# A Categorification of Hall Algebras

Christopher D. Walker
joint with John Baez

Department of Mathematics
University of California, Riverside

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**Groupoidification** is an attempt to reverse this process. As with any categorification process, this "reverse" direction is not systematic.

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- We use "spans" because they give us a way to describe the matrix of a linear operator.

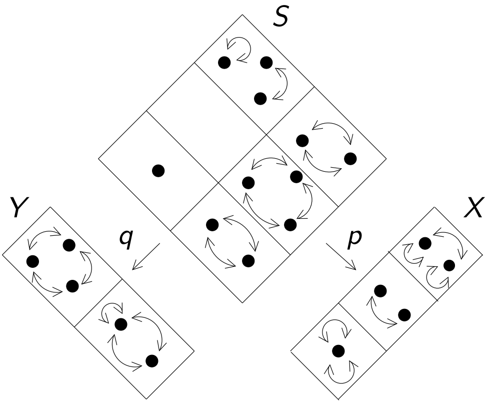
**Definition**

Given groupoids $X$ and $Y$, a **span** from $X$ to $Y$ is defined as

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

Where $S$ is another groupoid and $p : S \to X$ and $q : S \to Y$ are functors

A span is a way of describing 'how many' ways an element of one leg of the span is related to an element of the other leg of the span.


S
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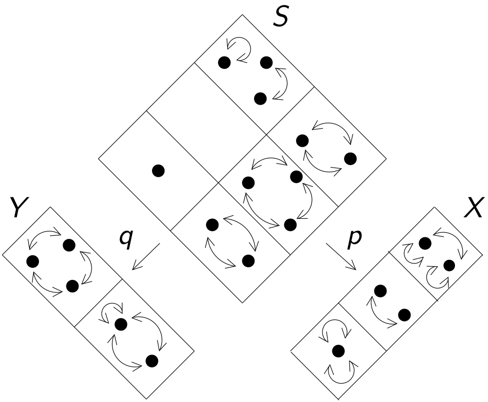


So formulaically we can describe the linear operator in terms of its matrix entries:

$$\underset{\sim}{S}_{[x][y]} = \sum_{[s]\in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{\#\mathrm{Aut}(y)}{\#\mathrm{Aut}(s)}$$

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The motivating example for groupoid cardinality is the weak quotient $S//G$ for a finite group $G$ acting on a finite set $S$. In this case, the cardinality becomes:

$$|S//G| = \frac{\#S}{\#G}$$

In order to produce a single vector (function) in $\mathbb{R}[\underline{X}]$, we consider a groupoid over $X$, $p : \Psi \to X$. We say $\Psi$ is **tame** if $p^{-1}(x)$ is tame for all $x$, where $p^{-1}(x)$ is the essential preimage of $x$. We then define the function:

$$\underset{\sim}{\Psi}([x]) = \#\mathrm{Aut}(x)|p^{-1}(x)|$$

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The multiplication by $\#\mathrm{Aut}(x)$ is simply a choice of convention (specifically the one which is appropriate for Hall algebras).

In the category of groupoids and spans, we can compose two spans using the "weak pullback" over the matching legs of the spans.

## Definition

Given two functors $f\colon S \to Y$ and $g\colon T \to Y$, we define the **weak pullback** as:

$$\begin{tikzcd} & TS \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & \\ T \arrow[dr, "g"'] & & S \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

where $TS$ is the groupoid whose objects are triples $(s, t, \alpha)$ where $\alpha\colon f(s) \to g(t)$.

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With this we have another description of the linear operator obtained from a span.

## Definition

Given a span of groupoids

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

the linear operator $\underset{\sim}{S} : \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{Y}]$ is given by $\underset{\sim}{S}\underset{\sim}{\Psi} = \underset{\sim}{S\Psi}$ where $\Psi$ is a groupoid over $X$, $v \colon \Psi \to X$, and $S\Psi$ is the weak pullback:

$$\begin{tikzcd} & & S\Psi \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_\Psi"] & \\ & S \arrow[dl, "q"'] \arrow[dr, "p"] & & \Psi \arrow[dl, "v"'] \\ Y & & X & \end{tikzcd}$$

Considering the definition of weak pullback, we get the previously mentioned formula for the matrix entries of $\underset{\sim}{S}$

$$\underset{\sim}{S}_{[x][y]} = \sum_{[s] \in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{\#\mathrm{Aut}(y)}{\#\mathrm{Aut}(s)}$$

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We can also write this using groupoid cardinality as follows:

$$\underset{\sim}{S}_{[x][y]} = \#\mathrm{Aut}(y)\ |(p \times q)^{-1}(x, y)|.$$

This version will be useful later with our example.

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- 1901 - E. Steinitz briefly described the idea of producing an algebra from isomorphism classes of finite abelian $p$-groups.
- 1959 - Phillip Hall rediscovered the idea, but also only gave a brief description.
- 1990 - Claude Ringel formalized the construction for certain abelian categories, and described the isomorphism of this algebra with (a piece of) a quantum group.
- The application of groupoidification to Hall algebras is very natural, since Hall algebras are constructed out of isomorphism classes of objects.

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- Let $Q$ be a quiver (i.e. a directed graph).

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- For a finite field $\mathbb{F}_q$, we form the category $Rep(Q)$ of finite dimensional representations of $Q$ over $\mathbb{F}_q$.

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- for three representations $M$, $N$, and $E$, define the set:

$$\mathcal{P}_{MN}^{E} = \{(f,g) \mid 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \text{ is exact}\}$$

- We then define an associative multiplication by:

$$u_M \cdot u_N = \sum_E \frac{\#\mathcal{P}_{MN}^{E}}{\#\mathrm{Aut}(M)\#\mathrm{Aut}(N)} u_E$$

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- Next we construct a multiplication span. We start by defining a new groupoid $SES(Rep(Q))$:
  - Objects - Short exact sequences of objects in $Rep(Q)$.
  - Morphisms - Isomorphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\alpha"] & E \arrow[r, "g"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] & E' \arrow[r, "g'"] & M' \arrow[r] & 0
\end{tikzcd}$$

We define the multiplication span:

$$\begin{tikzcd} & SES(Rep(Q)) \arrow[dl, "\pi_E"'] \arrow[dr, "\pi_M \times \pi_N"] & \\ Rep(Q)_\circ & & Rep(Q)_\circ \times Rep(Q)_\circ \end{tikzcd}$$

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$$

This makes sense, because given a pair of representations $(M, N)$ on the right, the span associates to it every short exact sequence with $N$ as the subrep and $M$ as the quotient. This is then projected down to representations $E$ which appear as extensions of $M$ by $N$.

We can then apply the degroupoidification to this span. Doing this, we get an operator:

$$m\colon \mathbb{R}[\underline{Rep(Q)_\circ}] \otimes \mathbb{R}[\underline{Rep(Q)_\circ}] \to \mathbb{R}[\underline{Rep(Q)_\circ}]$$

with

$$m(u_M \otimes u_N) = \sum_{E \in \mathcal{P}^E_{MN}} \#\mathrm{Aut}(E)\, |(p \times q)^{-1}(M, N, E)|\ u_E.$$

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We wish to show this matches the Hall algebra product $u_M \cdot u_N$.

For this, we must make a few observations.

- First, we note that the group $\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M)$ acts on the set $\mathcal{P}^{E}_{MN}$. This action is not necessarily free, but this is just the sort of situation groupoid cardinality is designed to handle.

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- First, we note that the group $\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M)$ acts on the set $\mathcal{P}^E_{MN}$. This action is not necessarily free, but this is just the sort of situation groupoid cardinality is designed to handle.
- Taking the weak quotient $\mathcal{P}^E_{MN}//(\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))$, we obtain a groupoid equivalent to one whose objects are short exact sequences of the form $0 \to N \to E \to M \to 0$ and morphisms are isomorphisms of short exact sequences (i.e. the subgroupoid $(p \times q)^{-1}(M, N, E)$ of $SES(Rep(Q))$).

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$$
\begin{aligned}
|(p \times q)^{-1}(M, N, E)| &= |\mathcal{P}^E_{MN} // (\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))| \\
&= \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(N)\,\#\mathrm{Aut}(E)\,\#\mathrm{Aut}(M)}
\end{aligned}
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$$\begin{aligned} |(p \times q)^{-1}(M,N,E)| &= |\mathcal{P}^E_{MN}//(\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))| \\ &= \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(N)\,\#\mathrm{Aut}(E)\,\#\mathrm{Aut}(M)} \end{aligned}$$

So, we obtain

$$m(u_M \otimes u_N) = \sum_{E \in \mathcal{P}^E_{MN}} \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(M)\,\#\mathrm{Aut}(N)}\, u_E.$$

which is precisely the Hall algebra product $u_M \cdot u_N$.

A similar process can be applied to the adjoint span:

$$\begin{tikzcd} & SES(X) \arrow[dl, "\pi_M \times \pi_N"'] \arrow[dr, "\pi_E"] & \\ X \times X & & X \end{tikzcd}$$

To obtain a coassociative comultiplication.

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To obtain a coassociative comultiplication.
Unfortunately, these are not compatible, in the sense that they do not form a bialgebra. Algebraically we can describe this as a bialgebra in a braided monoidal category, and we have been working on a way to describe this in terms of groupoids and spans.

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The Big Open Question: How do we include the "negative" part of the Quantum Group?