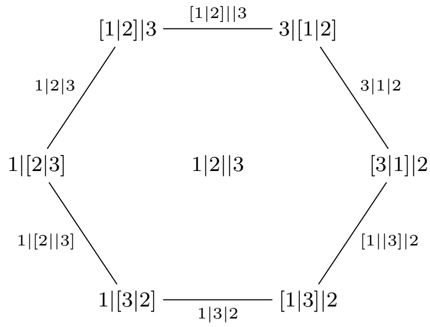
reminiscent of an associativity $a_{xyz} : (xy)z \to x(yz)$.

Example 5. It follows easily from Leibniz that $d(1|[2||3]) = \{\, 1|[2|3], 1|[3|2] \,\}$.

Example 6. For $1|2||3$, we first apply rule (1), where we remove the relation $2||3$. This leaves only the consequence $1|2$, so we perform all possible 1-shufflings of 3 with $1|2$. This produces $1|2|3$, $1|3|2$, $3|1|2$ as part of the boundary.

We can also apply rule (2), which is trickier. Let us start by writing down the complete set of consequences of $1|2||3$: they are $1|2$, $2||3$, $1||3$. Now, types involve two or more entries, so the substituted expression must be one of these consequences. Now $[1|2]||3$ is clearly a boundary cell: we have $C_1 = C_2 = 1|2$, and intercalating $1|2$ for $*$ in $*||3$ gives the original type expression; therefore this is OK. $1|[2||3]$ is also a boundary cell, which we verify by taking $C_1 = 2$ as the 1-component: intercalating 2 in $1|*$ gives the correct consequence $1|2$. Finally, $[1||3]|2$ is a boundary: by taking $C_1 = 1$, intercalation in $*|2$ gives a correct consequence. It is easy to check that no other boundary cells are possible.

Of course, these boundary cells also have non-trivial boundaries, which are calculated essentially as in our prior examples. The reader might like to verify that the resulting poset has a nerve where the star of $1|2||3$ looks like



(7)

which of course is an instance of a braiding hexagon.

Example 7. For $[1||2]||3$, the boundary is calculated using Leibniz: it is easy to check that this yields $\{ [1|2]||3, [2|1]||3, [1||2]|3, 3|[1||2] \}$. These cells also have non-trivial boundaries, and the resulting picture looks like an instance of naturality for the braiding.

Example 8. Here is our granddaddy example. We calculate the structure of $1||2||3$. Using rule (1), we can remove $1||2$ and 1-shuffle the 1-components of 1 and 2 (which are just 1 and 2 again), or remove $2||3$ and 1-shuffle the 1-components of 2 and 3 (which are 2 and 3 again). This yields the cells $1|2||3$, $2|1||3$, $1||2|3$, $1||3|2$.

Using rule (2), we first write the consequences $1||2$, $2||3$, $1||3$. Now $1||3$ is impossible as a substituted expression, since it equals its own 2-component, and intercalating it in $2||*$ or $*||2$ leads to a consequences false in $1||2||3$. The only correct possibilities are $[1||2]||3$ and $1||[2||3]$.

We have already done the essential calculations for obtaining the structure of these boundary cells. To save space, we will not exhibit the full-blown 3-cell structure of $1||2||3$: it turns out to be a 3-cell which mediates between the two ways of giving a commutative diagram proof of the Yang-Baxter identity in a braided monoidal category. In other words, it is a structure cell which obtains in a braided monoidal 2-category.

Part B. Bar constructions

In this section, we initially work over a base topos Sets; all of our constructions apply in a context where we work over a Grothendieck topos $\mathscr{E}$ as base. Later in this section, we take $\mathscr{E}$ to be the topos of Joyal species.

Classically, bar constructions have been used to build classifying bundles, free resolutions for group cohomology, and similar constructs. A common characteristic between these constructions is the production of an acyclic algebraic structure (*e.g.*, a contractible G-space EG, or a free resolution of a G-module). We begin by formalizing this characteristic in terms of a universal property for bar constructions.

B.4. Acyclic structures

We follow the algebraists' convention, taking the simplicial category to mean the category of finite ordinals (including the empty ordinal) and order-preserving maps. It is well-known that Δ is initial amongst strict monoidal categories equipped with a monoid, which in Δ is 1. This induces a monad $1 + -$ on Δ, called the *translation monad*. By composition, this in turn induces a pullback comonad P on simplicial sets; by the Kan construction, it has a left adjoint which is a monad C on simplicial sets, called the *cone monad*.

To explain this terminology, we recall the topologists' convention, where Δ is restricted to the full subcategory Δ_+ of non-empty ordinals. If S-Sets denotes the category of simplicial sets under the algebraists' convention, and S_+-Sets that under the topologists' convention, then by restriction we get a functor $S\text{-}\mathsf{Sets} \to S_+\text{-}\mathsf{Sets}$, which has a left adjoint. The left adjoint augments a S_+ set X by its set of path components $\pi_0(X)$. Starting with X, we can apply the left augmentation, followed by the cone monad, followed by restriction $S\text{-}\mathsf{Sets} \to S_+\text{-}\mathsf{Sets}$: this gives a monad which, passing to geometric realization, is the mapping cone of $X \to \pi_0(X)$. The right adjoint to this monad carries a comonad structure; the category of algebras over the monad is equivalent to the category of coalgebras over the comonad. This category of algebras/coalgebras could be called the "*acyclic topos*": the algebras X are acyclic as simplicial sets. More to the point, the algebra structure $CX \to X$ witnesses this acyclicity by providing a representative basepoint for each path component of X, together with a well-behaved simplicial homotopy which contracts each component down to its basepoint.

Definition B.4.1. An *acyclic structure* is an algebra over C (or coalgebra over P). An *S-acyclic structure* is one augmented over a set S.

A morphism between acyclic structures is just a C-algebra map; a morphism between S-acyclic structures is one whose component at S is the identity.

It doesn't matter whether the monad C is taken under the algebraists' or topologists' convention: the category of C-algebras in S-Sets is equivalent to the category of C-algebras in S_+-Sets, since given a C-algebra in S-Sets,

$$\cdots \; X_1 \rightrightarrows X_0 \longrightarrow X_{-1} \; , \tag{8}$$

it follows from acyclicity that this portion of the simplicial structure is a split coequalizer, so that the augmentation map is the usual augmentation to its set of path components.

In the sequel, it will be useful to regard an acyclic structure, as a functor $X : \Delta^{op} \to \mathsf{Sets}$, as a right coalgebra $X\,T \to X$ over the translation comonad $T : \Delta^{op} \to \Delta^{op}$.

B.5. Abstract bar constructions

Next, let us recall the formalism which leads to bar constructions. Let $U : A \to E$ be a monadic functor over a category E (in most of our applications, E will be a topos). Let $F : E \to A$ be the left adjoint, so that we have a monad $M : E \to E$ and a comonad $C : A \to A$. The comonad may be regarded as a comonoid in the endofunctor category $[A, A]$ (under the monoidal product given by endofunctor composition). Since Δ^{op} is initial amongst strict monoidal categories equipped with a comonoid, there exists a unique monoidal functor

$$\Delta^{op} \to [A, A] \tag{9}$$

sending the comonoid 1 to the comonoid C. Now if X is an object of A (*i.e.*, an M-algebra), we have an evaluation functor $ev_X : [A, A] \to A$. Thus we have the following composition which leads to a simplicial E-object:

$$\Delta^{op} \longrightarrow [A, A] \xrightarrow{ev_X} A \xrightarrow{U} E \tag{10}$$

and this defines what we mean by the bar construction, denoted $B(M, M, X)$. More explicitly, this is an augmented simplicial object of the form

$$\ldots \; MMX \underset{M\,\xi}{\overset{\mu_X}{\rightrightarrows}} MX \xrightarrow{\xi} X\ , \tag{11}$$

and this simplicial object admits an acyclic structure where the contracting homotopy is built out of components of the unit u of the monad M:

$$M^n\,X \xrightarrow{u_{M^n X}} M^{n+1}\,X\ . \tag{12}$$

(The "co-associativity" axiom for the right T-coalgebra structure holds by naturality of u.)

Definition B.5.1. An X-acyclic M-algebra is a simplicial M-algebra whose underlying simplicial E-object admits an X-acyclic structure.

Notice that we do not require any compatibility between the M-algebra structure and the acyclic structure: the acyclic structure map is not required, for example, to be a map of simplicial M-algebras.

A morphism of X-acyclic M-algebras is just a map of simplicial M-algebras whose underlying simplicial E-map is a morphism of X-acyclic structures. The category of X-acyclic M-algebras may thus be rendered as a (2-)pullback of

$$\begin{array}{ccc} & & X-\text{Acyclic}(E) \\ & & \Big\downarrow{\scriptstyle \text{"forget"}} \\ [\Delta^{op}, A] & \xrightarrow[{[1,U]}]{} & [\Delta^{op}, E] \end{array} \tag{13}$$

where the vertical arrow is the obvious forgetful functor from the category of X-acyclic structures in E.

The universal property of the bar construction is enunciated in the following

Theorem B.5.1. *$B(M, M, X)$ is initial in the category of X-acyclic M-algebras.*

Proof. The method of proof closely parallels that of the acyclic models theorem familiar from homological algebra. Let Y be an acyclic M-algebra augmented over X:

$$\ldots \; Y_1 \rightrightarrows Y_0 \longrightarrow X\ , \tag{14}$$

so that the T-coalgebra structure or contracting homotopy has the form $X \to Y_0 \to Y_1 \to \ldots$ In order to get the desired simplicial M-algebra map $B(M, M, X) \to Y$ which preserves the contracting homotopies, we are forced to use the diagram

$$\begin{array}{ccccccccc} X & \xrightarrow{u_X} & MX & \xrightarrow{u_{MX}} & MMX & \xrightarrow{u_{MMX}} & MMMX & \cdots\cdots\to & \ldots \\ \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_2} & & \Big\downarrow{\scriptstyle \phi_3} & & \\ X & \xrightarrow[h_0]{} & Y_0 & \xrightarrow[h_1]{} & Y_1 & \xrightarrow[h_2]{} & Y_2 & \cdots\cdots\to & \ldots \end{array} \tag{15}$$

where ϕ_{n+1} is defined as the unique M-algebra map which extends $(h_n)(\phi_n)$ along $u_{M^n X}$. Thus uniqueness is clear; what remains is to check that the ϕ_n form the components of a simplicial map ϕ. The proof of this is sketched in the lemma and corollary which follow. ☐

Let $\mathsf{Simp}(E)$ denote the category of simplicial objects in E. The pullback comonad P may be regarded as a comonoid in the endofunctor category $[\mathsf{Simp}(E), \mathsf{Simp}(E)]$, so that there is an induced monoidal functor

$$\Delta^{op} \to [\mathsf{Simp}(E), \mathsf{Simp}(E)] \tag{16}$$

which we may compose with evaluation at a simplicial object Y. If moreover Y carries an acyclic structure, then there is an induced acyclic structure on this composite, regarded as a simplicial object in $\mathsf{Simp}(E)$:

$$\Delta^{op} \longrightarrow [\mathsf{Simp}(E), \mathsf{Simp}(E)] \xrightarrow{ev_Y} \mathsf{Simp}(E) \tag{17}$$

We denote this bisimplicial object by $B(Y, T, T)$, where T is the aforesaid translation comonad. Since $PY = YT$, it has the form

$$\ldots \; YTT \rightrightarrows YT \longrightarrow YTT\,. \tag{18}$$

Observe that applying evaluation $\mathsf{Simp}(E) \to E$ at the augmented component (the component we had earlier indexed by the numeral -1), this yields Y again:

$$\ldots \; Y_1 \rightrightarrows Y_0 \longrightarrow Y_{-1} = X\,. \tag{19}$$

Lemma B.5.2. *Let Y be an acyclic M-algebra. Let $B(M, M, Y)$ be the bisimplicial object formed as the bar construction on Y. Then there exists a map of Y-acyclic M-algebras (valued in $\mathsf{Simp}(E)$)*

$$B(M, M, Y) \to B(Y, T, T) \tag{20}$$

whose value at the terminal object 0 in Δ^{op} is the identity map on Y.

Thinking of $B(M, M, Y) \to B(T, T, Y)$ as a transformation between functors of the form $\Delta^{op} \to \mathsf{Simp}(E)$, we may post-compose by evaluation $\mathsf{Simp}(E) \to E$ at the augmented component. This yields precisely

$$B(M, M, X) \to Y \tag{21}$$

whence follows a corollary which completes the proof of the theorem:

Corollary 1. *There exists an X-acyclic M-algebra map $B(M, M, X) \to Y$.*

Proof of lemma. We construct $B(M, M, Y) \to B(Y, T, T)$ inductively, beginning with the identity on Y. The next component is of the form $\theta_1 : MY \to YT$, namely the composite

$$MY \xrightarrow{Mh} MYT \xrightarrow{\xi T} YT \tag{22}$$

where h is the right T-coalgebra structure and ξ the M-algebra structure. It is immediate that θ_1 preserves the homotopy component (*i.e.*, $h = \theta_1(u_Y)$), and since θ_1 is an M-algebra map, it quickly follows that $(Y\,\mathsf{ep})\,\theta_1 = \xi$, where ep is the counit of the comonad T.

We leave to the reader to check that if we inductively define $\theta_n : M^nY \to YT^n$ as the composite

$$M^nY \xrightarrow{M^1\theta_{n-1}} MYT^{n-1} \xrightarrow{\theta_1 T^{n-1}} YT^n \tag{23}$$

then it easily follows that $(\theta_n)(u_{M^{n-1}Y}) = (hT^{n-1})(\theta_{n-1})$, *i.e.*, θ preserves the homotopies (preserves acyclic structure). The fact that θ_n preserves face and degeneracy maps follows by induction: since θ_n is an M-algebra map by construction, it suffices to check that the relevant diagrams which obtain by precomposing with a unit u commute, but since θ preserves homotopies, one can exploit the naturality of the homotopies to convert the diagrams into ones where the inductive assumption applies. In short, the argument is similar to the usual one for the acyclic models theorem, and this completes our sketch of the proof. □

B.6. Applications

A classical application of bar constructions is the Milgram bar construction of a classifying bundle (say of a discrete group G). As is well known, the total space EG is a contractible space on which G acts freely. What appears less well known is the following theorem.

Let us define a *contractive space* to be a space (in a suitable topological category, such as the category of compactly generated Hausdorff spaces) which is an algebra over the cone monad. Here, the cone monad means the mapping cone of the map $X \to 1$ into the one-point space, and this is the monad whose algebras are pointed spaces equipped with a continuous action by the unit interval I, the monoid whose multiplication is "inf", such that multiplication by 0 sends every point to the basepoint. An algebra structure may be viewed as a well-behaved homotopy which contracts the space to a point.

Theorem B.6.1. *EG is initial amongst G-spaces whose underlying space is equipped with a contractive structure.*

Proof. Let $R : S\text{-}\mathsf{Sets} \to \mathsf{Top}$ be geometric realization. EG is formed as $RB(G, G, 1)$, where the bar construction is applied to the monad $G \times -$ on Sets. This is a 1-acyclic space, *i.e.*, a contractive space. If X is any other contractive G-space, we wish to demonstrate that there is exactly one contractive G-map $EG \to X$.

If $S : \mathsf{Top} \to S\text{-}\mathsf{Sets}$ is the singularization functor right adjoint to R, then a map $EG \to X$ gives rise to $B(G, G, 1) \to SX$. Since $G \times -$ as a functor $\mathsf{Top} \to \mathsf{Top}$ is cocontinuous, it is easy to see that a G-map $EG \to X$ gives rise to a G-map $B(G, G, 1) \to S\,X$. Next, let C be the cone monad acting on the category of pointed spaces; then C is also cocontinuous (it has a right adjoint given by the path space functor), and it follows as before that a C-map $EG \to X$ gives rise to a C-map $B(G, G, 1) \to SX$. Indeed, contractive G-maps $EG \to X$ are in bijective correspondence with 1-acyclic G-maps $B(G, G, 1) \to SX$ in S-set, and there is exactly one of these by the theorem of the last section. The proof is complete. □

Now let E be the topos of Joyal species, $[P, \mathsf{Sets}]$, where P denotes the permutation category. If Op denotes the category of permutative operads, then the underlying functor $Op \to [P, \mathsf{Sets}]$ is monadic. Let O denote the monad for this adjunction. If M is an O-algebra (an operad), then there is an associated bar construction $B(O, O, M)$. Needless to say, it is acyclic.

Example 9 (Associahedra revisited)**.** Let t_+ be empty in degree 0, and terminal in higher degrees. There is a unique operad structure on t_+, as a suboperad of the terminal operad t.

Form the bar construction $B(O, O, t_+)$: this carries a simplicial O-algebra structure, i.e., a simplicial operad structure. If we form the operad quotient in which every unary operation is identified with the identity operation, then the operad which results is a permutative version of the associahedral operad given in part A.

Example 10. Now let t be the terminal operad, and form the simplicial operad $B(O, O, t)$. Again take the operad quotient in which the operation in t of degree 1 is identified with the operad identity. The result is a simplicial operad called the "*monoidahedral operad*", denoted M.

The principle is that we have included a generating operation in degree 0, so that we obtain not just higher associativity laws as in the associahedral operad, but higher unit laws as well. In contrast to the associahedral operad, the nerve components here are infinite-dimensional (again, due to the presence of a nullary operation).

This last example has interesting applications to Trimble's work on Grothendieck's fundamental n-groupoids. Working in the category of bipointed spaces and bipointed maps, let I denote the unit interval, bipointed by the pair $(0, 1)$. On the category of bipointed spaces, there is a monoidal product given by "wedges" $X \vee Y$, where the second point of X is identified with the first point of Y, and the wedge is bipointed by the first point of X followed by the second point of Y. Then the n-fold wedge of I is canonically identified with the interval $[0, n]$, bipointed by the pair $(0, n)$.

Now suppose we adapt the monoidahedral operad to the non-permutative setting. Let $\mathsf{Bip}(I, I^n)$ denote the space of bipointed maps from I to its n-fold wedge; this is the n-th component of a tautological non-permutative operad structure. Notice that these spaces are convex, so that if we take as basepoint in $\mathsf{Bip}(I, I^n)$ the map $I \to I^n$ given by multiplication by n, then by convexity there is an induced contractive structure on each of these spaces. Thus we get a contractive spatial operad $\mathsf{Bip}(I, I^*)$.

Then there is a 1-acyclic simplicial operad $S(\mathsf{Bip}(I, I^*))$, obtained by applying singularization to the aforesaid spatial operad. It follows that there is a unique 1-acyclic operad map $M \to$

$S(\mathsf{Bip}(I, I^*))$. This map is used to identify higher associativity and higher unit laws present in $\mathsf{Bip}(I, I^*)$.

Before giving the next few examples (which are relevant to weak n-functors and their geometry), we need a new definition. If X and Y are species, we use $X \cdot Y$ to denote their substitution product.

Definition B.6.1. Let A and B be operads. An $A - B$ bimodule is a species X equipped with structure maps $A \cdot X \to X$ and $X \cdot B \to X$, compatible in the usual sense.

It is tempting to try to define bimodule composition, where if X is an $A - B$ bimodule and Y is a $B - C$ bimodule, then the tensor product XY is an $A - C$ bimodule given by an evident coequalizer of the form

$$X \cdot B \cdot Y \rightrightarrows X \cdot Y \twoheadrightarrow XY \tag{24}$$

The trouble is that bimodule composition fails to be associative, because while $- \cdot Y$ preserves colimits, $X \cdot -$ does not. However, in practice many coequalizers of the type shown above split, and we can refer to triple products XYZ without essential ambiguity if the coequalizers ending with XY and YZ split.

If M is an operad, then the free M-bimodule monad F is given by the assignment $X \mapsto M \cdot X \cdot M$, and if X is itself a bimodule, we obtain a bar construction $B(F, F, X)$. Some examples follow.

Example 11. Let t_+ be the permutative version of the operad of example 1, viewed as a bimodule over itself. Then $B(F, F, t_+)$ is a contractive t-bimodule whose components are triangularized *cubes.*

Let us calculate this in detail. In the language of Joyal species and their analytic functors, we have the linear fractional transformation

$$t_+(X) = \frac{X}{1 - X}. \tag{25}$$

Let t_+^n denote the n-fold substitution power of t_+. Then the bar construction

$$\cdots \Rrightarrow t_+^5 \rightrightarrows t_+^3 \longrightarrow t_+\ , \tag{26}$$

has the form

$$\cdots \Rrightarrow \frac{X}{1 - 5X} \rightrightarrows \frac{X}{1 - 3X} \longrightarrow \frac{X}{1 - X}\ , \tag{27}$$

Looking at coefficients, a structure of species $X/(1 - kX)$ on n points is given by a "combing" or linear order on $\{1, \ldots, n\}$ together with a function $\{1, \ldots, n\} \to \{1, \ldots, k\}$. We abbreviate the set of such structures by k^n. Keeping the object n of the permutation category P fixed, we get a simplicial object

$$(\ 5^n \rightrightarrows 3^n \longrightarrow 1\ ) = (\ 5 \rightrightarrows 3 \longrightarrow 1\ )^n \tag{28}$$

where the latter power denotes n-fold cartesian product in S-set.

Passing now to the topologists' convention (*i.e.* truncating the augmented object 1), the claim is that

$$\cdots \Rrightarrow 7 \Rrightarrow 5 \rightrightarrows 3 \tag{29}$$

is the nerve of a once-subdivided 1-cube : $\cdot - \cdot - \cdot$.

Certainly there are 3 0-cells, and 5 minus 3 non-degenerate 1-cells. If n_k denotes the number of non-degenerate k-cells, then we have

$$1 \cdot (n_0) = 3 \tag{30}$$

$$1 \cdot (n_1) + 1 \cdot (n_0) = 5 \tag{31}$$

$$1 \cdot (n_2) + 2 \cdot (n_1) + 1 \cdot (n_0) = 7 \tag{32}$$

etc. in Pascal triangle fashion, so that $n_k = 0$ for $k > 1$. It is then very easy to demonstrate that we thus in fact get a subdivided 1-cube for the component $n = 1$. For higher n, we take powers of this 1-cube, and this leads to an n-cube, suitably triangularized.

Example 12. Let A be the associahedral operad of example 1, and consider t_+ as an $A - A$ bimodule. Then the bar construction $B(F, F, t_+)$ gives a canonical triangulation of the polyhedra which parametrize the operations of Stasheff's A_n maps (see Homotopy Associativity of H-Spaces I, II).

This example deserves further comments. There is a strong analogy between the cellular structure of the A_n maps, and the cellular structure of the data for bihomomorphisms, trihomomorphisms, etc., except that the A_n structures and A_n maps take account only of higher associativities and their weak preservations, but do not take account of units.

To take account of units, the geometry of A_n maps should be replaced by the geometry of the bar construction $B(F, F, t)$, where the terminal operad t is regarded as a bimodule over the monoidahedral operad M of example 2.

The polyhedra which result are again infinite-dimensional. I call these polyhedra "*functoriahedra*"; the claim/conjecture is that they carry all the cellular structure one desires of weak n-functors.