Groupoidified Linear Algebra

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November 22, 2008

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Groupoidification is an attempt to reverse this process. As with any categorification process, this "reverse" direction is not systematic.

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- We use groupoids because they give us a way of categorifying the positive real numbers.
- We use "spans" because the give us a way to describe the matrix of a linear operator.

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- Span- the category of groupoids as objects and 'spans' of groupoids as morphisms

What is a span?

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Definition

Given groupoids X and Y, a **span** from X to Y is defined as

$$Y \xleftarrow{\;q\;} S \xrightarrow{\;p\;} X$$

where S is another groupoid and $p : S \to X$ and $q : S \to Y$ are functors

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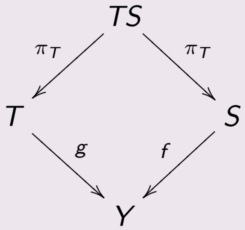
Definition

Given two functors $f\colon S \to Y$ and $g\colon T \to Y$, we define the **weak pullback** as:



where TS is the groupoid whose objects are triples (s, t, α) where $\alpha\colon f(s) \to g(t)$.

Where do the positive real numbers come from?

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Given a groupoid X, we define the **cardinality** of X to be:

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where x ranges over all isomorphism classes in X

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In order to produce a single vector (function) in $\mathbb{R}^{\underline{X}}$, we consider a groupoid over X, $p : \Psi \to X$. We say Ψ is **tame** if $p^{-1}(x)$ is tame for all x, where $p^{-1}(x)$ is the essential preimage of x. We then define the function:

$$\underset{\sim}{\Psi}([x]) = |p^{-1}(x)|$$

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We now produce a linear operator out of a span of groupoids.

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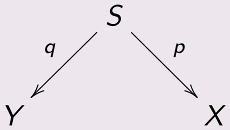
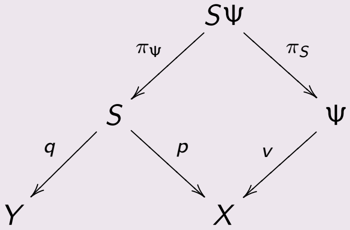
Definition

Given a tame span of groupoids



the linear operator $\underset{\sim}{S} : \mathbb{R}^{\underline{X}} \longrightarrow \mathbb{R}^{\underline{Y}}$ is given by $\underset{\sim}{S}\underset{\sim}{\Psi} = \underset{\sim}{S\Psi}$, where Ψ is a groupoid over X, $v\colon \Psi \to X$, and $S\Psi$ is the weak pullback:



Considering the definition of weak pullback, we get a nice formula for the matrix entries of $\underset{\sim}{S}$:

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$$S_{[x][y]} = \sum_{[s] \in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{|\mathrm{Aut}\ x|}{|\mathrm{Aut}\ s|}$$

Addition We can add vectors and operators as follows:

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Proposition

Give two groupoids Φ *and* Ψ *over* X*, the disjoint union* $\Phi + \Psi$ *forms a groupoid over* X*, and*

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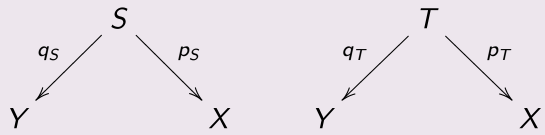
Give two spans:



the disjoint union $S + T$ forms a span from X to Y, and

$$\underset{\sim}{S + T} = \underset{\sim}{S} + \underset{\sim}{T}$$

Scalar Multiplication We can multiply both vectors and linear operators by scalars as follows:

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Proposition

Given a groupoid Λ and a span

$$Y \longleftarrow S \longrightarrow X$$

then

$$\underset{\sim}{\Lambda \times S} = |\Lambda| \underset{\sim}{S}.$$

We also have a concept of inner product, which turns our vector space into a Hilbert space.

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Definition

Given groupoids Φ *and* Ψ *over* X*, we define the* **inner product** $\langle \Phi, \Psi \rangle$ *to be this weak pullback:*

$$\begin{array}{ccc} & \langle \Phi, \Psi \rangle & \\ \swarrow & & \searrow \\ \Phi & & \Psi \\ \searrow & & \swarrow \\ & X & \end{array}$$

This definition holds to the standard properties of inner products.

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Proposition

Given a groupoid Λ *and groupoids* Φ, Ψ, *and* Ψ' *over* X, *the following properties hold:*

- $$\langle \Phi, \Psi \rangle \simeq \langle \Psi, \Phi \rangle.$$
- $$\langle \Phi, \Psi + \Psi' \rangle \simeq \langle \Phi, \Psi \rangle + \langle \Phi, \Psi' \rangle.$$
- $$\langle \Phi, \Lambda \times \Psi \rangle \simeq \Lambda \times \langle \Phi, \Psi \rangle.$$

Definition

A groupoid Φ *over* X *is called* **square-integrable** *if* $\langle \Phi, \Phi \rangle$ *is tame. We define* $L^2(\underline{X})$ *to be the subspace of* $\mathbb{R}^{\underline{X}}$ *consisting of finite real linear combinations of vectors* $\underset{\sim}{\Phi}$ *where* Φ *is square-integrable.*

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We then produce our Hilbert space.

Proposition

$L^2(\underline{X})$ *forms a Hilbert space with the inner product*
$\langle \underset{\sim}{\Psi}, \underset{\sim}{\Phi} \rangle = |\langle \Psi, \Phi \rangle|$.

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