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Workshop on TFT's at Northwestern
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1 Lecture 1

1.1 Overview.

We devote this lecture to 2-dimensional gauge theory with finite gauge group G[1]. A gauge theory is a quantum field theory in which the fundamental fields are principal G-bundles with connection defined over a space-time manifold M. In classical field theory we restrict attention to connections satisfying equations of motion (e.g. flat connections). In quantum field theory we study the space of all connections by attaching $\mathbb{C}$-linear data to it. In topological field theory we study coarse features of the above depending only on the topology of M.

Our goal for this lecture is to construct an extended 2-dimensional topological field theory $\mathcal{Z}_G$. By 'extended' we mean to say that $\mathcal{Z}_G$ extends down to a point. It assigns 'values' to 2, 1 and 0-dimensional manifolds. In what follows the path integral is used as a guiding principle.

Another approach to construction would be the Cobordism Hypothesis[2]. This would be done by showing $\mathbb{C}[G]$ is a fully dualizable object in a particular 2-category [3]. This would imply the existence of an extended 2-dimensional TFT with $\mathcal{Z}_G(\bullet) := \mathrm{Rep}_{\mathbb{C}}G$, and allow us to compute its value for any manifold of dimension less or equal to 2. This approach can be viewed as a 'generators and relations' type construction, while ours can be viewed as an a priori construction.

[1]The subject of finite group topological field theories started with Dijkgraaf and Witten in their paper on *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. 129 (1990) 393.

[2]The Cobordism Hypothesis was first stated by Baez and Dolan in their paper on *Higher-dimensional Algebra and Topological Quantum Field Theory*. Lurie has announced a proof of a very general form of the Cobordism Hypothesis, which was the subject of his lectures at Northwestern, and which we will use in the later lectures. A sketch of the proof can be found in his paper *On the Classification of Topological Field Theories*.

[3]The 2-category we have in mind is one in which objects are $\mathbb{C}$-algebras, 1-morphisms are bi-modules and 2-morphisms are maps of bi-modules. A finite dimensional, semi-simple $\mathbb{C}$-algebra is a fully dualizable object in this category, which is clearly the case for the group algebra $\mathbb{C}[G]$.

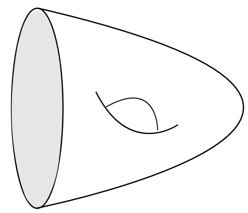


Figure 1: A surface with one boundary component.

1.2 Extended Topological Field Theory.

For the purpose of this lecture let M be a manifold of dimension 0, 1 or 2. We denote the space of fields on M by $\mathcal{M}_G(M)$ (since G is finite all connections are automatically flat). We have

$$\begin{aligned}\mathcal{M}_G(M) &= \{G\text{-bundles on M}\}\\ &= \{\text{maps } M \to BG\}\\ &= \{G\text{−Galois covers of } M\}\\ &= \{\text{representations } \pi_1(M) \to G\}/\sim\end{aligned}$$

where by $\sim$ we mean 'up to conjugation'. In particular, when M is a point, $\mathcal{M}_G(\bullet) = BG$. When M is a circle, $\mathcal{M}_G(S^1) = \frac{G}{G}$ (this is our notation for the groupoid with objects $g \in G$ and arrows $g \xrightarrow{x} xgx^{-1}$, $x \in G$).

An extended 2-dimensional TFT is a rule that assigns complex numbers to closed 2-manifolds, complex vector spaces to closed 1-manifolds and $\mathbb{C}$-linear categories to 0-manifolds. This functor has two key properties: it respects disjoint unions and gluing.

Adopting the path integral approach, $\mathcal{Z}_G$ assigns the following complex number to a closed surface Σ.

$$\mathcal{Z}_G(\Sigma) = \text{“}\int_{\mathcal{M}_G(\Sigma)} e^{iS(\phi)} d\phi \text{”}$$

The quotation marks are there to remind us that in more general contexts the measure $d\phi$ does not exist. In this finite setting all measures are well defined, and we take this integral to mean the sum over principal bundles, P, weighted by $\frac{1}{|\mathrm{Aut}P|}$. We can compute it explicitly[4].

$$\mathcal{Z}_G(\Sigma_g) = \sum_{\chi} \left(\frac{|G|}{\chi(e)}\right)^{2g-2}$$

g denotes the genus of Σ, $e \in G$ is the identity element, and the sum runs over all irreducible characters χ of G.

[4]See Rodriguez-Villegas in his *note on the representation theory of finite groups*.

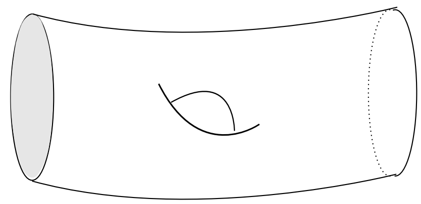


Figure 2: A surface with incoming and outgoing boundary components.

To understand what $\mathcal{Z}_G$ should assign to 1-dimensional manifolds we consider a surface Σ with one boundary component (see fig. 1). We fix a field $\phi_0 \in \mathcal{M}_G(\partial\Sigma)$, and consider all fields on Σ that restrict to ϕ_0. We expect

$$\mathcal{Z}_G(\Sigma)(\phi_0) = \text{``}\int_{\phi|_{\partial\Sigma}=\phi_0} e^{iS(\phi)} d\phi\text{''}$$

We consider $\mathcal{Z}_G(\Sigma)$ as an observable, namely, a function on fields defined on the boundary

$$\mathcal{Z}_G(\partial\Sigma) = \mathrm{Func}(\mathcal{M}_G(\partial\Sigma))$$

Given this interpretation,

$$\mathcal{Z}_G(S^1) = \mathrm{Func}(\mathcal{M}_G(S^1)) = \mathbb{C}[G]^G$$

By $\mathbb{C}[G]^G$ we mean the G-invariants of the group algebra $\mathbb{C}[G]$. These are the class functions on G with multiplication given by the convolution product.

Now consider the surface Σ in fig. 2 with two boundary components. We view it as a bordism bewteen an 'incoming' circle and an 'outgoing' one. By that we mean that we think of Σ as representing the time evolution of a geometric object, i.e., a closed string.

By restricting fields on Σ to the boundary we get a correspondence diagram

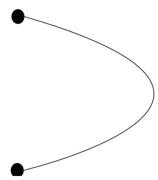
$$\begin{array}{ccc} & \mathcal{M}_G(\Sigma) & \\ \swarrow \pi_{\mathrm{in}} & & \pi_{\mathrm{out}} \searrow \\ \mathcal{M}_G(\partial\Sigma_{\mathrm{in}}) & & \mathcal{M}_G(\partial\Sigma_{\mathrm{out}}) \end{array} \qquad (1)$$

We construct $\mathcal{Z}_G(\Sigma)$ as a map between the two vector spaces associated to the incoming and outgoing boundaries. Given a function f on $\mathcal{M}_G(\partial\Sigma_{\mathrm{in}})$ we produce a function on $\mathcal{M}_G(\partial\Sigma_{\mathrm{out}})$ by first pulling back and then pushing forward.

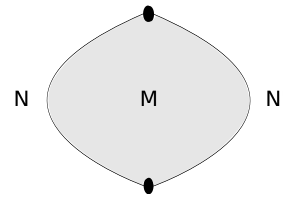
$$\mathcal{Z}_G(\Sigma) : f \mapsto \text{``}(\pi_{\mathrm{out}})_*((\pi_{\mathrm{in}})^*(f) \cdot e^{iS})\text{''}$$

So far we considered what $\mathcal{Z}_G$ associates to 2 or 1-dimensional manifolds. A "folk" theorem states that $(2,1)$-TFT is given by a commutative Frobenius algebra associated to the circle[5].

[5] By $(2,1)$-TFT we mean a TFT that takes values in Vect and is evaluated on 2 and 1-dimensional manifolds only. This theorem appeared first in print in Dijkgraaf's PhD thesis: *A Geometrical Approach to Two Dimensional Conformal Field Theory* (Utrecht, 1989). A complete proof appears in Lowell Abrams' thesis.



(a) 1-manifold with boundary.



(b) 2-manifold with corners

Figure 3

We expect $\mathbb{C}[G]^G$ to be as such and indeed it is. Multiplication is given by convolution, the unit is given by the delta function on $[e]$ and the trace is given by evaluation at the identity element $e \in G$ times $\frac{1}{|G|}$.

In a 'generators and relations' approach we construct $\mathcal{Z}_G(\Sigma)$ using the fact that any surface can be glued from a small set of generators given the 'pair of pants' in fig. 4(a), the 'thimble' in fig. 4(b) and the cylinder in fig. 4(c). These bordisms correspond to the multiplication, the trace and the identity in $\mathcal{Z}_G(S^1) = \mathbb{C}[G]^G$. Their reflections correspond to the co-multiplication and unit.

We continue by cutting our 1-manifold further. How should we interpret the invariant $\mathcal{Z}_g$ assigns to a 0-manifold? Let M be a 1-manifold with boundary. Once we fix a configuration of fields on ∂M we should get a vector space same as we got for a closed 1-manifold. This means a vector bundle over $\mathcal{M}_G(\partial M)$. Vector bundles over the space of fields form a category. Given this interpretation,

$$\mathcal{Z}_G(\bullet) = \mathrm{VBun}(\mathcal{M}_G(\bullet)) = \mathrm{VBun}(BG) = \mathrm{Rep}_{\mathbb{C}}G$$

To a 1-manifold with boundary, $\mathcal{Z}_G$ associates a functor between the categories associated to incoming and outgoing boundary components. In the case of figure 3(a), we get a functor $\mathrm{Rep}_{\mathbb{C}}G \otimes \mathrm{Rep}_{\mathbb{C}}G \to \mathrm{Vect}_{\mathbb{C}}$, given by mapping a pair of objects (A, B) to $\mathrm{Hom}_{\mathrm{Rep}G}(A, B)$.

We may consider a 2-manifold, M, with boundary containing non-closed embedded 1-manifolds, for example, see fig. 3(b) with $\partial M = N \coprod N'$. M is a manifold with corners thought of as a bordism between an incoming interval and an outgoing one. The intervals themselves are thought of as bordisms between an incoming point and an outgoing one. $\mathcal{Z}_G(N)$ and $\mathcal{Z}_G(N')$ are both given by the identity functor and $\mathcal{Z}_G(N)$ is its trivial endomorphism.

1.3 Open-Closed Topological Field Theory.

So far we constructed a fully extended $(2, 1, 0)$ TFT from gauge theory with finite gauge group G. Another possible construction is that of a 2-dimensional open-closed TFT and the two are closely related.

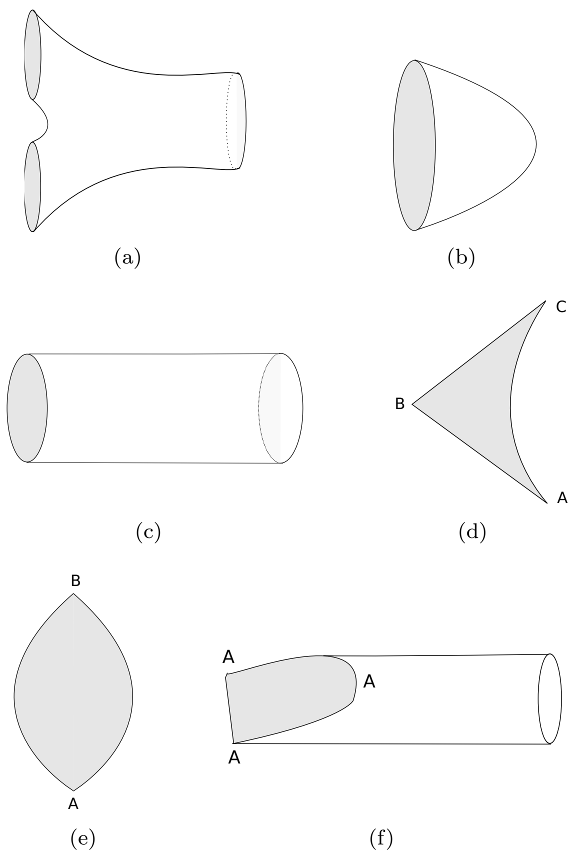


Figure 4: Bordisms in open-closed TFT.

A 2-dimensional TFT is a monoidal functor from some geometric category into $\mathrm{Vect}_{\mathbb{C}}$. This geometric category has as objects both open and closed 1-dimensional manifolds, i.e., intervals and circles. The endpoints of intervals are labeled by objects of a category of *boundary conditions*. In the literature these objects are referred to as *branes*. In our case the category of boundary conditions is $\mathrm{Rep}_{\mathbb{C}}G$. The morphisms of this category are oriented smooth 2-manifolds with circles and intervals embedded into their boundary. A few examples of such morphisms are given in figure 4. In fact, this is a complete set of generators, that is, any morphism can glued out of those morphisms and their duals.[6]

Let us discuss some features of the open-closed $2d$ TFT corresponding to gauge theory with finite gauge group G. The restriction of the open-closed

[6] For a detailed discussion of open-closed 2-dimensional TFT see Moore and Segal in their paper on *D-branes and K-theory in 2D topological field theory*.

TFT to the full subcategory consisting of closed objects is a $(2,1)$ TFT in the sense discussed before. It is identical to the $(2,1)$ part of the extended TFT we constructed.

An interval with labeled endpoint $I_{A,B}$ is assigned the vector space given by $\mathrm{Hom}_{\mathrm{Rep}G}(A,B)$. Note that this is just the evaluation of the functor associated with fig. 3(a) at (A,B). The triangle in fig. 4(d) is assigned the map

$$\mathrm{Hom}_{\mathrm{Rep}G}(A,B)\otimes\mathrm{Hom}_{\mathrm{Rep}G}(B,C)\to\mathrm{Hom}_{\mathrm{Rep}G}(A,C)$$

coming from composition.

When we glue figures 4(b) and 4(f) together we get a disc with an interval embedded into its boundary. We view it as a bordism between an incoming interval and an outgoing empty 1-dimensional manifold. To such a bordism we associate a map $\mathrm{End}_{\mathrm{Rep}G}(A)\to\mathbb{C}$ given by the trace. In particular, $\mathrm{tr}(\mathrm{id}_A)=\dim A$. We can dualize this picture by considering the same thimble diagram flipped, i.e., we have a bordism from an empty incoming 1-manifold and an outgoing interval. To such a bordism we associate a map $\mathbb{C}\to\mathrm{End}_{\mathrm{Rep}G}(A)$ taking $1\in\mathbb{C}$ to the identity.

The whistle diagram in fig. 4(f) deserves special attention. For every $A\in\mathrm{Rep}_{\mathbb{C}}G$ it induces a map $\mathrm{End}_{\mathrm{Rep}G}(A)\to\mathbb{C}[G]^G$. We can flip the whistle diagram so that we have a bordism from an incoming circle to an outgoing interval. This bordism induces a map $\mathbb{C}[G]^G\to\mathrm{End}_{\mathrm{Rep}G}(A)$. We denote the two by

$$\begin{aligned}\text{character} &: \mathrm{End}_G(A)\to\mathbb{C}[G]^G\\ \text{action} &: \mathbb{C}[G]^G\to\mathrm{End}_G(A)\end{aligned}$$

As a consequence of the Peter-Weyl theorem, we have

$$\mathbb{C}[G]\xrightarrow{\sim}\bigoplus_{A\text{ irrep}}\mathrm{End}(A)$$

This is an isomorphism of algebras given by $g\mapsto\bigoplus\rho_A(g)$. Note that

$$\mathrm{End}(\mathrm{id}_{\mathrm{Rep}G})\cong\bigoplus_{A\text{ irrep}}\mathrm{End}_G(A)$$

since an endomorphism of the identity functor is completely determined by a choice of endomorphisms of the irreducible objects. When we restrict the above isomorphism to the centers we get

$$\mathbb{C}[G]^G\xrightarrow{\sim}\mathrm{End}(\mathrm{id}_{\mathrm{Rep}G})\leftrightarrows\mathrm{End}_G(A)$$

The maps in both directions defined by this isomorphism are exactly the action and character maps. More explicitly

$$\frac{\dim A}{|G|}\sum_g\overline{\chi_A(g)}g\longleftrightarrow\mathrm{id}_A$$

Our example of an open-closed $2d$ TFT is "universal" in the sense that the commutative Frobenius algebra associated to the circle is given by $HH^*(\mathcal{C})$ where $\mathcal{C}=\mathrm{Rep}_{\mathbb{C}}G$ is the category of boundary conditions.