Notes on the Lie Operad

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The purpose of these notes is to compare two distinct approaches to the Lie operad. The first approach closely follows work of Joyal [10], which gives a species-theoretic account of the relationship between the Lie operad and homology of partition lattices. The second approach is rooted in a paper of Ginzburg-Kapranov [7], which generalizes within the framework of operads a fundamental duality between commutative algebras and Lie algebras. Both approaches involve a bar resolution for operads, but in different ways: we explain the sense in which Joyal's approach involves a "right" resolution and the Ginzburg-Kapranov approach a "left" resolution. This observation is exploited to yield a precise comparison, in the form of a chain map which induces an isomorphism in homology, between the two approaches.

1 Categorical Generalities

Let $\mathcal{FB}$ denote the category of finite sets and bijections. $\mathcal{FB}$ is equivalent to the permutation category $\mathbf{P}$, whose objects are natural numbers and whose set of morphisms is the disjoint union $\sum_{n\geq 0} S_n$ of all finite symmetric groups. $\mathbf{P}$ (and therefore also $\mathcal{FB}$) satisfies the following universal property: given a symmetric monoidal category $\mathcal{C}$ and an object A of $\mathcal{C}$, there exists a symmetric monoidal functor $\mathbf{P} \longrightarrow \mathcal{C}$ which sends the object $\mathbf{1}$ of $\mathbf{P}$ to A, and this functor is unique up to a unique monoidal isomorphism. (Cf. the corresponding property for the braid category in [11].)

Let V be a symmetric monoidal closed category, with monoidal product $\otimes$ and unit I. For the time being we assume V is complete and cocomplete; later we will need to relax this condition.

Definition 1 *A V-species is a functor $\mathcal{FB} \longrightarrow V$. The category of V-species and their natural transformations is denoted $V^{\mathcal{FB}}$.*

The category $V^{\mathcal{FB}}$ carries several monoidal structures. One is the Day convolution ([3]) induced by the monoidal product $\oplus$ on $\mathcal{FB}$. To set this up, we work in the context of V-enriched category theory (see [13]), and recall that any locally small category $\mathcal{C}$ (such as $\mathcal{FB}$) can be regarded as a V-category, by composing $\hom : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}et$ with the symmetric monoidal functor $\mathcal{S}et \longrightarrow V$ sending a set U to a U-indexed coproduct of copies of I. Then Day convolution is given abstractly by the formula

$$(F \otimes G)[S] = \int^{W,X \in \mathcal{FB}} F[W] \otimes G[X] \otimes \mathcal{FB}(S, W \oplus X).$$

Since each finite bijection $\phi : S \longrightarrow W \oplus X$ induces a decomposition of S as a disjoint union $T + U$ of subsets (setting $T = \phi^{-1}(W)$ and $U = \phi^{-1}(X)$), this coend formula may be simplified:

$$(F \otimes G)[S] = \sum_{S=T+U} F[T] \otimes G[U].$$

(For the remainder of this paragraph, categorical terms such as "category", "functor", "colimit", etc. refer to their V-enriched analogues.) The product $F \otimes G$ preserves colimits in the separate arguments F and G (i.e., $- \otimes G$ and $F \otimes -$ are cocontinuous for all F and G). Since F and G may be canonically presented as colimits of representables, one may define a symmetric monoidal structure on this product, uniquely up to monoidal isomorphism, so that the Yoneda embedding $y : \mathcal{FB}^{op} \longrightarrow V^{\mathcal{FB}}$ is a symmetric monoidal functor, i.e., so that there is a coherent isomorphism $\hom(W \oplus X, -) \cong \hom(W, -) \otimes \hom(X, -)$. In conjunction with the universal property of $\mathcal{FB}$, we may state a universal property of $V^{\mathcal{FB}}$: let $\mathcal{C}$ be a (V-) category which is symmetric monoidally cocomplete (meaning its monoidal product is separately cocontinuous), and let A be an object of $\mathcal{C}$. Then there exists a cocontinuous symmetric monoidal functor $V^{\mathcal{FB}} \longrightarrow \mathcal{C}$ sending $\hom(\mathbf{1}, -)$ to A, and this functor is unique up to monoidal isomorphism.

This universal property may be exploited to yield a second monoidal structure on $V^{\mathcal{FB}}$. Let $[V^{\mathcal{FB}}, \mathcal{C}]$ denote the category of cocontinuous symmetric monoidal functors $V^{\mathcal{FB}} \longrightarrow \mathcal{C}$; then the universal property may be better expressed as the assertion of an equivalence $\mathcal{C} \simeq [V^{\mathcal{FB}}, \mathcal{C}]$. In the case $\mathcal{C} = V^{\mathcal{FB}}$, the right-hand side of the equivalence carries a monoidal structure given by endofunctor composition. This monoidal structure transports across the equivalence to yield a monoidal product on $V^{\mathcal{FB}}$, denoted by $\circ$. An explicit formula for $\circ$ is given as follows. Under one set of conventions, a V-species F may be regarded as a right module over the permutation category $\mathbf{P}$, so that each component $F[n]$ carries an action $F[n] \otimes S_n \longrightarrow F[n]$. The n-fold Day convolution $G^{\otimes n}$ carries, under the same conventions, a left S_n-action $S_n \otimes G^{\otimes n} \longrightarrow G^{\otimes n}$. Then the coend formula for $F \circ G$ may be written as

$$(F \circ G)[S] = \sum_{n \geq 0} F[n] \otimes_{S_n} G^{\otimes n}[S].$$

A special case of this "substitution product" $\circ$ is the analytic functor construction. For each object X in V there is a V-species X such that $X[0] = X$

and $X[n] = 0$ otherwise. Letting X^n denote the n-fold tensor product in V, we have

$$(F \circ X)[0] = \sum_{n \geq 0} F[n] \otimes_{S_n} X^n$$

and $(F \circ X)[n] = 0$ otherwise; we write the right-hand side of the above equation as $F(X)$. The analytic functor $F(-) : V \longrightarrow V$ determines, up to isomorphism, its generating species F; we describe this determination for the category $V = \mathcal{V}ect_k$ of vector spaces over a ground field k. Let $F_n(X)$ denote the n^{th}-degree component $F[n] \otimes_{S_n} X^n$, and let X be a vector space freely generated from a set $\{x_1, \ldots, x_n\}$ whose cardinality equals that degree. Then the species value $F[n]$ can be recovered as the subspace of $F_n(X)$ spanned by equivalence classes of those expressions $\tau \otimes x_{i(1)} \otimes \ldots \otimes x_{i(n)}$ in which each x_i occurs exactly once. We use the notations $F[n]$ and $F(X)[n]$ interchangeably for these species values.

If $G[0] = 0$, we have $G^{\otimes n}[S] = 0$ whenever n exceeds the cardinality $|S|$, in which case $F \circ G$ makes sense for V *finitely* cocomplete. For general n we have in that case

$$G^{\otimes n}[S] = \sum_{S = T_1 + \ldots + T_n} G[T_1] \otimes \ldots \otimes G[T_n]$$

where the sum is indexed over ordered partitions of S into n nonempty subsets T_i. S_n permutes such ordered partitions in such a way that the orbits correspond to unordered partitions, which are tantamount to equivalence relations R on S. Let $Eq(S)$ denote the set of such equivalence relations, and let $\pi : S \longrightarrow S/R$ denote the canonical projection of S onto the set of R-equivalence classes. Then the substitution product may be rewritten as

$$(F \circ G)[S] = \sum_{R \in Eq(S)} F[S/R] \otimes \bigotimes_{x \in S/R} G[\pi^{-1}(x)].$$

2 Operads

Definition 2 *An operad in V is a monoid in the monoidal category $(V^{\mathcal{FB}}, \circ)$.*

Clearly an operad M induces a monad $M \circ - : V^{\mathcal{FB}} \longrightarrow V^{\mathcal{FB}}$, which in turn restricts to an analytic monad $V \longrightarrow V$. Many algebraic structures arising in practice are algebras of analytic monads. For $V = \mathcal{V}ect_k$, we have, e.g.,

1. The tensor algebra $T(X) = 1 + X + X^{\otimes 2} + \ldots$, denoted $\frac{1}{1-X}$. The algebras of $T(-)$ are associative algebras. The species value $T[n]$ is the space freely generated from the set of linear orders on an n-element set, with the evident S_n-action.
2. The symmetric algebra $S(X) = 1 + X + X^{\otimes 2}/S_2 + \ldots$, denoted $exp(X)$. Algebras of $S(-)$ are commutative associative algebras. The species value $exp[n]$ is the trivial 1-dimensional representation of S_n.

3. The Lie operad $L[-]$ may be presented as an operad generated by a binary operation $[,] \in L[2]$, subject to the Jacobi relation $[,[,]] + [,[,]]\sigma + [,[,]]\sigma^2 = 0$ and the alternating relation $[,] + [,]\tau = 0$, where σ is a 3-cycle and τ is a 2-cycle. The algebras of the analytic monad $L(-)$ are Lie algebras.

By way of contrast, Boolean algebras are not algebras of an analytic monad, since the equation $x \wedge x = x$ inevitably involves the use of a diagonal map not available in $\mathcal{FB}$.

Although analytic monads are obviously important, we stress that they are simply restrictions of monads $V^{\mathcal{FB}} \to V^{\mathcal{FB}}$, and that it is more flexible to work in the latter setting. For example, if V is only finitely cocomplete (e.g., the category of finite-dimensional vector spaces), then analytic monads cannot be defined in general. However, if M is an operad such that $M[0] = 0$, then there *is* a monad $M \circ -$ acting on the full subcategory of V-species G such that $G[0] = 0$. This situation occurs often: e.g., the algebras of the operad $exp(X) - 1$ are commutative monoids without unit. For other reasons, Markl ([14]) has also considered algebras over monads $M \circ -$ and $- \circ M$, more general than algebras of analytic monads; he refers to the former as "M-modules".

3 Poincaré-Birkhoff-Witt

From this section on, we fix a groundfield k of characteristic 0, and V henceforth denotes the category of finite-dimensional vector spaces over k.

To study the Lie operad $L[-]$, Joyal and others (e.g., [8]) take as starting point the Poincaré-Birkhoff-Witt (PBW) theorem. If L is a Lie algebra, then its universal enveloping algebra $\mathcal{U}(L)$ carries a canonical filtration, inherited as a quotient of the tensor algebra $T(L)$ equipped with the degree filtration. Embedded in $T(L)$ as a filtered subalgebra is the symmetric algebra $S(L)$, whose homogeneous components $S_n(L)$ may be realized as images of symmetrizing projectors acting on components $T_n(L) = L^{\otimes n}$:

$$\pi_n : L^{\otimes n} \to L^{\otimes n}$$

$$v_1 \otimes \ldots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

We obtain a composite of maps of filtered algebras

$$S(L) \to T(L) \to \mathcal{U}(L)$$

and the PBW theorem concerns the application of the associated graded algebra functor to this composite o:

Theorem 1 *(PBW) o_{gr} induces an isomorphism between $\mathcal{U}_{gr}(L)$ and $S(L)$ as graded algebras.*

If $L(X)$ is the free Lie algebra on X, then $\mathcal{U}(L(X)) \cong T(X)$ by an adjoint functor argument. Assembling some prior notation, it follows from PBW that there exists an isomorphism of analytic functors on $\mathcal{V}ect_k$:

$$\frac{1}{1-X} \cong (exp \circ L)(X)$$

which in turn determines a species-isomorphism, which componentwise is an isomorphism of S_n-representations:

$$\frac{1}{1-X}[n] \cong (exp \circ L)[n].$$

Notice both sides make sense as V-species. A guiding heuristic behind the species methodology is that the components of such species are structural analogues of coefficients of formal power series. This analogy can be made precise. Let $V[[x]]$ denote the rig (ring without additive inverses) of isomorphism classes of V-species, with $+$ given by coproduct and $\cdot$ given by $\otimes$. Let $\mathbf{N}[[x]]$ denote the rig of formal power series $\sum_{n\geq 0} \frac{a_n x^n}{n!}$ whose coefficients a_n are natural numbers. For both rigs there is a composition $\circ$, where $f \circ g$ is defined whenever $g(0) = 0$.

Proposition 1 *([9]) The function $dim : V[[x]] \longrightarrow \mathbf{N}[[x]]$, sending F to the list of coefficients $a_n = dim(F[n])$, is a rig homomorphism which preserves $\circ$.*

It follows from the proposition and the preceding species isomorphism that $dim(L[n]) = (n-1)!$, since $dim(L)(x)$ is the formal power series expansion of $-\log(1-x)$. One is interested in finding an appropriate lift of $-\log(1-x) \in \mathbf{N}[[x]]$ to a species $-\log(1-X)$ in $V[[x]]$ (and hence an identification between $-\log(1-X)$ and the Lie species $L[-]$).

4 Virtual species

Before we construct the species $\log(1-X)$, it is convenient to complete the rig $V[[x]]$ to a ring. One proceeds exactly as in K-theory, where one passes from vector bundles to virtual bundles.

Definition 3 *V is the category of $\mathbf{Z}_2$-graded finite-dimensional vector spaces, with monoidal product given by the formula*

$$(V \otimes W)_p = \sum_{m+n=p} V_m \otimes W_n$$

and symmetry given by the formula

$$\sigma(x_m \otimes y_n) = (-1)^{mn} y_n \otimes x_m$$

for $x_m \in V_m$, $y \in W_n$.

Definition 4 *Let F and G be $\mathcal{V}$-species. Then $F \sim G$ (F and G are virtually equivalent) if $F_0 \oplus G_1 \cong F_1 \oplus G_0$ as V-species.*

- **Remark:** Exact sequences in $V^{\mathcal{FB}}$ split. One can check this assertion componentwise, where exact sequences of kS_n-modules split since we assumed $char(k) = 0$.

Lemma 1 *$\sim$ is an equivalence relation.*

Proof: Transitivity follows from cancellation: $F \oplus H \cong G \oplus H$ implies $F \cong G$. This in turn follows easily from the remark. **QED**

Lemma 2 *The relation $\sim$ is respected by $\oplus$, $\otimes$, and $\circ$.*

The proof is left to the reader; see also [10] and [16].

A virtual species is a virtual equivalence class of $\mathcal{V}$-species. Many of our calculations refer to manipulations in the ring $\mathcal{V}[[x]]$ of virtual species, but methodologically it is useful to distinguish the various ways in which virtual equivalences arise. Part of the philosophy behind species is that clarity is promoted and calculations are under good combinatorial control when power series operations $+$, $\cdot$, and $\circ$ can be viewed as arising from functorial operations $\oplus$, $\otimes$, and $\circ$. Put differently, passage from $\mathcal{V}^{\mathcal{FB}}$ to $\mathcal{V}[[x]]$ loses categorical information, and it helps to recognize when a virtual equivalence $F \sim G$ comes from an isomorphism $F \cong G$ in $\mathcal{V}^{\mathcal{FB}}$.

In practice, many virtual equivalences which do not come from isomorphisms in $\mathcal{V}^{\mathcal{FB}}$ are rooted in the following

- **Remark:** If an object C of $\mathcal{V}^{\mathcal{FB}}$ carries a differential structure (meaning V-species maps $\partial : C_0 \longrightarrow C_1$ and $\partial : C_1 \longrightarrow C_0$ satisfying both instances of $\partial^2 = 0$), then C is virtually equivalent to its homology $H(C)$. This is tantamount to a structural Euler formula $C_0 - C_1 \sim H_0(C) - H_1(C)$, which obtains because exact sequences in $V^{\mathcal{FB}}$ split.

This principle can be quite powerful: its application does not commit one to any particular choice of differential structure on C, so that one is enabled to choose differentials to suit the local occasion. The downside is that because there is no canonical way to split exact sequences, it becomes harder to give precise formulas which exhibit such virtual equivalences $C \sim H(C)$.

5 Logarithmic species

Our goal is to lift the inversion

$$\frac{1}{1-x} = exp(L(x)) \text{ implies } -log(1-x) = L(x)$$

from the ring $\mathbf{Z}[[x]]$ to the ring $\mathcal{V}[[x]]$. In either ring, a necessary condition for $F(x)$ to have an inverse $F^{-1}(x)$ (with respect to $\circ$) is that the 0^{th} coefficient $F[0]$ be 0. Thus, instead of inverting $exp(X)$, we invert $exp(X) - 1$. Suppose then that $\log(1 + X)$ is a virtual inverse of $exp(X) - 1$:

$$\log(1 + X) \circ (exp(X) - 1) \sim X \sim (exp(X) - 1) \circ \log(1 + X).$$

Proposition 2 *If F and G are $\mathcal{V}$-species such that $F[0] = 0 = G[0]$, then $exp(F \oplus G) \cong exp(F) \otimes exp(G)$.*

Proof: It is immediate that $exp(F) = \sum_{n\geq 0} F^{\otimes n}/S_n$ is the free commutative monoid in $(\mathcal{V}^{\mathcal{FB}}, \otimes)$ generated from F, and the assertion says that the left adjoint exp preserves coproducts. **QED**

Defining $\log((1 + F) \otimes (1 + G))$ as $\log(1 + X) \circ (F + G + F \otimes G)$, it follows that $\log((1 + F) \otimes (1 + G)) \sim \log(1 + F) + \log(1 + G)$. In particular,

$$\log(\frac{1}{1 - X}) \sim -\log(1 - X)$$

where of course $-(F_0, F_1)$ is defined as (F_1, F_0). The species $\log(1 - X)$ is easily obtained from $\log(1 + X)$ by the following

Proposition 3 $F(-X)[S] \cong (-1)^{|S|}F[S] \otimes \Lambda[S]$, *where $\Lambda[S]$ denotes the top exterior power $\Lambda^{|S|}(kS)$ of the vector space kS freely generated from S.*

Proof: The species X, defined as the unit with respect to $\circ$, is given by $(X[1]_0, X[1]_1) = (k, 0)$ and $X[n] = 0$ otherwise. Thus $(-X[1]_0, -X[1]_1) = (0, k)$ and $-X[n] = 0$ otherwise. Hence

$$(F \circ (-X))[S] = \sum_{R \in Eq(S)} F[S/R] \otimes \bigotimes_{x \in S/R} (-X)[\pi^{-1}(x)] \cong F[S] \otimes (-X)[1]^{\otimes |S|}.$$

$(-X)[1]^n$ is 1-dimensional and is concentrated in degree = parity(n) (whence the sign $(-1)^{|S|}$). A transposition in S_n induces a sign change in $(-X)[1]^n$, by definition of symmetry: $\sigma(x_1 \otimes y_1) = -(y_1 \otimes x_1)$. This proves the claim. **QED**

This leaves us to compute the inverse $\log(1 + X)$ to $exp(X) - 1$. Recalling some earlier remarks, $F(X) = exp(X) - 1$ is the operad such that algebras of the monad $F \circ -$ (acting on $\mathcal{V}$-species G such that $G[0] = 0$) are commutative monoids without unit.

The underlying species F satisfies $F[0] = 0$, $F[1] = 1$ (i.e., $= (k, 0)$). Joyal quotes a general method due to G. Labelle for inverting such species. Introduce an operator

$$\delta_F : \mathcal{V}[[x]] \longrightarrow \mathcal{V}[[x]]$$
$$H \longmapsto H \circ F - H$$

so that $(1+\delta_F)(H) = H \circ F$ where 1 denotes the identity operator. Observe that δ_F preserves sums, because $- \circ F : \mathcal{V}^{\mathcal{FB}} \longrightarrow \mathcal{V}^{\mathcal{FB}}$ is the restriction of a cocontinuous monoidal functor.

Let $\hat{0}$ denote the bottom element of the lattice $Eq(S)$ ordered by inclusion of equivalence relations. $\hat{0}$ is the discrete equivalence relation whereby $S/\hat{0} \cong S$. We have

$$\delta_F(H)[S] = (\sum_{R \in Eq(S)} H[S/R] \otimes \bigotimes_{x \in S/R} F[\pi^{-1}(x)]) - H[S]$$

and since $H[S] \cong H[S/\hat{0}] \otimes \bigotimes_{x \in S/\hat{0}} F[1]$ by our assumptions on F, we may rewrite the right-hand side (up to virtual equivalence) as

$$\sum_{\hat{0}<R} H[S/R] \otimes \bigotimes_{x \in S/R} F[\pi^{-1}(x)] \quad (= \sum_{\hat{0}<R} H[S/R] \text{ if } F(X) = exp(X) - 1).$$

In the sequel we use δ_F to denote this functor on $\mathcal{V}$-species H, as well as for the operator on $\mathcal{V}[[x]]$.

The inverse $F^{-1}(X)$ is given as

$$(1+\delta_F)^{-1}(X) := \sum_{n \geq 0} (-1)^n \delta_F^n(X)$$

where the right-hand side makes sense when evaluated at $S \in Ob(\mathcal{FB})$, because $\delta_F^n(X)[S] = 0$ when $n \geq |S|$.

Proposition 4 $(F^{-1} \circ F)(X) \sim X$.

Proof:

$$(F^{-1} \circ F)(X) \sim (1+\delta_F)(\sum_{n \geq 0}(-1)^n \delta_F^n(X)) \sim \sum_{n \geq 0}(-1)^n \delta_F^n(X) + \sum_{n \geq 0}(-1)^n \delta_F^{n+1}(X)$$

which reduces to $\delta_F^0(X) = X$. **QED**

- **Remark:** One may prove by recursion that under our hypotheses on F, $F \circ G \sim F \circ H$ implies $G \sim H$. From this and the preceding proposition, it easily follows that $(F \circ F^{-1})(X) \sim X$.

6 A bar construction

When F carries an operad structure, this construction of F^{-1} admits a more categorical interpretation. Observe that there is an embedding

$$\delta_F^n(X) \hookrightarrow F \circ \ldots \circ F = F^{\circ(n)};$$

for $n > 0$. $\delta_F^n(X)$ is the n^{th} component of a $\mathbf{Z}$-graded subchain complex of the chain complex structure $\{F^{\circ(n)}\}_{n>0}$ obtained from the simplicial nerve of the monoid $(F, F \circ F \xrightarrow{m} F, X \xrightarrow{u} F)$. The augmented complex $B_r(F) = \{\delta_F^n(X)\}_{n\geq 0}$ is (in a formal sense) a bar construction for F.

To explain this, we first observe that there is a (necessarily unique) operad map $\epsilon : F \longrightarrow X$, called an *augmentation*, giving X a structure of F-algebra. The component $\epsilon[1] : F[1] \longrightarrow X[1]$ is an isomorphism, and the other components $\epsilon[n]$ are necessarily zero. Next, there is an exact sequence

$$0 \longrightarrow \delta_F \longrightarrow (-) \circ F \xrightarrow{-\circ\epsilon} (-) \circ X \longrightarrow 0$$

so that the functor δ_F for an operad F is analogous to tensoring on the right with an augmentation ideal IG of, say, a group ring $\mathbf{Z}G$. In forming

$$F^{-1}(X) = \sum_{n\geq 0} (-\delta_F)^n(X),$$

the expression $-\delta_F$ is a $\mathbf{Z}_2$-graded analogue of tensoring with the suspension $\Sigma(IG)$. Thus $F^{-1}(X)$ is an analogue of the normalized bar construction

$$\mathbf{Z} + \Sigma(IG) + (\Sigma(IG))^{\otimes 2} + \ldots$$

originally introduced by Eilenberg-Mac Lane [4].

- **Remark:** The analogy with classical bar constructions is not perfect, because unlike $-\otimes-$, the product $-\circ-$ does not preserve separate colimits: only $- \circ F$ preserves colimits. It is well-known that given a monoidal category whose product $-\otimes-$ preserves countable coproducts in separate variables, the free monoid on an object A is given by a geometric series $\sum_{n\geq 0} A^{\otimes n}$. For example, if IG is the augmentation ideal used in the standard bar construction for cohomology of a group G, then the bar construction is given by the tensor algebra, i.e., the free monoid, on $\Sigma(IG)$. The construction we gave of $F^{-1}(X)$ does not however yield a free monoid (i.e., a free operad). We return to this point later.

To give more weight to the sense in which $B_r(F)$ is a bar construction, we assemble the components $\delta_F^n(X) \circ F$ of $F^{-1} \circ F$ into a chain complex $E_r(F)$ of $\mathcal{V}$-species which will be a (right) F-free resolution of X. There is an embedding

$$\delta_F^n(X) \circ F \hookrightarrow F^{\circ(n+1)}$$

and one defines differentials on the right-hand components as before by taking alternating sums of face maps of the simplicial nerve; by restriction we get differentials on the left-hand components, and hence a chain complex structure $E_r(F)$ on $F^{-1} \circ F$. If X is regarded as a trivial chain complex concentrated in degree 0, there is a chain complex map

$$E_r(F) \longrightarrow X$$

which we claim is a homotopy equivalence. Since a complex is virtually equivalent to its homology, this will reprove the virtual equivalence $F^{-1} \circ F \sim X$.

It is trivial that $E_r(F)[1] \cong X[1]$. For $|S| > 1$, we construct a contracting homotopy for $E_r(F)[S]$. Each component $\delta_F^n(X) \circ F$ is a coproduct

$$(\delta_F^n(X) \circ F - \delta_F^n(X)) \oplus \delta_F^n(X)$$

where the first summand is of course $\delta_F^{n+1}(X) \cong \delta_F^{n+1}(X) \circ X$. The contracting homotopy

$$h_n : \delta_F^n(X) \circ F \longrightarrow \delta_F^{n+1}(X) \circ F$$

restricts to the summands as $h_{n,1}$ and $h_{n,2}$, defined as follows:

$$h_{n,1} = \delta_F^{n+1}(X) \circ u : \delta_F^{n+1}(X) \circ X \longrightarrow \delta_F^{n+1}(X) \circ F$$

$$h_{n,2} = 0 : \delta_F^n(X) \longrightarrow \delta_F^{n+1}(X) \circ F.$$

The proof that the h_n define a contracting homotopy (which uses only the simplicial identities) is standard and left to the reader.

- **Remark:** The reader may have wondered why we didn't use the simplicial nerve of the operad itself for an F-free resolution. The reason is that, regarded as a $\mathbf{Z}_2$-graded complex, the simplicial nerve has infinite-dimensional components, taking one outside the realm of virtual species.

- **Remark:** The problem of giving a similar categorical interpretation of the virtual equivalence $(F \circ F^{-1})(X) \sim X$ (in terms of contracting homotopies, for example) is difficult if we use the Joyal-Labelle construction of F^{-1}. Later we follow the construction of Ginzburg-Kapranov and call it a left bar construction $B_l(F)$: a chain complex which gives an alternative construction of the virtual species F^{-1}. The relationship between $B_r(F)$ and $B_l(F)$ is also categorical: there is a chain map $B_l(F) \longrightarrow B_r(F)$ which induces an isomorphism in homology.

7 The Lie species

Returning to our example $F(X) = exp(X) - 1$, we may iterate the formula

$$\delta_F(H)[S] = \sum_{\hat{0}<R} H[S/R]$$

to obtain the following characterization: $\delta_F^n(X)[S]$ is a space with basis elements given as n-fold chains of strict inclusions of equivalence relations

$$0 < R_1 < \ldots < R_n = 1$$

where $\hat{1}$ denotes the top element of the lattice $Eq(S)$ (the indiscrete equivalence relation). Each such chain may be identified with an $(n-1)$-fold chain in the poset $Eq(S) - \{\hat{0}, \hat{1}\}$, or as cells of dimension $n-2$ in the simplicial complex underlying the nerve of this poset. Let $C_i[S]$ denote the set of cells of dimension i. Then for $|S| > 2$, we may identify the complex $B_r(F)[S]$,

$$0 \to \delta_F^{|S|-1}(X)[S] \to \dots \delta_F^2(X)[S] \to \delta_F^1(X)[S] \to X[S] \to 0,$$

with

$$0 \to C_{|S|-3}[S] \to \dots \to C_0[S] \to k \to 0 \to 0$$

so that $\log(1+X)[S]$ is virtually equivalent to *reduced* homology of the simplicial complex C of $Eq(S)$.

It is known that $Eq(S)$ is a geometric lattice:

Definition 5 *A finite lattice is geometric if every element x is a join of atoms, if every maximal chain beginning at $\hat{0}$ and ending at x has the same length $\rho(x)$, and if $\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$. (A finite lattice is modular iff this last inequality is an equality.)*

Theorem 2 *([5]) If L is a finite geometric lattice, then the reduced homology of $L - \{0, 1\}$ is trivial except in top dimension, where the Betti number is the value $\mu(0, 1)$ of Rota's Moebius function on the lattice.*

Corollary 1 $\log(1+X)[S] \sim (-1)^{|S|-1} H_{|S|-1}(B_r(F)[S])$

Corollary 2 *The Lie species $L[S]$ is isomorphic to $H_{|S|-1}(B_r(F)[S]) \otimes \Lambda[S]$.*

Proof: $L[-] \sim -\log(1-X)$, and

$$-\log(1-X)[S] \sim -(-1)^{|S|-1}(-1)^{|S|} H_{|S|-1}(B_r(F)[S]) \otimes \Lambda[S]$$

so that both the right-hand side and $L[S]$ are concentrated in degree 0. For such $\mathcal{V}$-species, virtual equivalence implies isomorphism. **QED**

This concludes our rendition of Joyal's calculation of $L[S]$. This calculation does not specify the operad structure of $L[-]$, but in view of the formula just given for $L[n]$, we make the following remark: given an operad in chain complexes with components $C[n]$, the homology operad $H_*(C)$ contains a suboperad with components $H_{n-1}(C[n])$. Unfortunately, this remark does not apply to $C[n] = B_r(F)[n] \otimes \Lambda[n]$, since the bar construction $B_r(F)[n]$ carries no obvious operad structure. A different bar construction $B_l(F)$, or rather its dual cobar construction, is better suited for applying the present remark.

8 Free operads

Let F be a $\mathcal{V}$-operad such that $F[0] = 0$ and $F[1] = 1$. Our earlier construction for $F^{-1}(X)$, namely $\sum_{n\geq 0}(-\delta_F)^n(X)$, is not a free operad on the suspension $-\delta_F(X)$ of the augmentation ideal $\delta_F(X)$, due to the failure of $- \circ -$ to preserve separate colimits.

To construct free operads, it is convenient to use the language of trees [2].

Definition 6 *A rooted tree is an (undirected) acyclic connected finite graph together with a distinguished node called the root.*

Let N be the set of nodes of a rooted tree T with root r, and E the set of edges. There is a bijection $\phi_- : E \xrightarrow{\sim} N - \{r\}$, uniquely determined by the requirement that the node $\phi_-(e)$ lie on the edge e. Let $\phi_+(e)$ denote the other node lying on e. One may give a rooted tree a canonical directed graph structure, with source-target map $\langle \phi_-, \phi_+ \rangle : E \longrightarrow N \times N$. The free category on a directed graph has a terminal object r iff the directed graph comes from a tree rooted at r. The free category on a rooted tree T defines a partial order on its set of nodes: an atomic element in this poset is called a *leaf* of T. A directed subgraph of T whose free category possesses a terminal object is called a *subtree*. In particular, the slice over an object x comes from a subtree T_x called the *tree over* x.

A rooted tree T may be characterized up to isomorphism by a function $f_T : N - \{r\} \longrightarrow N$, sending a node to its unique successor in the induced partial order. Identifying functions of this form with based endofunctions $(N, r) \longrightarrow (N, r)$, the endofunctions which arise in this way are precisely those whose iterates converge to the constant function at r. If x is a node in T, the set $\sigma_x = f_T^{-1}(x)$ is called the *sprout over* x.

Up to isomorphism, a rooted tree whose set of leaves is S is characterized in terms of S by the following data: an element C of the free commutative monoid $exp(\mathcal{P}S)$ on the power set of S (i.e. a multiset of subsets of S), together with a linear order on each set of repetitions in C of a given subset. To obtain such a structure C_T from a rooted tree T, assign to each node x the set $\lambda(x)$ of leaves $\leq x$, and let C_T be the multiset $\{\lambda(x) : x \text{ is a node of } T\}$. There is a repetition $\lambda(y) = \lambda(x)$ whenever $\sigma_x = \{y\}$; in that case impose the order $\lambda(y) < \lambda(x)$. If on the other hand $\{y\}$ is a proper subset of σ_x, there is a proper inclusion $\lambda(y) \subset \lambda(x)$. The set C_T contains S as $\lambda(r)$, and contains each singleton $\{s\}$ in S as $\lambda(s)$. The absence of cycles in the tree T implies a *trichotomy law*: for all $U, V \in C_T$, either $U \subseteq V$ or $V \subseteq U$ or $U \cap V = \phi$. In view of the essential equivalence between trees and such multisets, we make the following definition.

Definition 7 *A tree on a finite set S is a finite multiset C of subsets of S containing S and each singleton of S, and satisfying the trichotomy law, together with a linear ordering on each set of repetitions. The set of trees on S is denoted by $\mathcal{T}[S]$. For $T \in \mathcal{T}[S]$, the set of singletons $\{s\} \in T$ is identified with S.*

$\mathcal{T}[-]$ carries a structure of $\mathcal{S}et$-species: to each bijection $\phi : S \longrightarrow T$ we form a map $\mathcal{T}[S] \longrightarrow \mathcal{T}[T]$ by relabelling along ϕ.

Next, we introduce a grafting operation on trees. Suppose given a tree $T \in \mathcal{T}[X]$ and for each $x \in X$ a tree T_x rooted at x. Then the discrete graph on X (having no edges and denoted again by X) is obviously embedded in T and in the disjoint union of the T_x; the pushout of these two embeddings in the category of undirected graphs gives an acyclic connected graph. Defining the root of the pushout as the image of the root of T, the result is a rooted tree called a *grafting*, denoted by $m(T; T_x : x \in X)$. Let R be an equivalence relation on S; put $X = S/R$, and let $\pi : S \longrightarrow X$ be the canonical quotient. Then the grafting operation just described induces a map

$$m[S] : \sum_{R \in Eq(S)} \mathcal{T}[S/R] \oslash \bigotimes_{x \in S/R} \mathcal{T}[\pi^{-1}(x)] \longrightarrow \mathcal{T}[S]$$

There is a map $u : X \longrightarrow \mathcal{T}$ which sends the element r of $X[1]$ to the rooted tree consisting only of r. The triple $(\mathcal{T}, m, u)$ defines a $\mathcal{S}et$-operad: satisfaction of the operad axioms follows easily from universal properties of pushouts. In fact, it is not difficult to show that $\mathcal{T}$ is the free operad generated by the $\mathcal{S}et$-species $exp(X) - 1$. The unit map $\eta : (exp(X) - 1) \longrightarrow \mathcal{T}$ which generates it is given componentwise by maps $(exp(X) - 1)[S] \longrightarrow \mathcal{T}[S]$ each of which, for $|S| > 0$, sends the unique element in the domain to the "sprout" on S: the multiset $S + \sum_{s \in S}\{s\}$. It is clear that every structure of species $\mathcal{T}$ is obtained by grafting together a collection of such sprouts and instances of $u[1]$.

More generally, letting $\mathcal{C}$ be a complete cocomplete symmetric monoidal closed category, we describe the free operad $\mathcal{O}(G)$ on a $\mathcal{C}$-species G such that $G[0] = 0$. Let $N(T)$ denote the set of nodes in a tree $T \in \mathcal{T}[S]$. As a species,

$$\mathcal{O}(G)[S] = \sum_{T \in \mathcal{T}[S]} \bigotimes_{x \in N(T) - S} G[\sigma_x]$$

for $|S| > 0$, and $\mathcal{O}(G)[0] = 0$. If in addition $G[1] = 0$, a summand corresponding to a tree T is 0 whenever there are nodes $x \in N(T) - S$ such that σ_x is a singleton; i.e., when there are repetitions of subsets. So if $G[1] = 0$, the summation can be restricted to multisets without repetitions, i.e., to rooted trees where each node is the join of the leaves below it. These shall be called *proper* trees.

- **Remark:** The description we have given of $\mathcal{O}(G)$ can be formulated as a "wreath product" $\mathcal{T} \int G$, as in unpublished notes of Kelly, similar to the wreath product constructions pertaining to Kelly's "clubs" [12]. The essential point is that the multiplication on the operad $\mathcal{O}(G)$ is induced from the grafting multiplication on $\mathcal{T}$. The unit map $u : X \longrightarrow \mathcal{O}(G)$ is also induced from that of $\mathcal{T}$; the map $u[1]$ factors through the summand where the iterated tensor product is indexed over an empty set (empty products are interpreted as the unit I for the tensor).

There is a familiar free-forgetful adjunction arising from the free operad construction; it will be useful to describe the counit of this adjunction using a kind of term-rewrite system. To begin, let $T \in \mathcal{T}[S]$ be a tree described as a multiset, and U a subset of S occurring in T neither as $\lambda(r)$ nor as $\lambda(s)$ for $s \in S$. Then we may delete U from T to obtain another tree $T/U \in \mathcal{T}[S]$, called the contraction of T along U (deletion corresponds to contraction of an internal edge in the tree). Contractions may be continued until one reaches a sprout, i.e., the element in the image of the unit $\eta : (exp(X) - 1)[S] \longrightarrow \mathcal{T}[S]$. This iterated contraction reproduces the unique $\mathcal{S}et$-species map $\mathcal{T} \longrightarrow exp(X) - 1$, necessarily a morphism of operads.

If a tree is regarded as the result of grafting together a collection of sprouts, then each contraction may be regarded as a tree surgery, where a subtree T' obtained by grafting together two sprouts is replaced by a sprout having the same leaves as the subtree. More formally, suppose given two subsprouts with leaf-sets of the form σ_x and σ_y, where $y \in \sigma_x$ and $\lambda(y) = U$, the deleted set. T', and the sprout which replaces it in T/U, both have x as root: we use the notation x/y to denote the root in the latter case. Both have the same set of leaves, which we may write as $\sigma_{x/y}$. Observe there is an inclusion $\sigma_y \hookrightarrow \sigma_{x/y}$ and a surjection $\pi_y : \sigma_{x/y} \longrightarrow \sigma_x$: indeed, defining an equivalence relation R on the set $\sigma_{x/y}$ by zRw if $z = w$ or $z, w \in \sigma_y$, the set σ_x is canonically identified with the set of R-equivalence classes on $\sigma_{x/y}$.

Now suppose G carries an operad structure. In $(G \circ G)[\sigma_{x/y}]$ there is a summand of the form

$$G[\sigma_x] \odot \bigotimes_{z \in \sigma_x} G[\pi_y^{-1}(z)]$$

where $|\pi_y^{-1}(z)| = 1$ if $z \neq y$. There is a unit map $I \cong X[\pi_y^{-1}(z)] \longrightarrow G[\pi_y^{-1}(z)]$ for $z \neq y$, and of course $\pi_y^{-1}(y) = \sigma_y$: therefore there is an induced map

$$G[\sigma_x] \odot G[\sigma_y] \longrightarrow (G \circ G)[\sigma_{x/y}]$$

whence, after composing with the multiplication on G, a map

$$G[\sigma_x] \odot G[\sigma_y] \longrightarrow G[\sigma_{x/y}].$$

Expand this by tensoring with identity maps to obtain a "G-contraction" map

$$m_{T/U} : \bigotimes_{x \in N(T) - S} G[\sigma_x] \longrightarrow \bigotimes_{x \in N(T/U) - S} G[\sigma_x]$$

on the T- summand of $\mathcal{O}(G)[S]$. Such G-contractions may be iterated until a sprout is reached, leading to a map

$$\bigotimes_{x \in N(T) - S} G[\sigma_x] \longrightarrow G[S]$$

and by the operad axioms, this map is independent of the order in which contractions are performed. Finally, all of these maps assemble to give a map

$$\epsilon_G[S] : \mathcal{O}(G)[S] \longrightarrow G[S]$$

which is the component at S of the counit ϵ_G of the adjunction.

We remark that if F is an operad and $F[1] = 1$, then ϵ_F restricts to a map $\mathcal{O}(\delta_F(X)) - X \longrightarrow \delta_F(X)$. In particular there are contraction maps

$$\mu_{T/U} : \bigotimes_{x \in N(T)-S} \delta_F(X)[\sigma_x] \longrightarrow \bigotimes_{x \in N(T/U)-S} \delta_F(X)[\sigma_x].$$

9 The left bar construction

From this section on, F denotes a $\mathcal{V}$-operad with $F[0] = 0$ and $F[1] = 1$. We recall (see [7], [6]) a chain complex structure $B_l(F)$ on the free operad generated from the suspension $G = -\delta_F(X)$. (In this case $\mathcal{O}(G)[S]$ is a coproduct over *proper* trees on S, so that both its $\mathbf{Z}_2$-graded components are finite-dimensional.)

The grading on $B_l(F)[S]$ is determined from the construction of $\mathcal{O}(-\delta_F(X))$ as a $\mathcal{V}$-species: for example, if F is the $\mathcal{V}$-operad $exp(X) - 1$, and T is a proper tree, then the summand

$$\bigotimes_{x \in N(T)-S} (-\delta_F(X))[\sigma_x]$$

is concentrated in degree given by the number n of sprouts σ_x (taken modulo 2). Taking $F(X) = exp(X) - 1$ as basic, we define differentials for $B_l(F)$. Let $\mathcal{T}_j[S]$ denote those proper trees on S consisting of exactly j subsets of S aside from S and $\{s\}$ for $s \in S$. Arguing as we did in the proof of proposition 3, $\mathcal{O}(-\delta_F(X))$ lifts to a $\mathbf{Z}$-graded species $\mathcal{O}(\Sigma\delta_F(X))$ whose $(j+1)^{st}$ component is given by

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T]$$

If $\dot{\mathcal{P}}S$ denotes the set $\mathcal{P}S - \{S\} - \cup_{s \in S}\{\{s\}\}$, and $\Lambda^j[S]$ the j^{th} exterior power of a space freely generated from S, there is a monomorphism

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \longrightarrow \Lambda^j[\dot{\mathcal{P}}S].$$

To see this, first observe that every tree on S contains S and all singletons $\{s\}$, so no essential information is lost if we ignore these; thus there is a species isomorphism

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \cong \sum_{T \in \mathcal{T}_j[S]} \Lambda[T - \{S\} - \cup_{s \in S}\{\{s\}\}].$$

to work with species valued in the category of $\mathbf{Z}_2$-graded spaces (rather than just the finite-dimensional ones). We have inclusions

$$-F \circ \mathcal{O}(-\delta_F(X)) \overset{\eta \circ i}{\hookrightarrow} \mathcal{O}(-F) \circ \mathcal{O}(-F) \overset{m}{\hookrightarrow} \mathcal{O}(-F)$$

and a differential structure on $\mathcal{O}(-F)$ from the preceding discussion; by restriction we get a differential structure on $-F \circ \mathcal{O}(-\delta_F(X))$, and hence a complex $E_l(F)$. To show $E_l(F) \longrightarrow X$ is a homotopy equivalence, it suffices to exhibit a contracting homotopy for $E_l(F)[S]$ when $|S| > 1$. We have

$$-F \circ \mathcal{O}(-\delta_F(X)) \cong (-\delta_F(X) \circ \mathcal{O}(-\delta_F(X))) \oplus (-X \circ \mathcal{O}(-\delta_F(X)))$$

and now we write down the restriction of the contracting homotopy h to each summand. For the first summand we have

$$-\delta_F(X) \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{\eta \circ 1} \mathcal{O}(-\delta_F(X)) \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{m} \mathcal{O}(-\delta_F(X))$$

which we further compose with

$$\mathcal{O}(-\delta_F(X)) \xrightarrow{i} -X \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{-u \circ 1} -F \circ \mathcal{O}(-\delta_F(X))$$

to obtain the first restriction h_1; here the map i is a degree 1 shift of the canonical isomorphism $\mathcal{O}(-\delta_F(X)) \cong X \circ \mathcal{O}(-\delta_F(X))$. The restriction h_2 of the contracting homotopy to the second summand $-X \circ \mathcal{O}(-\delta_F(X))$ is just a zero map. The proof that $h = h_1 + h_2$ is a contracting homotopy (which only uses simplicial identities) is standard and left to the reader.

The virtual equality $F \circ \mathcal{O}(-\delta_F(X)) \sim X$ follows from the fact that $E_l(F) \longrightarrow X$ induces an isomorphism in homology. Thus $\mathcal{O}(-\delta_F(X))$ and $\sum_{n \geq 0}(-\delta_F)^n(X)$ both represent the virtual species $F^{-1} \in \mathcal{V}[[x]]$. In the next section we exhibit a chain map between their associated bar constructions $B_r(F)$, $B_l(F)$ which is a quasi-isomorphism.

10 A quasi-isomorphism

Under the hypotheses on an operad F used in the previous section, we describe a chain map $B_l(F) \longrightarrow B_r(F)$ which induces an isomorphism in homology.

As usual, it is easiest to begin with the case $F(X) = exp(X) - 1$. First we exhibit a map

$$\mathcal{O}(-\delta_F(X)) \longrightarrow \sum_{n \geq 0}(-\delta_F)^n(X)$$

which comes from a map of $\mathbf{Z}$-graded species with components

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \longrightarrow (\delta_F)^{j+1}(X)[S].$$

Let T be a proper tree on S whose elements, aside from S and the singleton sets, are subsets $U_1, \ldots, U_j$ of S. To each non-empty subset $U \subseteq S$ we may associate an equivalence relation R_U on S defined by xR_Uy if $x = y$ or $x, y \in U$. If $R, R' \in Eq(S)$, let the sum $R + R'$ denote their join in this lattice. Suppose the subsets U_i are arranged so that $U_i \subseteq U_j$ implies $i \leq j$, and let $R^{(i)}$ denote $\sum_{k \leq i} R_{U_k}$. Then there are strict inclusions $R^{(i)} < R^{(j)}$ when $i < j$, and the trichotomy law guarantees that $\hat{0} < R^{(i)} < \hat{1}$ for all i. We thus obtain a chain in $\delta_F^{j+1}(X)[S]$ of the form

$$\hat{0} < R^{(1)} < \ldots < R^{(j)} < R^{(j+1)} := \hat{1}.$$

It is sometimes convenient to omit the last member from this chain, which carries no essential information.

The space $\Lambda[T]$ may be regarded as being generated by a j-fold exterior product $U_1 \wedge \ldots \wedge U_j$. Let $\dot{T}$ denote the poset whose elements are the sets U_i ordered by inclusion, and let $[j]$ denote the poset $\{1 < \ldots < j\}$. Consider the set $Ord_b(\dot{T}, [j])$ of order-preserving bijections $\dot{T} \longrightarrow [j]$: to each $\phi \in Ord_b(T, [j])$ we define a sign $sgn(\phi)$ as the sign of the permutation $[j] \longrightarrow [j]$ which sends the element i to $\phi(U_i)$, and define equivalence relations

$$R_\phi^{(i)} = \sum_{k \leq i} R_{\phi^{-1}(k)}$$

for $1 \leq i \leq j$, which form a chain

$$R_\phi = [\hat{0} < R_\phi^{(1)} < \ldots < R_\phi^{(j)} < \hat{1}].$$

Finally, define a map

$$\iota_j[S] : \sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \longrightarrow (\delta_F)^{j+1}(X)[S]$$

$$U_1 \wedge \ldots \wedge U_j \longmapsto \sum_{\phi \in Ord_b(\dot{T}, [j])} sgn(\phi) \cdot R_\phi.$$

This is clearly well-defined and natural in $S \in \mathcal{FB}$.

Proposition 5 *The map $\iota : B_l(F) \longrightarrow B_r(F)$ is a monomorphism of chain complexes.*

Proof: First we remark that chains of the form R_ϕ for $\phi \in Ord_b(T, [j])$ are chains $[\hat{0} < R_1 < \ldots < R_j < \hat{1}]$ such that for all i, the equivalence relation R_{i+1}/R_i on S/R_i (given by the kernel pair of the quotient $S/R_i \longrightarrow S/R_{i+1}$, with $R_0 = \hat{0}$ and $R_{j+1} = \hat{1}$ for convenience) is an equivalence relation R_{V_i} associated with a subset $V_i \subseteq S/R_i$. Let us call chains which satisfy this property "good". To prove that ι is monic, it is enough to see that we can retrieve T and

$\phi : \hat{T} \longrightarrow [j]$ from the data of a good chain. This is easy: letting $\pi_i : S \longrightarrow S/R_i$ be the canonical quotient, $\hat{T}$ is the set of subsets $U_i = \pi_i^{-1}(V_i)$, and ϕ is defined by $\phi(U_i) = i$. To show that ι preserves differentials, consider the differential of R_ϕ:

$$\sum_{i=1}^{j}(-1)^i[\cdots < R_\phi^{(i-1)} < R_\phi^{(i+1)} < \cdots].$$

By the trichotomy property for trees, either the i^{th} summand is a good chain, or $\phi^{-1}(i) \cap \phi^{-1}(i+1)$ is empty. In the latter case, consider $\phi' \in Ord_b(\hat{T}, [j])$ obtained by composing ϕ with the transposition $(i\ i+1)$. Clearly $sgn(\phi') = -sgn(\phi)$, and one easily checks that the i^{th} summands in $\partial(R_\phi)$ and $\partial(R_{\phi'})$ are the same. Thus, in computing $\partial(\iota(U_1 \wedge \cdots \wedge U_j))$, the "bad" chains cancel: what remains is a linear combination of good chains R_{ϕ_i} where ϕ_i denotes the restriction of ϕ to $\hat{T} - \phi^{-1}(i)$. Specifically, we have

$$\partial(\iota(U_1 \wedge \cdots \wedge U_j)) = \sum_{\phi \in Ord_b(\hat{T},[j])} sgn(\phi) \cdot \sum_{i=1}^{j}(-1)^i R_{\phi_i}$$

and some painstaking care with signs shows this equals

$$\sum_{i=1}^{j}(-1)^i \sum_{\phi \in Ord_b(\widehat{T/U_i},[j-1])} sgn(\phi) \cdot R_\phi = \iota(\partial(U_1 \wedge \ldots \wedge U_j))$$

so that ι preserves differentials. This completes the proof. **QED**

Theorem 3 *For $F(X) = exp(X) - 1$, $B_r(F)$ and $B_l(F)$ are quasi-isomorphic.*

Proof: From theorem 2, $H_*(B_r(F)[n])$ is concentrated in the top degree $n-1$, and from the results of [7], so is $H_*(B_l(F)[n])$. Since this is top degree, both of these homologies are given by the cycle groups in that degree, and by proposition 5, the chain map ι restricts to a monomorphism

$$Z_{n-1}(B_l(F)[n]) \longrightarrow Z_{n-1}(B_r(F)[n])$$

between these cycle groups. On the other hand, both of these homologies are virtually equivalent since both represent the S_n-character $\log(1+X)[n]$. Since both are concentrated in the same degree, this virtual equivalence implies they are isomorphic. Therefore the monomorphism above must itself be an isomorphism. This completes the proof. **QED**

Now consider more generally an operad F such that $F[0] = 0$, $F[1] = 1$. To construct a chain map $\iota : B_l(F) \longrightarrow B_r(F)$, we couple the construction for the special case $F(X) = exp(X) - 1$ with a few simple observations. In this special case there are inclusions

$$\delta_F^{j+1}(X) \hookrightarrow F^{\circ(j+1)} \hookrightarrow \mathcal{O}(F)$$

so that each chain $\hat{0} < R_1 < \ldots < R_j < \hat{1}$ of equivalence relations on S may be regarded as a (non-proper) tree on S. Abbreviating $\delta_{exp(X)-1}$ to δ, we have for general F

$$\delta_F^{j+1}(X)[S] = \sum_{T\in\delta^{j+1}(X)[S]} \bigotimes_{x\in N(T)-S} F[\sigma_x]$$

Let $T \in \mathcal{T}_j[S]$ be a proper tree and consider a map $\phi \in Ord_b(\hat{T}, [j])$; earlier we produced a chain R_ϕ of equivalence relations, which we now regard as a tree. Observe that there is an isomorphism

$$\bigotimes_{x\in N(T)-S} \delta_F(X)[\sigma_x] \cong \bigotimes_{x\in N(R_\phi)-S} F[\sigma_x]$$

since we assumed $F[1] = 1$. From this isomorphism, it is clear that there is an induced chain map

$$\sum_{T\in\mathcal{T}_j[S]} \Lambda[T] \otimes \bigotimes_{x\in N(T)-S} \delta_F(X)[\sigma_x] \longrightarrow \sum_{T\in\delta^{j+1}(X)[S]} \bigotimes_{x\in N(T)-S} F[\sigma_x]$$

which gives the desired map $\iota : B_l(F) \longrightarrow B_r(F)$.

It is not difficult to show, using theorem 3, that ι induces an isomorphism in homology. This may suggest new techniques for deciding whether a given quadratic operad satisfies Koszul duality ([7]). For example, in certain cases one can establish Koszul duality through an appeal to theorem 2.

11 The Lie operad

Let $F(X) = exp(X) - 1$, and let $B_l(F)'$ denote the dual of the complex $B_l(F)$, i.e., the cochain complex with differentials

$$\sum_{T\in\mathcal{T}_{j-1}[S]} \Lambda[T]' \longrightarrow \sum_{T\in\mathcal{T}_j[S]} \Lambda[T]'$$

obtained by transposing those of $B_l(F)[S]$. In [7] it is shown that the operad multiplication and unit on $\mathcal{O}(-\delta_F(X))$, which is the graded species underlying $B_l(F)'$, respect these differentials, so that $B_l(F)'$ carries a structure of an operad valued in the category of $\mathbf{Z}_2$-graded cochain complexes. It is called the *cobar* construction of F, and may be regarded as a lift of the virtual species $F^{-1}(X) = \log(1 + X)$ to an operad.

Since the operation of additive inverse on $\mathcal{V}[[x]]$ lifts to the $\mathbf{Z}_2$-graded suspension (as an involution on $\mathcal{V}^{\mathcal{FB}}$ or on species valued in $\mathbf{Z}_2$-graded complexes), we can transport the monad $B_l(F)' \circ (-)$ across the suspension to lift the virtual species $-F^{-1}(-X) = -\log(1-X)$ to an operad. We can describe this operad using proposition 3. First, there is a $\mathcal{V}$-operad with components

$$(-1)^{|S|-1}\Lambda[S],$$

obtained by transporting the monad *exp* for free commutative monoids across the $\mathbf{Z}_2$-graded suspension. This is described in detail in [7], where it is called the determinantal operad. Next, if F and G are operads, then there is an obvious operad structure on the species with components $F[S] \odot G[S]$. Putting this together, there is an identification between operads given componentwise as

$$-B_l(F)'(-X)[S] \cong B_l(F)'[S] \odot (-1)^{|S|-1}\Lambda[S]$$

and since cohomology with coefficients in a field preserves coproducts and tensor products, there is an operad whose components are

$$H^{n-1}(B_l(F)[n]) \odot \Lambda[n]$$

in degree 0 (compare with corollary 2 and the remarks which follow it).

Theorem 4 *([7]) The Lie operad L is isomorphic to this operad.*

Sketch of proof: The cohomology space above is a cokernel of a space generated by the set of binary trees in $\mathcal{T}[n]$, so that the operad above is an operad generated by a single binary operation (represented by a sprout with two leaves, called a 2-sprout). The kernel of this cokernel is the space of trees in $\mathcal{T}[n]$ obtained by grafting a collection of 2-sprouts with a 3-sprout (which represents a ternary operation), so that the operad $H^*(B_l(F))$ is generated by a binary operation and subject to a single equation in which the ternary operation is set equal to zero. Upon twisting with the determinantal operad in degree 3, one may verify by hand that this equation is the Jacobi identity. Twisting with the determinantal operad in degree 2 gives the alternating identity, and these give a complete set of identities for the operad. But these are exactly the identities governing the Lie operad. **QED**

- **Remark:** In [1], an explicit description is given for the correspondence between certain elements in free Lie algebras (Lyndon basis elements) and homology classes for partition lattices. Our notes here provide a conceptual framework for this description, which can be extracted from theorems 3 and 4 (compare with corollary 2).

Definition 8 *A (Z_2-graded) homotopy Lie algebra is an algebra over the operad*

$$(-1)^{n-1}B_l(F)'[n] \odot \Lambda[n]$$

(which is valued in the category of Z_2-graded complexes).

- **Remark:** If $F(X) = \frac{X}{1-X}$ is the operad governing associative algebras without unit, then there is a virtual equivalence $F(X) \sim -F^{-1}(-X)$. The right-hand side is represented by an operad with components

$$(-1)^{|S|-1} B_l(F)'[S] \otimes \Lambda[S],$$

which is easily (and classically) shown to be homotopy-equivalent to the operad $\frac{X}{1-X}$. Algebras over this (right-hand) operad are called homotopy associative algebras; they are equivalent to the A_∞-algebras of [15].

References

[1] Helene Barcelo. *On the Action of the Symmetric Group on the Free Lie Algebra and the Partition Lattice.* PhD thesis, U.C. San Diego, 1988.

[2] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lecture Notes in Mathematics*. Springer-Verlag, 1973.

[3] B. J. Day. On closed categories of functors. In *Lecture Notes in Mathematics 137*, pages 1–38. Springer-Verlag, 1970.

[4] S. Eilenberg and S. Mac Lane. On the groups $H(\pi, n)$, I. *Ann. Math.*, 58(1):55–106, 1953.

[5] J. Folkman. The homology groups of a lattice. *J. Math. Mech.*, 15:631–636, 1966.

[6] E. Getzler and J. D. S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces. Available at hep-th/9403055, 1994.

[7] V. A. Ginzburg and M. M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76:203–272, 1994.

[8] V. Hinich and V. Schechtman. Homotopy Lie algebras. *Adv. Soviet Math.*, 16(2):1–28, 1993.

[9] A. Joyal. Une théorie combinatoire des séries formelles. *Advances in Mathematics*, 42(1):1–82, 1981.

[10] A. Joyal. Foncteurs analytiques et especes de structure. In *Lecture Notes in Mathematics 1234*, pages 126–159. Springer-Verlag, 1985.

[11] A. Joyal and R. Street. Braided tensor categories. *Advances in Mathematics*, 102:20–78, 1993.

[12] G. M. Kelly. On clubs and doctrines. In *Lecture Notes in Mathematics 420*, pages 181–250. Springer-Verlag, 1974.

[13] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Math. Soc. Lecture Note Series*. Cambridge University Press. 1982.

[14] M. Markl. A compactification of the real configuration space: resolution of diagonals is the operadic completion. Preprint, 1996.

[15] J. D. Stasheff. Homotopy associativity of H-spaces. II. *Trans. Amer. Math. Soc.*, 108:293–312, 1963.

[16] Y. N. Yeh. The calculus of virtual species and K-species. In *Lecture Notes in Mathematics 1234*, pages 351–369. Springer-Verlag, 1985.