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Workshop on TFT's at Northwestern
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1 Lecture 2

1.1 Overview.

The first part of this lecture will be dedicated to a discussion of the character and action map that relate the open and closed sectors of our topological field theory. We will restrict attention to the case of representations of the form $A = \mathbb{C}[G/K]$ where $K \subset G$. Here we will define ‘char’ and ‘act’ via a pull-back and push-forward construction.

The second part of this lecture will be dedicated to a discussion of topological field theories arising from gauge theory with symmetry group G where G is a complex reductive groups. The first problem we run into is that moduli spaces of fields on space-time manifolds, i.e., G -bundles with connections, are no longer discrete and do not allow for the same counting invariants we were able to define when G was finite. We solve the problem by going up one dimension, namely, constructing 3-dimensional TFTs.

We still encounter the issue of defining an appropriate theory of functions on moduli spaces of fields which are generally stacks. We discuss two natural categorified theories of ‘functions’. One is quasi-coherent sheaves and the other is \mathcal{D} -modules on X . We discuss some of their properties that are essential for constructing the 3d TFT.

1.2 Hecke Algebras.

We ended our previous lecture with a discussion of the whistle diagram in fig. 1. It represents a bordism between an incoming open string and an outgoing closed string. Its mirror image is a bordism between an incoming closed string and an outgoing open string. The maps arising from this bordism and its mirror image, once we apply our field theory functor, are denoted by

$$\begin{aligned} \text{character} & : \text{End}_G(A) \rightarrow \mathbb{C}[G]^G \\ \text{action} & : \mathbb{C}[G]^G \rightarrow \text{End}_G(A) \end{aligned}$$

A natural source of representations A comes from considering subgroups $K \subset G$ and inducing their trivial representation. We denote these by

$$V_{G,K} := \text{Ind}_K^G \mathbb{C} = \mathbb{C}[G/K]$$

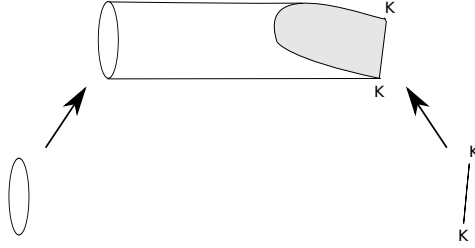


Figure 1: Correspondence diagram.

Note that $V_{G,K}$ represents the functor of K -invariants.

$$\begin{aligned} \mathrm{Hom}_G(V_{G,K}, A) &= \mathrm{Hom}_G(\mathrm{Ind}_K^G \mathbb{C}, A) \\ &= \mathrm{Hom}_K(\mathbb{C}, \mathrm{Res}_K^G A) \\ &= A^K \end{aligned}$$

The *Hecke algebra*, $\mathcal{H}_{G,K}$, is defined to be the algebra of endomorphisms

$$\mathcal{H}_{G,K} := \mathrm{End}_G(V_{G,K})$$

By the above remark

$$\begin{aligned} \mathcal{H}_{G,K} &= \mathbb{C}[G/K]^K \\ &= \mathbb{C}[K \backslash G / K] \end{aligned}$$

The Hecke algebra is the sub-algebra of $\mathbb{C}G$ consisting of K -bi-invariant functions.

In the open-closed TFT, discussed in the first lecture, the vector space associated to the interval with endpoints labeled by $V_{G,K}$ is by definition $\mathcal{H}_{G,K}$. We would like to understand the character and action map in this case using gauge theory.

The moduli space of G -bundles on the interval with K -reduction of the structure group at the endpoints is given by

$$\mathcal{M}_G(\bullet \xrightarrow{G} \bullet)_K = K \backslash G / K$$

In the extreme case where $K = \{e\}$, we have $\mathcal{M}_G(\bullet \xrightarrow{G} \bullet) = G$. The Hecke algebra is now viewed as the algebra of functions over this moduli space. This fits well with our previous discussion of \mathcal{Z}_G evaluated on closed 1-manifolds. Whether these 1-manifolds are open or closed, \mathcal{Z}_G associates to both the vector space of observables, namely, the vector space functions on fields.

Consider the following correspondence arising from fig. 1 by restriction of fields to the boundary.

$$\begin{array}{ccc} & \frac{G}{K} & \\ \swarrow & & \searrow \\ \frac{G}{G} & & K \backslash G / K \end{array}$$

By pulling back a function on $\frac{G}{K}$ and pushing it forward by averaging over left and right K -action, we get a K -bi-invariant function on $K \backslash G / K$. This gives us the action map for $A = V_{G,K}$. By pulling back a K -bi-invariant function and pushing it forward to a function on $\frac{G}{G}$ we get the character map. The formula for the character of the induced representation is deduced from this correspondence diagram.

1.3 Gauge Theory for Complex Lie Groups

We replace a finite group by a complex reductive Lie group G such as $GL_n(\mathbb{C})$. Fields on a manifold M are principal G -bundles with flat connections, that is, G -local systems¹

$$\begin{aligned} \mathcal{M}_G(M) &= \{G\text{-local systems on } M\} \\ &= \{\text{representations } \pi_1(M) \rightarrow G\} / \sim \end{aligned}$$

For example, if Σ_g is a closed surface of genus g then

$$\mathcal{M}_G(\Sigma_g) = \left\{ \begin{array}{l} A_1, \dots, A_g \\ B_1, \dots, B_g \end{array} \in G \mid \prod [A_i, B_i] = 1 \right\} / G$$

and $\mathcal{M}_G(\Sigma_g)$ is given as a quotient of a complex affine algebraic variety by some group action.

When G was finite we counted the number of points of the groupoid $\mathcal{M}_G(\Sigma_g)$ weighted by the number of automorphisms. For a complex group G this can no longer be done². One possible solution to the problem is to pass from $2d$ extended TFT to $3d$ extended TFT. Following our analysis in lecture 1, the value of a $3d$ TFT on Σ_g should be “ $\text{Func}(\mathcal{M}_G(\Sigma_g))$ ”. But what should be the right function theory on a space like $\mathcal{M}_G(\Sigma_g)$?

Let X be a scheme or more generally a stack. We have two natural candidates for a categorified space of functions:

- I. The category $\mathcal{Q}(X)$ of quasi-coherent sheaves on X .

It is a dg-category (a category whose Hom-spaces are differential graded vector spaces). Its ‘basic’ objects are algebraic vector bundles, possibly of infinite rank. Other objects are generated by taking kernels, co-kernels etc.

For example, if X is affine then $X = \text{Spec}R$ and $\mathcal{Q}(X)$ is a dg version of the derived category of R -modules, it consists of complexes of R -modules, but we have to correct the morphisms, i.e., we have to invert quasi-isomorphisms. For a general X we can define $\mathcal{Q}(X)$ by gluing $\mathcal{Q}(U)$ for an open cover $\{U\}$ of X .

¹An exposition on G -local system can be found at the Secret Blogging Seminar.

²We may consider counting \mathbb{F}_q -points of $\mathcal{M}_G(\Sigma_g)$. This is done in Hausel and Rodriguez-Villegas’s paper on *Mixed Hodge polynomials of character varieties*.

II. The category $\mathcal{D}(X)$ of \mathcal{D} -modules on X .

It is also a dg-category. Its objects are pairs (\mathfrak{F}, ∇) with $\mathfrak{F} \in \mathcal{Q}(X)$ and $\nabla : \mathfrak{F} \rightarrow \mathfrak{F} \otimes \Omega^1(X)$ where ∇ satisfies Leibnitz rule and flatness means $\nabla^2 = 0$.

\mathcal{D} -modules are close analogues of functions or distributions on X . For example, we can recover $f \in C^\infty(X)$ as a \mathcal{D} -module $\mathcal{D} \cdot f \subset C^\infty(X)$ where \mathcal{D} is the sheaf of polynomial differential operators on X . When $X = \mathbb{C}$ and $f = e^{\lambda x}$ then $\mathcal{D} \cdot e^{\lambda x} = \mathcal{D}/\mathcal{D} \cdot (\partial - \lambda)$. We can also recover a distribution $\delta \in C^{-\infty}(X)$ as a \mathcal{D} -module $\mathcal{D} \cdot \delta \subset C^{-\infty}(X)$. For example, $\delta = \delta_\lambda$, $\lambda \in \mathbb{C}$, then $\mathcal{D} \cdot \delta_\lambda = \mathcal{D}/\mathcal{D} \cdot (x - \lambda)$.

Both $\mathcal{Q}(X)$ and $\mathcal{D}(X)$ are commutative algebra objects in the category dg-categories over \mathbb{C} where multiplication is given by tensor product of vector bundles or sheaves. Another name for that is symmetric monoidal $(\infty, 1)$ -categories. Both admit pull-backs and push-forwards. Given $\pi : X \rightarrow Y$ we have

$$\begin{aligned} \pi^* : \mathcal{Q}(Y) &\rightarrow \mathcal{Q}(X) & \pi_* : \mathcal{Q}(X) &\rightarrow \mathcal{Q}(Y) \\ \pi^* : \mathcal{D}(Y) &\rightarrow \mathcal{D}(X) & \pi_* : \mathcal{D}(X) &\rightarrow \mathcal{D}(Y) \end{aligned}$$

We can use pull-back and push-forward to define a notion of *integral transform*. Given a correspondence diagram

$$\begin{array}{ccc} & X \times Y & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & & Y \end{array}$$

and an object $\mathfrak{K} \in \mathcal{D}(X \times Y)$ we can define

$$\begin{aligned} \mathcal{D}(X) &\rightarrow \mathcal{D}(Y) \\ \mathfrak{F} &\mapsto \mathfrak{K} * \mathfrak{F} \\ \mathfrak{K} * \mathfrak{F} &:= (\pi_Y)_*((\pi_X)^*(\mathfrak{F}) \otimes \mathfrak{K}) \end{aligned}$$

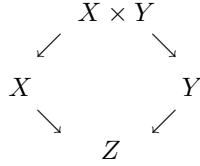
This is the analogue of the analytic integral transform

$$Kf(y) := \int_X f(x)K(x, y)dx$$

Before we continue our discussion of geometric function theory we would like to make a digression and discuss function theory on finite sets. Let X, Y be finite sets.

$$\begin{aligned} \text{Func}(X \times Y) &= \text{Func}(X) \otimes \text{Func}(Y) \\ &= \text{Hom}_{\mathbb{C}}(\text{Func}(X), \text{Func}(Y)) \\ &= Y \times X \text{ matrices} \end{aligned}$$

Consider the diagram



The subspace $X \times_Z Y \subset X \times Y$ consists of pairs (x, y) that map to the same element in Z , and $\text{Func}(X \times_Z Y)$ are functions supported on $X \times_Z Y$. When $Z = \{\bullet\}$ we recover $X \times Y$.

$$\begin{aligned}
 \text{Func}(X \times_Z Y) &= \text{Func}(X) \otimes_{\text{Func}(Z)} \text{Func}(Y) \\
 &= \text{Hom}_{\text{Func}(Z)}(\text{Func}(X), \text{Func}(Y)) \\
 &= Y \times X \text{ block matrices with blocks} \\
 &\quad \text{labeled by elements of } Z
 \end{aligned}$$

Back to schemes or stacks. We have the following theorem due to Toen³ for the case where X, Y are schemes and $Z = \{\bullet\}$, and due to Ben-Zvi, Francis and Nadler⁴ for the case where X, Y, Z are “perfect stacks”, e.g., any scheme or most common stacks in characteristic zero.

Theorem 1.

$$\begin{aligned}
 \mathcal{Q}(X \times_Z Y) &\cong \mathcal{Q}(X) \otimes_{\mathcal{Q}(Z)} \mathcal{Q}(Y) \\
 &\cong \text{Func}_{\mathcal{Q}(Z)}(\mathcal{Q}(X), \mathcal{Q}(Y))
 \end{aligned}$$

The corresponding theorem for \mathcal{D} -modules only holds for smooth schemes, it is false for stacks⁵.

³in *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (2007), no. 3, 615–667.

⁴in *Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry*.

⁵Ben-Zvi and Nadler in *The Character Theory of a Complex Group*