YOUNG DIAGRAMS AND SCHUR FUNCTORS

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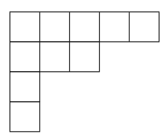
The natural numbers $\mathbb{N}$ are the free commutative monoid on one generator. Suppose someone told you to create something even cooler than the natural numbers. This would be a tall order, but here's a simple-minded thing you could try: the free commutative monoid on the set of natural numbers! This would be a commutative monoid with elements like $1 \oplus 3 \oplus 5 \oplus 1$: that is, *formal* sums of natural numbers, not to be confused with ordinary sums.

Since these formal sums are commutative, we can always order their summands in increasing order, or—as is traditionally done—in decreasing order, like $5 \oplus 3 \oplus 1 \oplus 1$. Such a thing can be drawn as a 'Young diagram': in this case, a bunch of boxes with 5 boxes in the first row, 3 in the second row, 1 in the third row and 1 in the fourth:



The definition I just gave allows Young diagrams with *no* boxes in certain rows, but so far people don't think about these (a bit like how some people don't recognize zero as a natural number). So, the usual definition goes like this:

Definition 1. *A* **Young diagram** *is a finite sequence of natural numbers $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. We call k the number of* **rows** *in the Young diagram, n_1 the number of* **columns***, and $n = \sum_i n_i$ the number of* **boxes** *in the Young diagram.*

Young diagrams are not as important as natural numbers, obviously, but they really are quite ubiquitous. They classify lots of things. For example:

(1) Young diagrams with n boxes classify partitions of n-element sets up to isomorphism.
(2) Young diagram with n boxes classify conjugacy classes in S_n.
(3) Young diagrams with n boxes classify irreps of S_n up to isomorphism.
(4) Young diagrams with at most k rows classify polynomial irreps of $\mathrm{GL}(k, \mathbb{C})$, the group of all linear transformations of $\mathbb{C}^k$, up to isomorphism.
(5) Young diagrams with at most $k-1$ rows classify irreps of $\mathrm{SL}(k, \mathbb{C}) = \{g \in \mathrm{GL}(k, \mathbb{C}) : \det(g) = 1\}$ up to isomorphism.

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(6) Young diagrams with at most $k-1$ rows classify irreps of $\mathrm{SU}(k) = \{g \in \mathrm{SL}(k, \mathbb{C}) : g \text{ is unitary}\}$ up to isomorphism.

Item (1) is obvious. First, any partition of an n-element set will have blocks with sizes given by some list of natural numbers $n_1 \geq \cdots \geq n_k > 0$. Second, these classify the partition up to isomorphism. That is, two partitions of two n-element sets have the same list of block sizes if and only if they are related by some bijection between those sets.

Item (2) is also quite easy. A permutation of an n-element set is made up of a bunch of cycles, which give a partition of that set, and thus a Young diagram. Two permutations lie in the same conjugacy class if and only if they give the same Young diagram.

Item (3) is a bit deeper. Here the **symmetric group** S_n is the group of permutations of the n-element set $\{1, \ldots, n\}$. A **representation** of a group G is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of some vector space V. We call ρ a representation of G **on** V. A representation is **irreducible** if V contains no subspaces that are mapped to themselves by all transformations $\rho(g)$, except for $\{0\}$ and V itself. We call a representation a **rep** for short, and an irreducible representation an **irrep**.

There's a fairly obvious concept of isomorphism for representations, and also a way to take direct sums of representations. For the groups in items (4)–(6), every finite-dimensional rep is isomorphic to a direct sum of irreps. So, for these groups, the project of classifying representations (up to isomorphism) boils down to classifying irreps. Irreps are like the 'atoms' of group representation theory; from these atoms we can build bigger molecules, but we should start by understanding the atoms.

Why should irreps of S_n correspond to Young diagrams with n boxes? For this it helps to know a bit of character theory. Suppose G is a finite group. Any representation $\rho\colon G \to \mathrm{GL}(V)$ on a finite-dimensional vector space V has a **character** $\chi_\rho\colon G \to \mathbb{C}$ given by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. This is a **class function**, meaning that it's constant on each conjugacy class of G, or in other words,

$$\chi_\rho(hgh^{-1}) = \chi_\rho(g)$$

for all $g, h \in G$. There are a couple of great theorems in group representation theory that explain why characters are so important.

Theorem 2. *Two finite-dimensional representations of G are isomorphic iff they have the same character.*

Theorem 3. *Characters of irreducible representations of G form a basis of the vector space of class functions on G.*

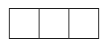
As a result, we get:

Corollary 4. *For any finite group G, the number of conjugacy classes of G equals the number of isomorphism classes of irreps of G.*

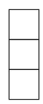
In modern mathematics, we like 'bijective proofs'. That is, if two sets have the same cardinality, we like to find an explicit bijection between them. Of course there *exists* a bijection between the set of conjugacy classes of G and the set of isomorphism classes of irreps of G, because they have the same cardinality. But is there 'natural' bijection between these things?

It's actually hard to make this question precise: what does 'natural' mean here? We usually make naturality precise using the concept of natural isomorphism between functors, but it's hard to figure out what functors to talk about in this particular case.

Still, one can ask if there's a bijection between conjugacy classes and irreps that seems 'natural' in some intuitive sense: one that can be constructed by some systematic recipe. It seems the answer is *no* in general: at least nobody has found one that works for all finite groups. But the answer is *yes* for the groups S_n. There is systematic recipe to get an irrep of S_n from an n-box Young diagram. I won't describe this recipe: this is one of the main things you'd typically learn in a course on Young diagrams, or representations of S_n. Instead, I'll just give some examples. The Young diagram

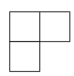


gives the 'trivial rep' of S_3. This is representation $\rho\colon S_3 \to \mathrm{GL}(1, \mathbb{C})$ with $\rho(g) = 1$. The Young diagram



gives the 'sign rep' $\rho\colon S_n \to \mathrm{GL}(2, \mathbb{C})$, with $\rho(\sigma) = \mathrm{sgn}(\sigma)$.

Both these examples easily generalize to any dimension: the Young diagram with n boxes and just one row gives the trivial rep of S_n, while the diagram with n boxes and just one column gives the sign representation. A more interesting representation of S_3 comes from this, the only other Young diagram with 3 boxes:



The gives a 2d irrep of S_3, $\rho\colon S_3 \to \mathrm{GL}(2, \mathbb{C})$, which actually maps S_3 to $\mathrm{GL}(2, \mathbb{R})$. To get this representation, just think of S_3 as the symmetries of an equilateral triangle centered at the origin of the plane, and extend these symmetries to linear transformations of the plane.

Item (4) is even deeper. Why should Young diagrams give polynomial representations of $\mathrm{GL}(k, \mathbb{C})$? The basic idea is this. There is an obvious representation of $\mathrm{GL}(k, \mathbb{C})$ on $\mathbb{C}^k$. This is a polynomial irrep. All other polynomial irreps of $\mathrm{GL}(k, \mathbb{C})$ can be built from this one using Young diagrams!

How? First, for any finite group G there's an algebra $\mathbb{C}[G]$ called the **group algebra** of G. This is the vector space with G as its basis, with multiplication defined on basis elements to be multiplication in G. Any representation $\rho\colon G \to \mathrm{GL}(V)$ makes V into a $\mathbb{C}[G]$-module. In fact V become both a left $\mathbb{C}[G]$-module, as follows:

$$gv = \rho(g)v$$

and also a right $\mathbb{C}[G]$-module, as follows:

$$vg = \rho(g^{-1})v.$$

Next, by item (3) any Young diagram with n boxes gives a representation of S_n, say $\rho\colon S_n \to \mathrm{GL}(R)$ for some vector space R. For any vector space V, there is also a representation of S_n on the n-fold tensor product

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}},$$

where a permutation acts by permuting the factors.

So, R becomes a right $\mathbb{C}[G]$-module and $V^{\otimes n}$ becomes a left $\mathbb{C}[G]$-module. We can tensor these and get a vector space

$$R \otimes_{\mathbb{C}[S_n]} V^{\otimes n}.$$

But because $\mathrm{GL}(V)$ has a representation on V, it gets a representation on this vector space.

Taking $V = \mathbb{C}^k$, this how a Young diagram gives a representation of $\mathrm{GL}(k, \mathbb{C})$. It's easy to check that this is a polynomial representation. Less obviously, it's an irrep of $\mathrm{GL}(k, \mathbb{C})$. Even less obviously, we get *all* polynomial irreps of $\mathrm{GL}(k, \mathbb{C})$, up to isomorphism, this way. And still less obviously, this construction gives a one-to-one correspondence between Young diagrams with at most k rows polynomial irreps of $\mathrm{GL}(k, \mathbb{C})$.

There is a lot more to say about this, obviously. But the key to understanding these matters at a deeper level is to create a category that has Young diagrams as objects. More precisely, we'll find a category Schur where every object is a direct sum of objects coming from Young diagrams. Objects of this category will act to give functors from Vect to itself, called Schur functors. I won't prove it here, but if we apply these functors to $\mathbb{C}^k$ we get all the representations of $\mathrm{GL}(k, \mathbb{C})$. In fact, we get them all from Young diagrams with at most k rows. Items (5) and (6) work similarly, but for these smaller groups we only need Young diagrams with at most $k - 1$ rows.

The first step is to stop thinking of the S_n's separately and define the **permutation groupoid** to be

$$S = S_0 + S_1 + S_2 + S_n + \cdots .$$

The sum here stands for the 'disjoint union' or technically 'coproduct' of the groups S_n. A **groupoid** is a category with all morphisms invertible. A group gives a groupoid with one object: the group elements are the morphisms from this object to itself. The disjoint union of a bunch of groups is a groupoid! The groupoid S has one object for each natural number n, all the morphisms are endomorphisms, and the morphisms $\sigma\colon n \to n$ are elements of S_n. This is a very important groupoid, because it's equivalent to the groupoid of finite sets and bijections.

We can define a **representation** of a groupoid X to be a functor $R\colon X \to \mathsf{Vect}$. When our groupoid has one object this reduces to a representation of a group. But a representation $R\colon S \to \mathsf{Vect}$ sends each object n to a vector space $R(n)$, and each morphism $\sigma\colon n \to n$ to a linear map $R(\sigma) \in \mathrm{GL}(R(n))$ in a way that preserves composition and identities. So, a representation $R\colon S \to \mathsf{Vect}$ is just a list of reps of all the groups S_n.

For what we'll do next it's nice to impose some 'finiteness' conditions. So, we'll say that a **polynomial** representation $R\colon S \to \mathsf{Vect}$ is one with $\dim R(n) < \infty$ for all n and $\dim R(n) = 0$ except for finitely many n. There's a category Schur of these polynomial representations of S, where the morphisms are natural transformations. Every object of Schur is a direct sum of finitely many functors coming from Young diagrams.

We call Schur the category of **Schur functors**, for the following reason. Given any vector space V, and $R \in \mathsf{Schur}$, we get a new vector space

$$\widetilde{R}(V) = \bigoplus_{n=0}^{\infty} R(n) \otimes_{\mathbb{C}[S_n]} V^{\otimes n}.$$

As before, the group S_n has a a representation on $R(n)$ but also on $V^{\otimes n}$, where it acts by permuting the factors. Thus, $R(n)$ becomes a right $\mathbb{C}[S_n]$-module and $V^{\otimes n}$ becomes a left $\mathbb{C}[S_n]$-module. This lets us define the vector space $R(n) \otimes_{\mathbb{C}[S_n]} V^{\otimes n}$.

Notice that the above equation looks a lot like a Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

The variable x was a number, but its analogue V is a vector space! The coefficients a_n were numbers too, but the coefficients $R(n)$ are representations of S_n. The process of dividing by $n!$ has been replaced by tensoring over the group algebra $\mathbb{C}[S_n]$. Finally, the sum has been replaced by a direct sum. So, the definition of the Schur functor $\widetilde{R}$ *categorifies* the concept of a Taylor series.

Indeed, any representation $R\colon S \to \mathsf{Vect}$ gives a functor $\widetilde{R}\colon \mathsf{Vect} \to \mathsf{Vect}$ via the above formula. But a polynomial representation R is a bit better: it also gives a functor $\widetilde{R}\colon \mathsf{FinVect} \to \mathsf{FinVect}$, where $\mathsf{FinVect}$ is the category of *finite-dimensional* vector spaces. We use the same formula:

$$\widetilde{R}(V) = \bigoplus_{n=0}^{\infty} R(n) \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \tag{1}$$

but now we note that each summand is finite-dimensional, and only finitely many are nonzero.

So, being a bit more careful to distinguish these functors, given any $R \in \mathsf{Schur}$ we get a functor

$$\widetilde{R}_{\mathsf{Vect}}\colon \mathsf{Vect} \to \mathsf{Vect}$$

but also a functor

$$\widetilde{R}_{\mathsf{FinVect}}\colon \mathsf{FinVect} \to \mathsf{FinVect}.$$

We call both these functors **Schur functors**, since they are defined by the same formula.

In fact, Schur functors know how to act on any category C that has enough structure for Equation (1) to make sense! For example, the formula works fine whenever $C = \mathsf{Rep}(G)$ is the category of representations of some group G, so for any $R \in \mathsf{Schur}$ we get a functor

$$\widetilde{R}_{\mathsf{Rep}(G)}\colon \mathsf{Rep}(G) \to \mathsf{Rep}(G).$$

If we take $G = \mathrm{GL}(k, \mathbb{C})$, and apply all these functors to the obvious representation of $\mathrm{GL}(k, \mathbb{C})$ on $\mathbb{C}^k$, we get all polynomial representations of $\mathrm{GL}(k, \mathbb{C})$. In fact, as mentioned, the objects $R \in \mathsf{Schur}$ coming from Young diagrams with at most k rows give precisely the polynomial irreps of $\mathrm{GL}(k, \mathbb{C})$. Similar remarks hold for items (5) and (6).

Schur functors know how to act on a category C whenever this category is:

- symmetric monoidal,

- $\mathbb{C}$-linear (meaning the hom-sets are complex vector spaces, with composition and the tensor product of morphisms being linear in each argument), and
- Cauchy complete, meaning:
  - it has direct sums (or in more categorical terms, biproducts)
  - it has kernels of idempotents (that is, morphisms $\rho\colon x \to x$ with $p^2 = p$).

Any abelian category is Cauchy complete, but the generalization to Cauchy complete categories is useful: for example, the category of vector bundles on a manifold doesn't have kernels of arbitrary morphisms, but it does have kernels of idempotents. There are also deeper theoretical reasons for being interested in Cauchy complete $\mathbb{C}$-linear functors: for starters, any $\mathbb{C}$-linear functor automatically preserves biproducts and kernels of idempotents.

Todd Trimble and I proved a theorem that says roughly this:

Theorem. *Schur functors are precisely the functors that know how to act on all Cauchy complete symmetric monoidal $\mathbb{C}$-linear categories.*

When we say that a Schur functor 'knows how to act' on every Cauchy complete symmetric monoidal $\mathbb{C}$-linear category C, we mean more than merely that each $R \in \mathsf{Schur}$ gives a functor

$$\widetilde{R}_C\colon C \to C$$

for each such C. We also mean that given a suitable map between such categories, say $f\colon C \to D$, we get a square

$$\begin{tikzcd} C \arrow[r, "\widetilde{R}_C"] \arrow[d, "f"'] & C \arrow[d, "f"] \\ D \arrow[r, "\widetilde{R}_D"'] & D \end{tikzcd}$$

that commutes up to a natural isomorphism $\widetilde{R}_f$. If the square commuted, we would say $\widetilde{R}$ is a natural transformation from U to itself, where

$$U\colon \mathsf{SymmMonCauch} \to \mathsf{Cat}$$

maps any Cauchy complete symmetric monoidal $\mathbb{C}$-linear category to its underlying category. Since it commutes only up to a natural isomorphism, the most we can hope for is that $\widetilde{R}$ is a 'pseudonatural' transformation. This is a 2-categorical analogue of a natural transformation, to be expected here because Cat is not just a category but a 2-category. There is also a 2-category $\mathsf{SymMonCauch}$, with:

- Cauchy complete symmetric monoidal $\mathbb{C}$-linear categories as objects,
- symmetric monoidal $\mathbb{C}$-linear functors as morphisms,
- monoidal natural transformations as 2-morphisms.

There is a 2-functor

$$U\colon \mathsf{SymmMonCauch} \to \mathsf{Cat}$$

mapping any Cauchy complete symmetric monoidal $\mathbb{C}$-linear category to its underlying category. Each object $R \in \mathsf{Schur}$ gives a pseudonatural transformation from U to itself.

In the world of 2-categories we have not only 2-functors and pseudonatural transformations between these, but 'modifications' between pseudonatural transformations. All these are involved in the precise statement of the theorem:

Theorem 5. Schur *is equivalent to the category with*

- *pseudonatural transformations* $\alpha\colon U \Rightarrow U$ *as objects,*
- *modifications between these as morphisms.*

Since Young diagrams correspond precisely to the objects of Schur that aren't direct sums of other objects in a nontrivial way, this result gives yet another sort of entity classified by Young diagrams. For the proof see:

- John Baez and Todd Trimble, Schur Functors I.

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