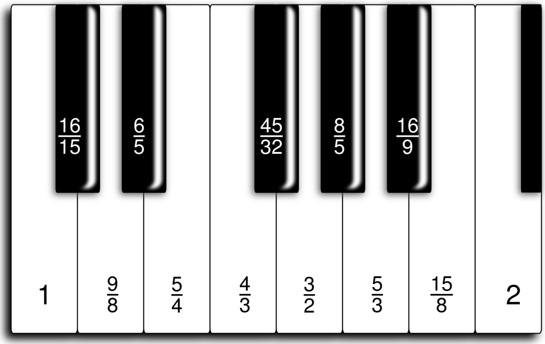
THE MATHEMATICS OF TUNING SYSTEMS

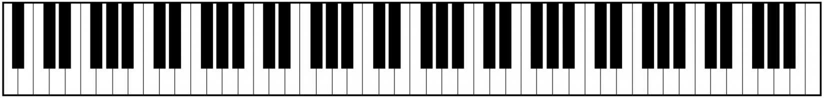


John Baez

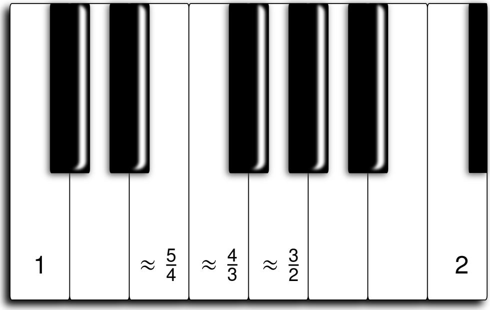
January 30, 2026

Mathematics Colloquium of the Claremont Colleges

The modern 88-key piano says a lot about how we think about music.



We enjoy pairs of tones — called “intervals” — whose frequency ratios are simple fractions.



The ratio of 2 plays a special role in music. We call it an “octave”. Two tones differing by an octave sound “the same, only different”. The piano keyboard repeats every octave.

Around 500 AD, the philosopher Boethius wrote *De institutione musica*. He used Latin letters A, B, C, D, E, . . . , O for the notes that a male voice could sing. So A was the lowest note a male voice could sing, supposedly.

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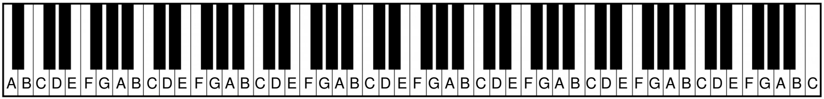
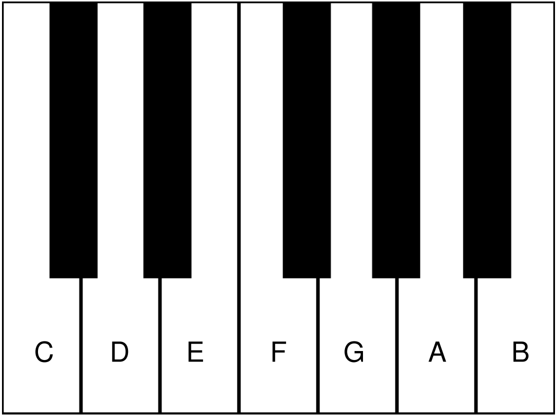
The lowest note on a modern piano is still called A, but now we repeat the letters after each octave.

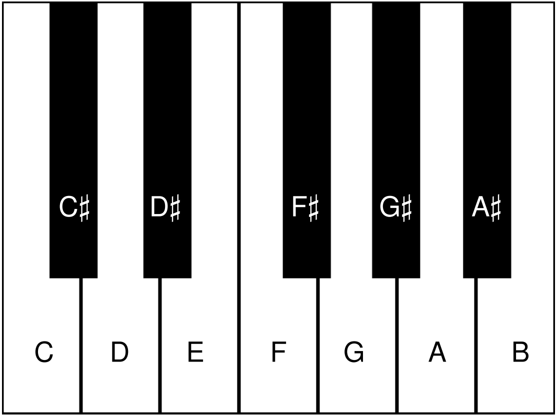


Thus, our notes are called A, B, C, D, E, F, G.

But playing the white notes starting at A produces a “minor scale”. Around the 1700s, the “major scale” became dominant, and we get that if we start with C:



Starting around 1200, people began playing more notes outside the 7-tone scale. In the key of C, they are the black keys here.



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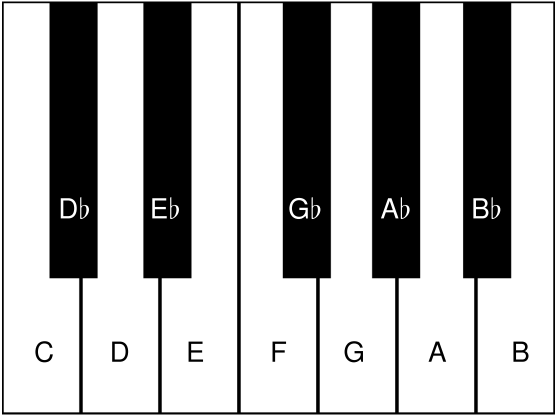

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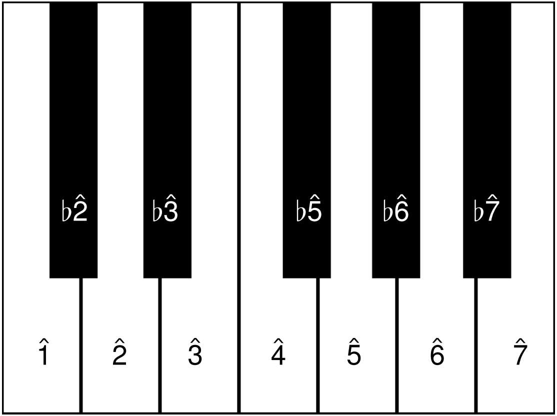
Musicians also use numbers for notes, and I’ll do that.



Equal temperament

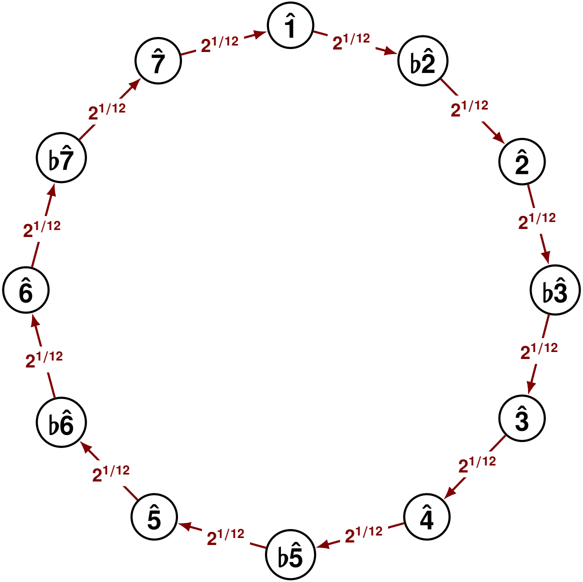
In “*N*-tone equal temperament” we choose notes that divide the octave equally into N pieces.

What matters are frequency *ratios*, so this means that each note has a frequency $2^{1/N}$ times that of the previous (lower) note.

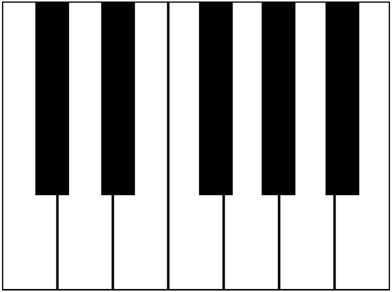
Right now almost all keyboard instruments and digital audio workstations use 12-tone equal temperament by default — though some can handle other tuning systems.

String instruments are different!

12-tone equal temperament



Why do we use a 12-tone scale? The *historical* reasons are hard to unearth. Many cultures have scales with 5, 7, or 12 tones.



Our modern scale actually combines all three!

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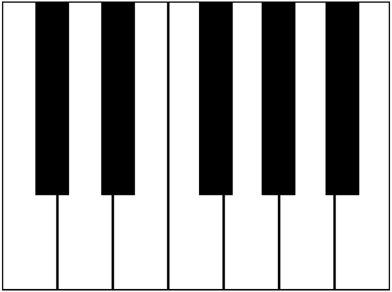
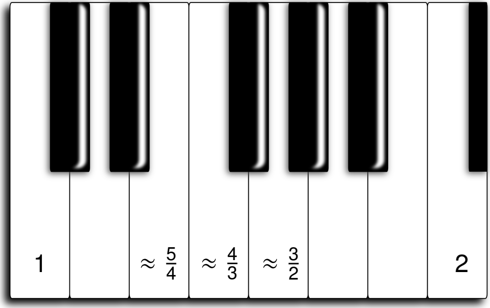


Our modern scale actually combines all three!

We can understand why this is good using math.

After the octave, the simplest interval is the “fifth”, with frequency ratio of 3/2.

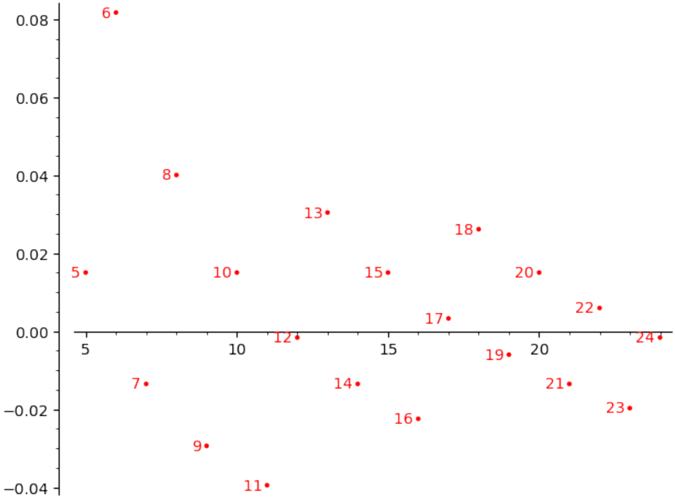


Equal-tempered scales with 5, 7 or 12 notes give especially good approximations to 3/2.

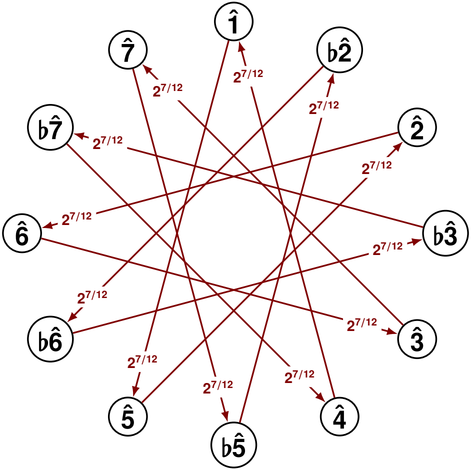
Best approximations to 3/2 in equal-tempered scales.
Scales in red are better than any scale above.

				error
2-TET	$2^{1/2}$	=	1.41421	–5.7%
3-TET	$2^{2/3}$	=	1.58740	+5.8%
4-TET	$2^{2/4}$	=	1.41421	–5.7%
5-TET	$2^{3/5}$	=	1.51572	+1.1%
6-TET	$2^{4/6}$	=	1.58740	+5.8%
7-TET	$2^{4/7}$	=	1.48599	–0.9%
8-TET	$2^{5/8}$	=	1.54221	+2.8%
9-TET	$2^{5/9}$	=	1.46973	–2.0%
10-TET	$2^{6/10}$	=	1.51572	+1.1%
11-TET	$2^{6/11}$	=	1.45948	–2.7%
12-TET	$2^{7/12}$	=	1.49831	–0.1%
29-TET	$2^{17/29}$	=	1.50129	+0.09%

Scott Centoni worked out how close you can get to 3/2 with a number of the form $2^{M/N}$.

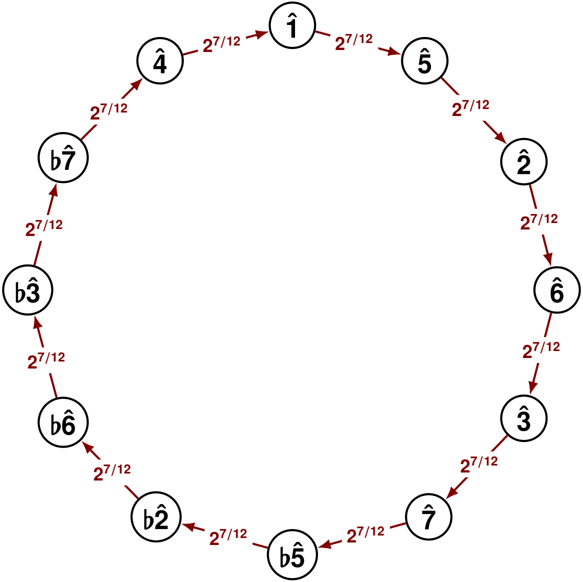


12-tone equal temperament, showing fifths



equal tempered fifth = $2^{7/12} \approx 1.49831$

12-tone equal temperament - circle of fiths

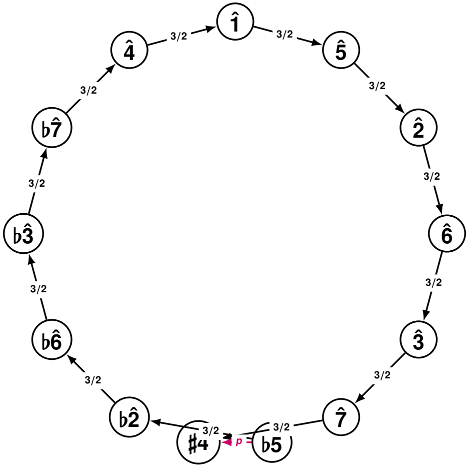


Starting with medieval music, the main Western tuning systems are these:

- **Pythagorean tuning: 800 – 1300 AD.** This uses only powers of $\frac{3}{2}$.
- **Just intonation: 1300 – 1550 AD.** This uses products of powers of $\frac{3}{2}$ and $\frac{5}{4}$.
- **Quarter-comma meantone: 1550 – 1690 AD.** This uses $\sqrt[4]{5}$ to get lots of thirds that are exactly $5/4$.
- **“Well-tempered” systems: 1690 – 1850 AD.** These are quite diverse and complicated.
- **Equal temperament: 1850 – now.** This uses only powers of $2^{1/12}$.

These dates are just rough indications, of course.

Pythagorean tuning with both $\sharp\hat{4}$ and $\flat\hat{5}$

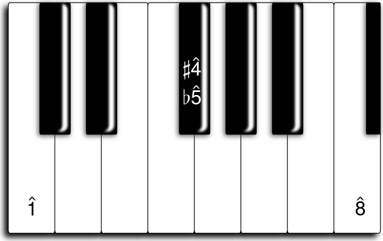


$$p = \textbf{Pythagorean comma} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1.0136$$

In equal temperament we have

$$\sharp\hat{4} = \flat\hat{5} = \sqrt{2} \approx 1.4142$$

Halfway up the octave, this is called the “tritone” or “*diabolus in music*”.



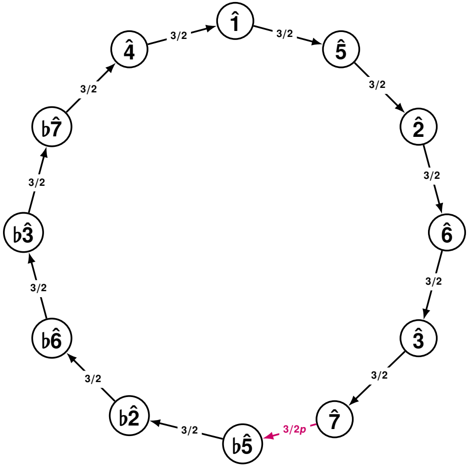
In Pythagorean tuning we have

$$\flat 5 = \left(\frac{3}{2}\right)^{6} \cdot 2^{-3} = \frac{729}{512} \approx 1.4238$$

$$\sharp 4 = \left(\frac{3}{2}\right)^{-6} \cdot 2^{4} = \frac{1024}{729} \approx 1.4047$$

and the ratio of these is the Pythagorean comma.

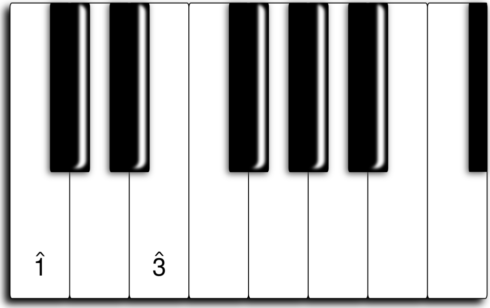
Pythagorean tuning with only $\flat\hat{5}$



$$3/2p = \textbf{Pythagorean wolf fifth} = \frac{2^{18}}{3^{11}} = \frac{262144}{177147} \approx 1.4798$$

The “wolf fifth” in Pythagorean tuning goes from the $\hat{7}$ to the $\flat\hat{5}$. This was easily avoided in medieval music. A bigger problem: some important intervals get complicated fractions.

Ideally the “third” ($\hat{3}$) has a frequency ratio of $\frac{5}{4} = 1.25$.



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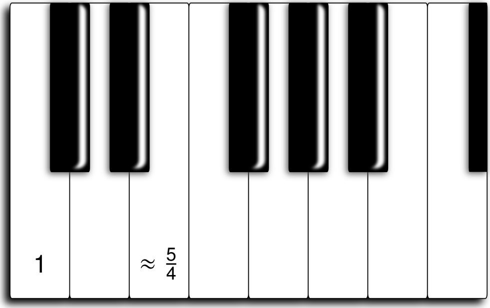

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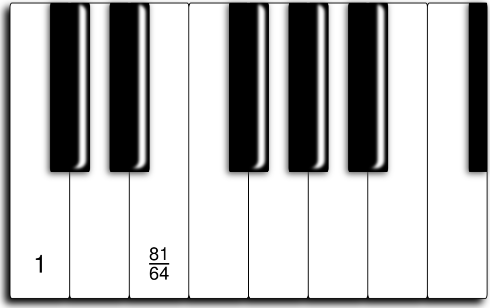

But in Pythagorean tuning the third has a ratio of $\frac{81}{64} \approx 1.266$ — noticeably too high.

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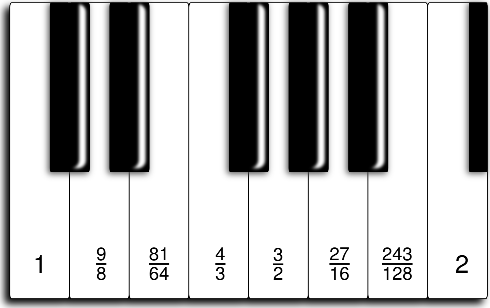

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By around 1300, music with thirds became so important that "just intonation" replaced Pythagorean tuning. This system, discussed by Ptolemy already in 140 AD, puts the fractions $\frac{3}{2}$ and $\frac{5}{4}$ on an equal footing.

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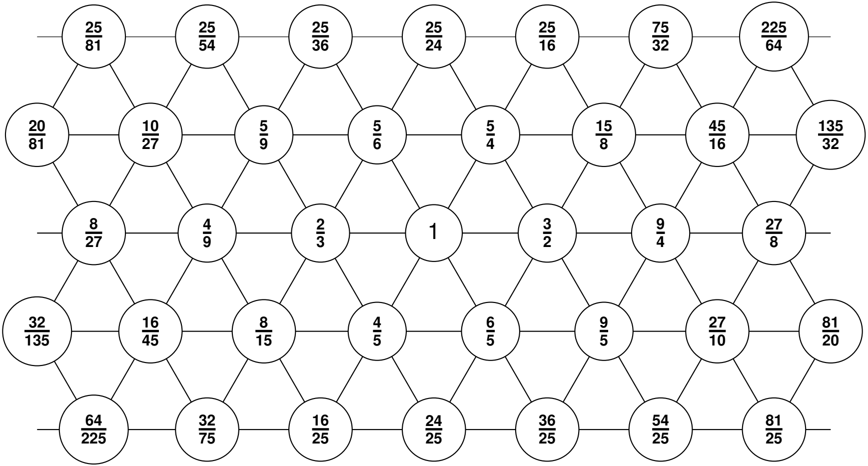
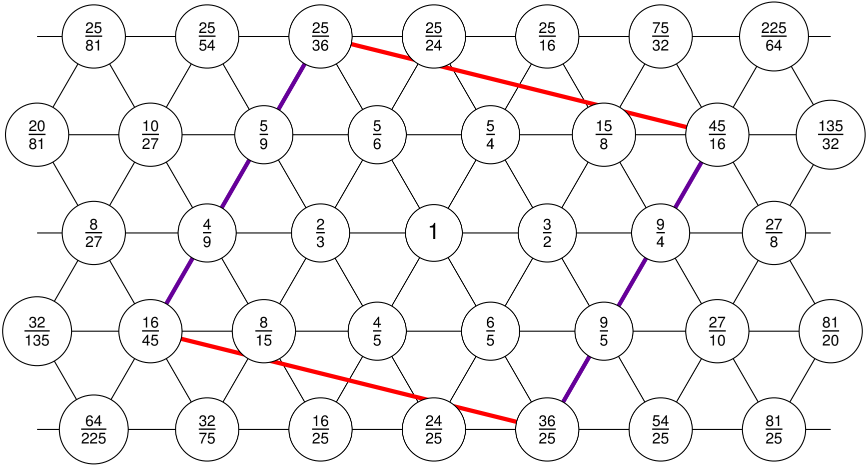
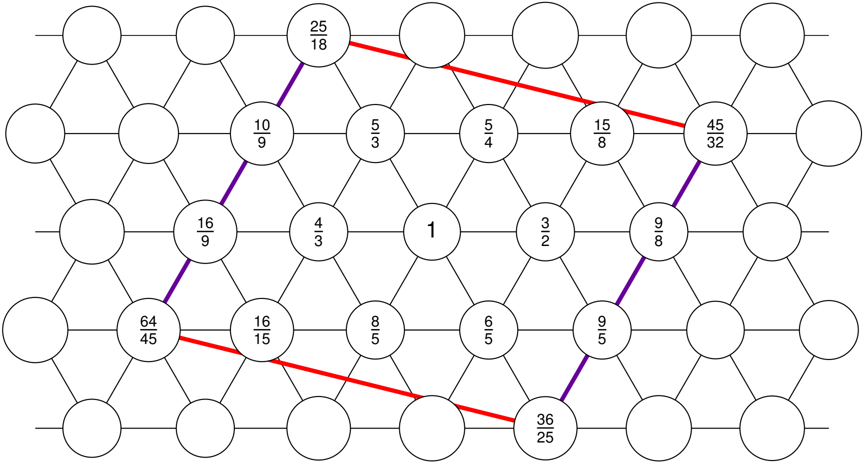


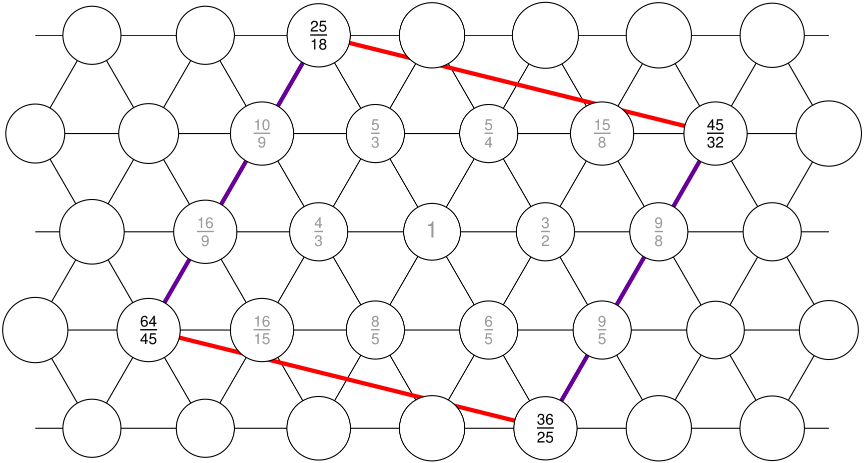
We need to cut out a portion of this infinite grid to use for our scale.



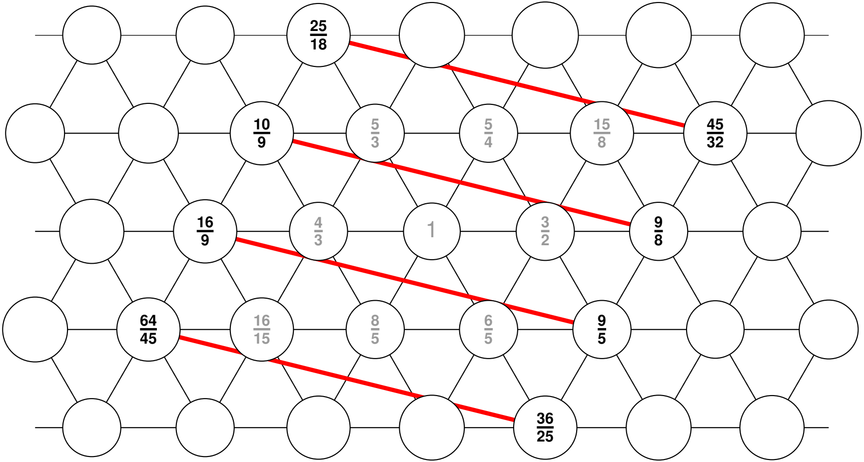
Multiplying by suitable powers of 2, we get numbers between 1 and 2.



Numbers at the corners are all close to $\sqrt{2}$. We must pick one to be our tritone: we have 4 choices.



Numbers on the left and right edges differ by a factor of 81/80. For each of the two middle pairs we must pick one to be in our scale, so again we have 4 choices.



Choices of notes in just intonation

$\hat{1}$	1
$\flat\hat{2}$	16/15
$\hat{2}$	10/9 or 9/8
$\flat\hat{3}$	6/5
$\hat{3}$	5/4
$\hat{4}$	4/3
$\flat\hat{5}$	25/18 or 45/32 or 65/45 or 36/25
$\hat{5}$	3/2
$\flat\hat{6}$	8/5
$\hat{6}$	5/3
$\flat\hat{7}$	16/9 or 9/5
$\hat{7}$	15/8
$\hat{8}$	2

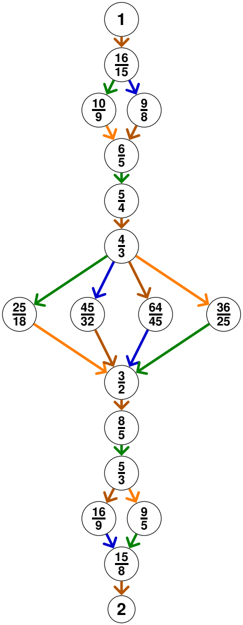
All 16 scales in just intonation:

LESSER CHROMATIC SEMITONE = $\frac{25}{24}$

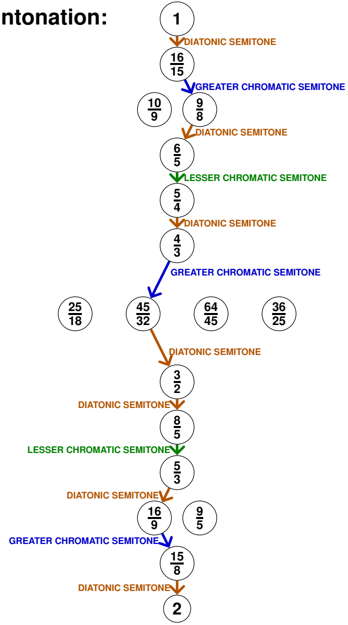
GREATER CHROMATIC SEMITONE = $\frac{135}{128}$

DIATONIC SEMITONE = $\frac{16}{15}$

LARGE DIATONIC SEMITONE = $\frac{27}{25}$



The most commonly used scale in just intonation:



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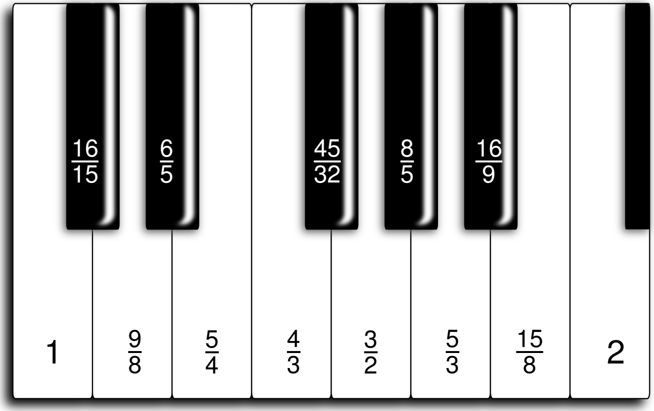
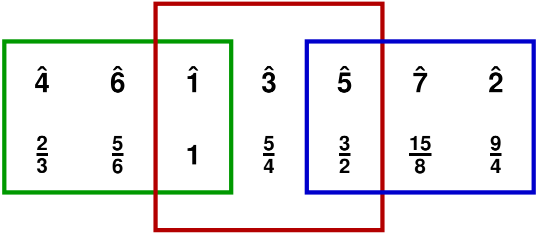
This particular just intonation scale is the most widely used:



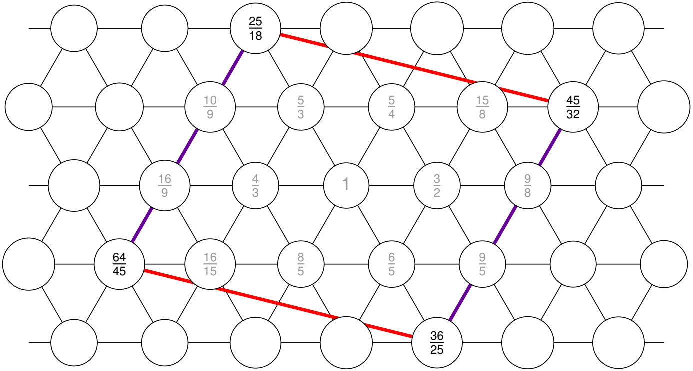
The white notes are Ptolemy's "intense diatonic scale".

For this scale, the notes without sharps or flats form three "major triads", each with frequency ratio $1 : \frac{5}{4} : \frac{3}{2}$.

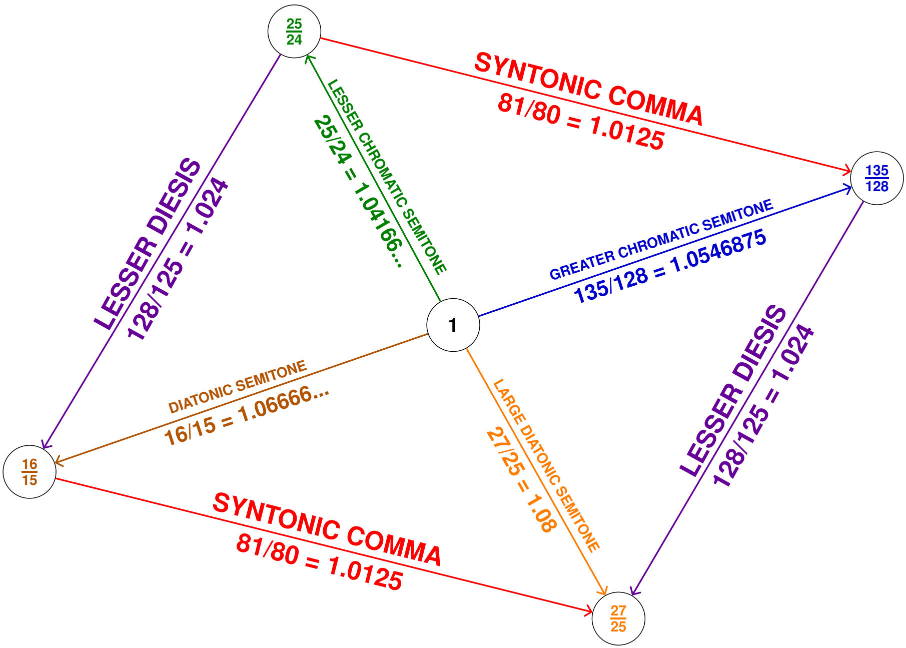


Of the four choices of tritone:

- two pairs are separated by the syntonic comma, $\frac{81}{80}$
- two are separated by the lesser diesis, $\frac{128}{125}$.



These numbers show up, secretly, in all of the 16 just intonation scales!


25/24
135/128
1
16/15
27/25
SYNTONIC COMMA
81/80 = 1.0125
LESSER CHROMATIC SEMITONE
25/24 = 1.04166...
GREATER CHROMATIC SEMITONE
135/128 = 1.0546875
LESSER DIESIS
128/125 = 1.024
DIATONIC SEMITONE
16/15 = 1.06666...
LARGE DIATONIC SEMITONE
27/25 = 1.08
LESSER DIESIS
128/125 = 1.024
SYNTONIC COMMA
81/80 = 1.0125

Starting in the Renaissance, the increasing desire to change keys pushed musicians to embrace tuning systems that aren't optimized for just one key:

- **Quarter-comma meantone: 1550 – 1690 AD.**
- **"Well-tempered" systems: 1690 – 1850 AD.**
- **Equal temperament: 1850 – now.**

All these are mathematically interesting.

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All these are mathematically interesting. What's next?

- Xenharmonics — tuning systems outside 12-tone equal temperament.
- Adaptive tuning — music where the tuning changes as it plays.
- ??? — things we haven't imagined yet.