Higher Gauge Theory: 2-Connections

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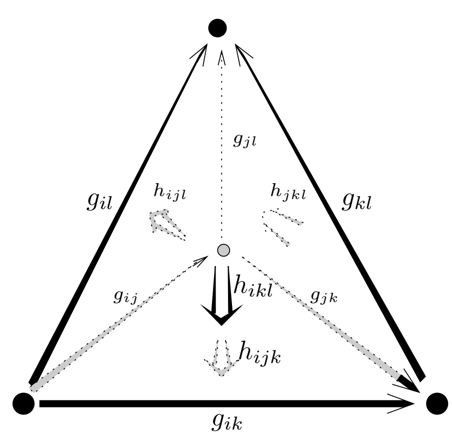


More details at:

http://math.ucr.edu/home/baez/union/

Gauge Theory

Ordinary gauge theory describes how point particles transform as they move along paths in spacetime:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$$

It is natural to assign a group element to each path:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

since composition of paths then corresponds to multiplication:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \end{tikzcd}$$

while reversing the direction of a path corresponds to taking inverses:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g^{-1}"'] \end{tikzcd}$$

and the associative law makes the holonomy along a triple composite unambiguous:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \arrow[r, bend left, "g''"] & \bullet \end{tikzcd}$$

In short: *the topology dictates the algebra!*

The electromagnetic field is described using the group U(1). Other forces are described using other groups.

Higher Gauge Theory

Higher gauge theory describes not just how point particles transform, but also how 1-dimensional strings transform as we move them. For this we must categorify the notion of a group! A '2-group' has objects:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

and also morphisms:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "f"] \end{tikzcd}$$

We can multiply objects:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \end{tikzcd}$$

multiply morphisms:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"{name=U1}] \arrow[r, bend right, "g_1'"'{name=D1}] & \bullet \arrow[r, bend left, "g_2"{name=U2}] \arrow[r, bend right, "g_2'"'{name=D2}] & \bullet \arrow[from=U1, to=D1, Rightarrow, "f_1"'] \arrow[from=U2, to=D2, Rightarrow, "f_2"'] \end{tikzcd}$$

and also compose morphisms:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g"{name=U}] \arrow[r, "g'"{name=M}] \arrow[r, bend right=60, "g''"'{name=D}] & \bullet \arrow[from=U, to=M, Rightarrow, "f"] \arrow[from=M, to=D, Rightarrow, "f'"] \end{tikzcd}$$

Various laws should hold... all obvious from the pictures.

We can make this precise and categorify all of gauge theory! But first, let's review ordinary gauge theory.

The Holonomy Along a Path

Let's work in a convenient category of 'smooth spaces', including smooth manifolds as a full subcategory, but cartesian closed, with all limits and colimits. For details, see the paper with Urs Schreiber on my website!

Let M be a smooth space and let G be a smooth group (e.g. a Lie group). Let $\mathfrak{g}$ be the Lie algebra of G.

We want to compute a group element for each path. We can do this using a $\mathfrak{g}$-valued 1-form A on M. For any path $\gamma\colon [t_0, t_1] \to M$, this gives an element $\mathrm{hol}(\gamma) \in G$ called the **holonomy** of A along this path, as follows.

Solve this differential equation:

$$\frac{d}{dt} g(t) = A(\gamma'(t)) g(t)$$

with initial value $g(t_0) = 1$. Then let:

$$\mathrm{hol}(\gamma) = g(t_1).$$

Holonomy as a Functor

The holonomy along a path doesn't depend on its parametrization. When we compose paths, their holonomies multiply:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$$

When we reverse a path, we get a path with the inverse holonomy:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right] \end{tikzcd}$$

So, let $\mathcal{P}_1(M)$ be the **path groupoid** of M:

- objects are points $x \in M$: $\quad \bullet_x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$:

$$\begin{tikzcd} x\,\bullet \arrow[r, bend left, "\gamma"] & \bullet\, y \end{tikzcd}$$

This is a **smooth groupoid**: it has a smooth space of objects, a smooth space of morphisms, and all the groupoid operations are smooth.

Theorem. There is a one-to-one correspondence between smooth functors

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G$$

and $\mathfrak{g}$-valued 1-forms A on M.

Bundles

The story so far is oversimplified. It's evil to demand that holonomies *are* group elements – we should only demand that each point in M have a *neighborhood* in which holonomies *can be regarded as* group elements.

So, define a **bundle** over M to be:

- a smooth space P (the **total space**),
- a smooth space F (the **standard fiber**),
- a smooth map $p\colon P \to M$ (the **projection**),

such that for each point $x \in M$ there exists an open neighborhood U equipped with a diffeomorphism

$$f\colon p^{-1}U \to U \times F$$

(the **local trivialization**) such that

$$\begin{tikzcd} p^{-1}U \arrow[rr, "f"] \arrow[dr, "p|_{p^{-1}U}"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commutes.

Principal Bundles

If F is a smooth space, $\mathrm{Aut}(F)$ is a smooth group. If $P \to M$ is a bundle with standard fiber F, the local trivializations over neighborhoods U_i covering M give smooth maps (**transition functions**):

$$g_{ij} \colon U_i \cap U_j \to \mathrm{Aut}(F)$$

such that:

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

for any $x \in U_i \cap U_j \cap U_k$. In other words, this diagram commutes:

$$\begin{tikzcd} & \bullet \arrow[dr, "g_{jk}"] & \\ \bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & & \bullet \end{tikzcd}$$

For any smooth group G, we say the bundle $P \to M$ has G as its **structure group** when the maps g_{ij} factor through an action $G \to \mathrm{Aut}(F)$.

If furthermore $F = G$ and G acts on F by right multiplication, we say P is a **principal G-bundle**.

Connections

Suppose $P \to M$ is a principal G-bundle equipped with local trivializations over neighborhoods U_i covering M. A **connection** on P consists of: a smooth functor

$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$

for each i, such that the transition function g_{ij} defines a smooth natural isomorphism:

$$\mathrm{hol}_i|_{\mathcal{P}_1(U_i \cap U_j)} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

for all i, j. In other words, this diagram commutes:

$$\begin{tikzcd} \bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "\mathrm{hol}_i(\gamma)"'] & \bullet \arrow[d, "\mathrm{hol}_j(\gamma)"] \\ \bullet \arrow[r, "g_{ij}(y)"'] & \bullet \end{tikzcd}$$

for any path $x \xrightarrow{\gamma} y$ in $U_i \cap U_j$.

Theorem. There is a one-to-one correspondence between connections on the principal G-bundle $P \to M$ and $\mathfrak{g}$-valued 1-forms A_i on the open sets U_i satisfying

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}$$

on the intersections $U_i \cap U_j$.

Higher Gauge Theory

Now let's categorify everything in sight and get a theory of *2-connections on principal 2-bundles!*

The crucial trick is 'internalization'. Given a familiar gadget x and a category K, we define an 'x in K' by writing the definition of x using commutative diagrams and interpreting these in K.

We need examples where $K = \mathrm{C}^\infty$ is the category of smooth spaces:

- A **smooth group** is a group in C^∞.
- A **smooth groupoid** is a groupoid in C^∞.
- A **smooth category** is a category in C^∞.
- A **smooth 2-group** is a 2-group in C^∞.
- A **smooth 2-groupoid** is a 2-groupoid in C^∞.

A category with all morphisms invertible is a groupoid. A groupoid with one object is a group. A 2-category with all morphisms and 2-morphisms invertible is a **2-groupoid**. A 2-groupoid with one object is a **2-group**.

Here 2-groups and 2-groupoids come in two flavors: *strict* and *weak*. In the former all laws hold as equations; in the latter, they hold up to specified isomorphisms which satisfy coherence laws of their own. For details, see the paper with Aaron Lauda on my website!

2-Bundles

Toby Bartels has developed a theory of 2-bundles, which we sketch here.

We can think of a smooth space M as a smooth category with only identity morphisms. A **2-bundle** over M consists of:

- a smooth category P (the **total space**),
- a smooth category F (the **standard fiber**),
- a smooth functor $p\colon P \to M$ (the **projection**),

such that each point $x \in M$ is equipped with an open neighborhood U and a smooth equivalence:

$$f\colon p^{-1}U \to U \times F$$

(the **local trivialization**) such that:

$$\begin{tikzcd} p^{-1}U \arrow[rr, "f"] \arrow[dr, "p|_{p^{-1}U}"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commutes.

Principal 2-Bundles

Theorem. Let F be a smooth category and $\mathcal{G} = \mathrm{Aut}(F)$ its smooth 2-group of self-equivalences. Given a 2-bundle $P \to M$ with standard fiber F, the local trivializations over open sets U_i covering M give:

- smooth maps
$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$
- smooth maps
$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$
  with
$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

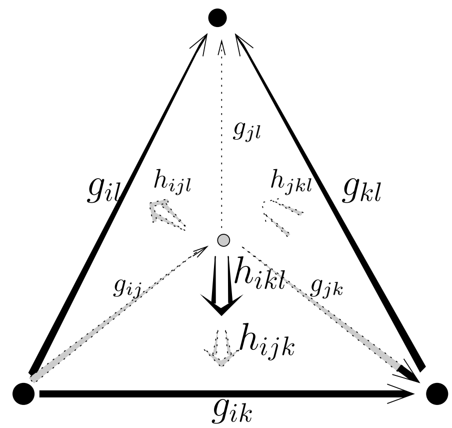
$$\begin{tikzcd}
 & \bullet \arrow[dr, "g_{jk}"] \arrow[d, phantom, "h_{ijk}\Downarrow"] & \\
\bullet \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & {} & \bullet
\end{tikzcd}$$

such that:

$$\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] \arrow[dr, phantom, "h_{ijk}\Searrow" near start] & \bullet \arrow[d, "g_{kl}"] \\
\bullet \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] \arrow[ur, phantom, "h_{ikl}\Searrow"' near end] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"'] \arrow[dr, phantom, "\Swarrow h_{jkl}" near start, shift left=3] & \bullet \arrow[d, "g_{kl}"] \\
\bullet \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"'] \arrow[ur, phantom, "\Swarrow h_{ijl}" near start] & \bullet
\end{tikzcd}$$

on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

In other words, this diagram commutes:



This is a 'nonabelian 2-cocycle condition'. In general, we expect that $\mathcal{G}$-n-bundles will be classified by the nth nonabelian Čech cohomology with coefficients in the smooth n-group $\mathcal{G}$.

For any smooth 2-group $\mathcal{G}$, we say a 2-bundle $P \to M$ has $\mathcal{G}$ as its **structure 2-group** when g_{ij} and h_{ijk} factor through an action $\mathcal{G} \to \mathrm{Aut}(F)$.

If furthermore $F = \mathcal{G}$ and $\mathcal{G}$ acts on F by left multiplication, we say P is a **principal $\mathcal{G}$-2-bundle**.

The Path 2-Groupoid

Just as connections on bundles involve the path groupoid, 2-connections on 2-bundles involve the **path 2-groupoid** $\mathcal{P}_2(M)$ of a smooth space M:

- objects are points of M: $\bullet_x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$:

$$\begin{tikzcd} {}_x\bullet \arrow[r, bend left, "\gamma"] & \bullet_y \end{tikzcd}$$

- 2-morphisms are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ such that $f(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:

$$\begin{tikzcd} {}_x\bullet \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow f"] & \bullet_y \end{tikzcd}$$

This is a strict smooth 2-groupoid!

Holonomy as a 2-Functor

Now for simplicity let's assume $\mathcal{G}$ is *strict*. A strict smooth 2-group $\mathcal{G}$ is determined by:

- the smooth group G consisting of all objects of $\mathcal{G}$,
- the smooth group H consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t\colon H \to G$ sending each morphism in H to its target,
- the action α of G on H defined using conjugation in the group $\mathrm{Mor}(\mathcal{G})$ via

$$\alpha(g)h = 1_g h {1_g}^{-1}$$

The system (G, H, t, α) satisfies equations making it a 'crossed module'. Conversely, any crossed module of smooth groups gives a strict smooth 2-group. Differentiating everything in a crossed module we get $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$.

Theorem. When $\mathcal{G}$ is strict, there is a one-to-one correspondence between smooth 2-functors

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

and pairs (A, B) consisting of a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M satisfying the **fake flatness** condition:

$$dA + A \wedge A + dt(B) = 0$$

2-Connections

Let $\mathcal{G}$ be a strict smooth 2-group and let $P \to M$ be a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets U_i covering M. Then a **2-connection** on E consists of the following data:

- For each i a smooth 2-functor:

$$\begin{array}{ccccc} \mathrm{hol}_i & : & \mathcal{P}_2(U_i) & \to & \mathcal{G} \\ & & \begin{tikzcd} x \arrow[r, bend left=40, "\gamma"{name=U}] \arrow[r, bend right=40, "\eta"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\Sigma"] & y \end{tikzcd} & \mapsto & \begin{tikzcd} \bullet \arrow[r, bend left=40, "\mathrm{hol}_i(\gamma)"{name=U}] \arrow[r, bend right=40, "\mathrm{hol}_i(\eta)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mathrm{hol}_i(\Sigma)"] & \bullet \end{tikzcd} \end{array}$$

- For each i, j a pseudonatural isomorphism:

  $$\mathrm{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \xrightarrow{g_{ij}} \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

  extending the transition function g_{ij}. In other words, for each path $x \xrightarrow{\gamma} y$ in U_{ij} a morphism in $\mathcal{G}$:

  $$g_{ij}(\gamma)\colon \mathrm{hol}_i(\gamma)\, g_{ij}(y) \to g_{ij}(x)\mathrm{hol}_j(\gamma)$$

  depending smoothly on γ, such that this diagram commutes for any $\Sigma\colon \gamma \Rightarrow \eta$ in $\mathcal{P}_2(U_i \cap U_j)$:

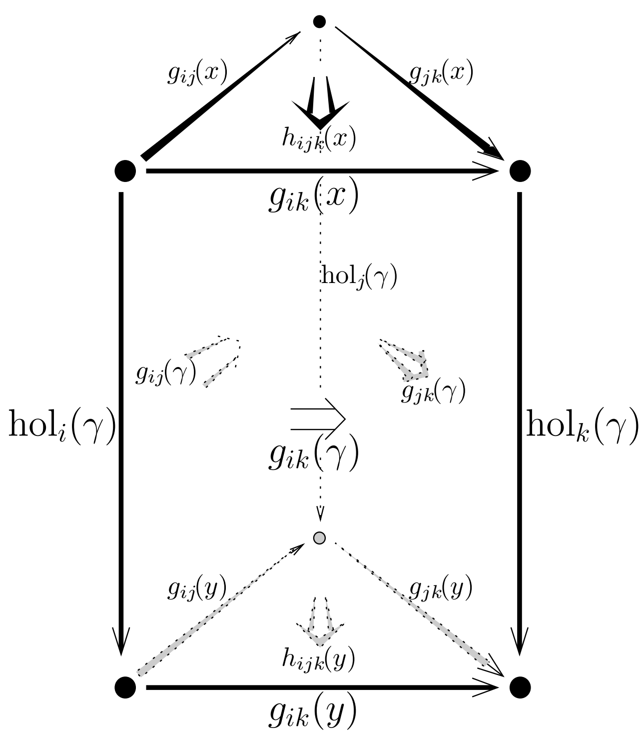
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, bend right=40, "\mathrm{hol}_i(\gamma)"'{name=A}] \arrow[d, bend left=40, dotted, "\mathrm{hol}_i(\eta)"{name=B}] \arrow[Rightarrow, dotted, from=A, to=B, "\mathrm{hol}_i(\Sigma)"] \arrow[dr, phantom, "\Nearrow g_{ij}(\gamma)"] \arrow[dr, phantom, shift right=4ex, "\Nearrow g_{ij}(\eta)"']
& \bullet \arrow[d, bend right=40, "\mathrm{hol}_j(\gamma)"'{name=C}] \arrow[d, bend left=40, "\mathrm{hol}_j(\eta)"{name=D}] \arrow[Rightarrow, from=C, to=D, "\mathrm{hol}_j(\Sigma)"] \\
\bullet \arrow[r, "g_{ij}(y)"'] & \bullet
\end{tikzcd}$$

such that:

- for each i, j, k the function h_{ijk} defines a modification:

$$\begin{tikzcd}
& \mathrm{hol}_j \arrow[dr, "g_{jk}"] \arrow[d, phantom, "h_{ijk}\Downarrow"] & \\
\mathrm{hol}_i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & {} & \mathrm{hol}_k
\end{tikzcd}$$

  In other words, this diagrams commutes for any $\Sigma\colon \gamma \Rightarrow \eta$ in $\mathcal{P}_2(U_i \cap U_j \cap U_k)$:



Theorem. For any strict smooth 2-group $\mathcal{G}$, suppose that $P \to M$ is a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets U_i covering M. Then there is a one-to-one correspondence between 2-connections on P and Lie-algebra-valued differential forms (A_i, B_i, a_{ij}) satisfying certain equations, as follows:

- The holonomy 2-functor hol_i is specified by an $\mathfrak{g}$-valued 1-form A_i and an $\mathfrak{h}$-valued 2-form B_i on U_i, satisfying the fake flatness condition:

$$dA_i + A_i \wedge A_i + dt(B_i) = 0$$

- The pseudonatural isomorphism $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ is specified by the transition functions g_{ij} together with an $\mathfrak{h}$-valued 1-form a_{ij} on $U_i \cap U_j$, satisfying the equations:

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}$$

$$B_i = \alpha(g_{ij})(B_i) + da_{ij} + a_{ij} \wedge a_{ij} + d\alpha(A_i) \wedge a_{ij}$$

- For $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$ to be a modification, the differential forms a_{ij} must satisfy the equation:

$$a_{ij} + \alpha(g_{ij}) a_{jk} = h_{ijk} a_{ik} h_{ijk}^{-1} + (dh_{ijk} + d\alpha(A_i) h_{ijk}) h_{ijk}^{-1}$$

  on $U_i \cap U_j \cap U_k$.

Punchline. Except for fake flatness, these weird-looking equations show up already in Breen and Messing's definition of a *connection on a nonabelian gerbe!* So, 2-bundles and gerbes give closely related approaches to higher gauge theory... a big world waiting to be explored.