Wilson loop dynamics without regularization*

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October 29, 2004

Abstract

It is widely believed that the dynamics of electromagnetism can only be described in terms of Wilson loops if the loops are regularized. Here we show this is not the case, as unsmeared Wilson loops give well-defined "quasioperators" on the space spanned by "smooth" coherent states, and we give an explicit formula for the time evolution of these Wilson loop quasioperators.

*Poster for *Quantum Gravity in the Americas 2004* workshop at PI
†Joint work with John C. Baez

1 Classical electromagnetism

We work on a $(3+1)$-dimensional static, globally hyperbolic spacetime $M \cong \mathbb{R} \times S$ with metric

$$g_M = e^{2\Phi}(-\mathrm{d}t^2 + g).$$

We denote by Ω^p the space of square-integrable p-forms on S. The phase space of vacuum electomagnetism on M is then the real inner-product space

$$\mathbf{P} = \mathrm{dom}\{\mathrm{d}\colon \Omega^1 \to \Omega^2\}/\overline{\mathrm{d}\Omega^0} \oplus \ker\{\mathrm{d}^*\colon \Omega^1 \to \Omega^0\}.$$

A typical point of $\mathbf{P}$ is denoted $x = [A] \oplus E$. The Hamiltonian is given by

$$H[x] = \frac{1}{2}\int_S [g(E,E) + g(\mathrm{d}A, \mathrm{d}A)]\mathrm{vol},$$

where g denotes the induced metric on Ω^1 or Ω^2, as appropriate. Assuming that the kernel of the Laplacian on Ω^1 is trivial (so that there are no Bohm-Aharonov modes), then time evolution is given by

$$\begin{pmatrix} A(t) \\ E(t) \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\Delta}t & \frac{1}{\sqrt{\Delta}}\sin\sqrt{\Delta}t \\ -\sqrt{\Delta}\sin\sqrt{\Delta}t & \cos\sqrt{\Delta}t \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

2 Classical observables

The phase space $\mathbf{P}$ has a dense subspace $\mathbf{P}_0$ consisting of smooth field configurations $[A] \oplus E$. Densely-defined linear functionals on $\mathbf{P}$ with dense domain $\mathbf{P}_0$ include:

- Circulation integrals: for each smooth closed path γ in S, we have

  $$\oint_\gamma A \quad \text{(holonomy)} \qquad \text{and} \qquad \oint_\gamma E \quad \text{(e.m.f.)}$$

- Flux integrals: for each smooth surface Σ in S,

  $$\iint_\Sigma E \qquad \text{and} \qquad \iint_\Sigma A$$

  For electric flux integrals, Σ should be understood as a homology equivalence class of surfaces modulo boundaries since, if Σ is a boundary, $\iint_\Sigma E$ is zero. Under this interpretation we need to fix the gauge to make A divergenceless too, in onder for the flux of A to be well-defined on homology classes.

The symplectic structure on $\mathbf{P}$ corresponds to the following Poisson brackets on functions of $\mathbf{P}_0$:

$$\{\oint_\gamma A, \iint_\Sigma E\} = \text{intersection number of } \gamma \text{ and } \Sigma.$$

Ashtekar and Corichi have shown that

$$\{\iint_\Sigma A, \oint_\gamma E\} = \text{linking number of } \partial\Sigma \text{ and } \gamma.$$

3 Fock quantization

We base our construction of Fock space on coherent states, which have a classical interpretation. Each field configuration $x \in \mathbf{P}$ corresponds to a coherent state $|x\rangle$, and

$$\langle x' \mid x \rangle = e^{i\omega(x,x')/2} e^{-\frac{1}{4}h(x-x',x-x')}$$

where

$$\omega(x, x') = \int_S [g(A, E') - g(A', E)]\mathrm{vol}$$

is the symplectic structure on $\mathbf{P}$ and

$$h(x, x') = \int_S [g(A, \sqrt{\Delta}A') + g(E, \tfrac{1}{\sqrt{\Delta}}E')]\mathrm{vol}$$

is the unique real inner product such that

$$\langle x, x' \rangle = h(x, x') + i\omega(x, x')$$

is a time-invariant complex inner product on $\mathbf{P}$. The span of the coherent states is dense in Fock space, denoted $\mathbf{K}$. The **smooth coherent states** $\{|x\rangle : x \in \mathbf{P}_0\}$ span a dense subspace $\mathbf{K}_0$ of the Fock space $\mathbf{K}$.

4 Holonomies as quasioperators

If γ is a smooth loop in S, we can show that $\oint_\gamma \hat{A}$ exists as a **quasioperator**, meaning that its matrix elements are not defined on all of Fock space, but they are on the dense subspace $\mathbf{K}_0$ spanned by the smooth coherent states. In particular,

$$\langle x| \oint_\gamma \hat{A} \,|x\rangle = \oint_\gamma A,$$

when $x = [A] \oplus E$ is a smooth field configuration. It is less obvious but also true that

$$\frac{\langle x'| \oint_\gamma \hat{A} \,|x\rangle}{\langle x' \mid x\rangle} = \oint_\gamma \Big(\frac{A + A'}{2}\Big) + i \oint_\gamma \frac{1}{\sqrt{\Delta}} \Big(\frac{E - E'}{2}\Big)$$

when x, x' are smooth field configurations. It can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_\gamma \hat{A} = \oint_\gamma \hat{E},$$

as a quasioperator equation, not surprisingly.

5 Wilson loops as quasioperators

Since electromagnetism is a $U(1)$ gauge theory, we should also study the Wilson loop operator $e^{i\oint_\gamma \hat{A}}$ quantizing the exponentiated holonomy. Unfortunately, it can be shown that all its matrix elements vanish:

$$\langle x'|\, e^{i\oint_\gamma \hat{A}}\, |x\rangle = 0$$

because a current concentrated on the loop γ is too singular. However, the normal-ordered holonomy $:e^{i\oint_\gamma \hat{A}}:$ is a well-defined quasioperator on $\mathbf{K}_0$ *even without smearing the loop* γ and, in fact,

$$\frac{\langle x'|\, :e^{i\oint_\gamma \hat{A}}:\, |x\rangle}{\langle x' \mid x\rangle} = \exp i \frac{\langle x'|\, \oint_\gamma \hat{A}\, |x\rangle}{\langle x' \mid x\rangle}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle x'|\, :e^{i\oint_\gamma \hat{A}}:\, |x\rangle}{\langle x' \mid x\rangle} = i \frac{\langle x'|\, \oint_\gamma \hat{E}\, |x\rangle}{\langle x' \mid x\rangle} \exp i \frac{\langle x'|\, \oint_\gamma \hat{A}\, |x\rangle}{\langle x' \mid x\rangle}$$

so we can explicitly understand the time evolution of Wilson loops in the Fock representation of quantum electromagnetism without the need for any regularization of the loops.