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Workshop on TFT's at Northwestern
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# 1 Lecture 3

## 1.1 Overview.

In the first lecture we discussed an example of 2-dimensional TFT constructed from a finite group $\Gamma$ by assigning to the point the category of modules of the group algebra $\mathbb{C}[\Gamma]$. In the second lecture we discussed categorical versions of the group algebra for a complex reductive group $G$[1]. This was in preparation for a discussion of topological field theories associated to $G$. These require higher categorical constructions, namely, 2-categories of $G$-module categories assigned to the point.

This lecture focuses on two versions of $G$-module categories: algebraic $G$-categories and smooth $G$-categories. By a result of Ben-Zvi, Francis and Nadler, assigning the 2-category of algebraic $G$-categories to the points defines a 2-dimensional TFT. Assigning the 2-category of smooth $G$-categories to the point only defines a 1-dimensional TFT. A modified version extends up to 2-manifolds. This modification is defined by assigning to the point the 2-category of $\mathcal{H}$−mod where $\mathcal{H}$ is the finite Hecke category. From a physics perspective all of these are part of a 3-dimensional gauge theory.

## 1.2 $3d$ TFT from a Finite Group.

Passing to a 3-dimensional TFT, we need a notion of an action of a finite group $\Gamma$ on a category $\mathcal{C}$ . This means to any $\gamma \in \Gamma$ we should have a functor $a_\gamma : \mathcal{C} \to \mathcal{C}$ and for every pair of elements in $\Gamma$ we should have a natural isomorphism $a_{\gamma_1\gamma_2} \Rightarrow a_{\gamma_1} \circ a_{\gamma_2}$ satisfying some coherence condition.

Let $\mathrm{Vect}_\mathbb{C}\Gamma$ be the category of complex vector bundles over $\Gamma$. It is a monoidal category via convolution defined by the push-forward of vector bundles along the multiplication map $\mu : \Gamma \times \Gamma \to \Gamma$. Let $\mathcal{C}$ be a ($\mathbb{C}$-linear, abelian) $\mathrm{Vect}_\mathbb{C}\Gamma$−module category. A $\Gamma$−action is defined on $\mathcal{C}$ . The functor $a_\gamma$ is given by $\mathbb{C}_\gamma \otimes -$, where $\mathbb{C}_\gamma$ is the 1-dimensional vector bundle supported at $\gamma \in \Gamma$.

[1] Throughout this lecture we will often write $\Gamma$ when we mean a finite group and $G$ when we mean a complex group.

$\mathrm{Vect}_{\mathbb{C}}\Gamma$ may be viewed as a categorification of the group algebra $\mathbb{C}[\Gamma]$. Its complexified Grothendieck ring is exactly $\mathbb{C}[\Gamma]$. When $\mathrm{Vect}_{\mathbb{C}}$ is considered as a $\mathrm{Vect}_{\mathbb{C}}\Gamma-$module category, its category of endo-functors, $\mathrm{End}_{\mathrm{Vect}\Gamma}(\mathrm{Vect})$, is isomorphic to $\mathrm{Rep}_{\mathbb{C}}\Gamma$. Therefore we can recover our set-theoretic notion of a $\Gamma-$action from $\mathrm{Vect}_{\mathbb{C}}\Gamma-$mod.

We can construct a fully extended 3-dimensional TFT associated to a finite group $\Gamma$, by assigning to closed $(0,1,2,3)-$manifolds:

$$\begin{array}{rcll} \mathcal{Z}_\Gamma : \quad \bullet & \rightsquigarrow & \mathrm{Vect}_{\mathbb{C}}\Gamma\text{-mod} & \text{(2-category)} \\ S^1 & \rightsquigarrow & \mathrm{Vect}(\frac{\Gamma}{\Gamma}) & \text{(category)} \\ \Sigma^2 & \rightsquigarrow & \mathrm{Func}(\mathcal{M}_\Gamma(\Sigma)) & \text{(vector space)} \\ N^3 & \rightsquigarrow & \int_{\mathcal{M}_\Gamma(N)} e^{iS(\phi)} d\phi & \text{(number)} \end{array}$$

where $\mathrm{Vect}(\frac{\Gamma}{\Gamma})$ denotes the category of $\Gamma$-equivariant vector bundles over $\Gamma$. Note that $\mathrm{Vect}(\frac{\Gamma}{\Gamma})$ can be identified with the Drinfeld center of $\mathrm{Vect}_{\mathbb{C}}\Gamma-$mod. We will have more to say about that later.

## 1.3 TFT from a Complex Group.

In section 1.2 we were working in the category of $\mathbb{C}-$linear abelian categories. From now on we will be working in the category of dg-categories or $\infty-$categories. In particular, everything in sight is derived.

When $G$ is a complex reductive group we have several variants of the group algebra to choose from. Accordingly, we have several notions of a $G$-action on a category depending on what categorical version of the group algebra we choose (conveniently enough, when the group was finite all those different notions agreed with $\mathrm{Vect}_{\mathbb{C}}\Gamma-$mod).

In the second lecture we introduced two versions of categorical function theories on $G$. One was the category $\mathcal{Q}(G)$ of quasi-coherent sheaves on $G$, and the other was the category $\mathcal{D}(G)$ of $\mathcal{D}$-modules on $G$. Both are monoidal dg-categories via convolution defined by the push-forward along the multiplication map

$$\mathcal{F} * \mathcal{G} := \mu_*(\pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G})$$

We refer to $\mathcal{Q}(G)-$module categories as *algebraic* $G$-categories and to $\mathcal{D}(G)-$module categories as *smooth* $G$-categories.

A construction of a $3d$ TFT associated to a complex reductive group $G$ may start by assigning a 2-category to the point

$$\mathcal{Z}_G : \bullet \rightsquigarrow \{G-\text{categories}\}$$

Considering our two versions of $G$-categories, we ask to what extent we can extend this assignment to higher dimensional manifolds.

By a theorem of Ben-Zvi, Francis and Nadler, the assignment of algebraic $G$-categories to a point

$$\mathcal{Z}_G^{\mathcal{Q}} : \bullet \rightsquigarrow \mathcal{Q}(G) - \mathrm{mod}$$

extends to a 2-dimensional TFT. As for smooth $G$-categories, the assignment

$$\mathcal{Z}_G^{\mathcal{D}} : \bullet \rightsquigarrow \mathcal{D}(G) - \mathrm{mod}$$

only extends only up to 1-dimensional TFT. Later on we will consider a modified version of $\mathcal{Z}_G^{\mathcal{D}}$ that extends to 2-manifolds.

## 1.4 Algebraic $G$-categories.

A natural source of $\mathcal{Q}(G)$–module categories comes from algebraic $G$-actions. By pushing forward along the action map $G \times X \to X$, we get a functor

$$\mathcal{Q}(G) \otimes \mathcal{Q}(X) \to \mathcal{Q}(X)$$

In particular, given a subgroup $K \subset G$, the category $\mathcal{Q}(G/K)$ is an algebraic $G$-category. It is the analogue of the induced representation

$$\mathrm{Ind}_K^{\Gamma}\mathbb{C} = \mathbb{C}[\Gamma/K] \in \mathbb{C}[\Gamma] - \mathrm{mod} = \mathcal{Z}_{\Gamma}(\bullet)$$

for a finite group $\Gamma$. The *Hecke category* is the category of endo-functors

$$\begin{aligned} \mathcal{H}_{G,K}^{\mathcal{Q}} &:= \mathrm{End}_{\mathcal{Q}(G)}(\mathcal{Q}(G/K)) \\ &= \mathcal{Q}(K\backslash G/K) \end{aligned}$$

and is the analogue of the Hecke algebra discussed in the second lecture. When $K = G$, we have $\mathcal{Q}(G/G) = \mathrm{Vect}_{\mathbb{C}}$ as an algebraic $G$-category and

$$\begin{aligned} \mathcal{H}_{G,G}^{\mathcal{Q}} &:= \mathrm{End}_{\mathcal{Q}(G)}(\mathrm{Vect}_{\mathbb{C}}) \\ &= \mathcal{Q}(G\backslash G/G = \bullet/G) \\ &= \mathrm{Rep}_{\mathbb{C}} G \end{aligned}$$

where representations are not necessarily finite dimensional. As in the finite case, we recover our set-theoretic $G$-action.

## 1.5 Morita Theory.

In the second lecture we discussed function theory of finite sets and used it as an analogy for categorical function theory of schemes. We continue to pursue this analogy here. Let $X$ be a finite set and consider the algebra of (complex valued) functions $\mathrm{Func}(X)$.

$$\begin{aligned} \mathrm{End}_{\mathbb{C}}(\mathrm{Func}(X)) &= \mathrm{Func}(X \times X) \\ &= X \times X \text{ matrices} \end{aligned}$$

The algebra of endomorphisms is Morita equivalent to $\mathrm{Func}(\bullet)$:

$$\begin{aligned} \mathrm{Func}(X \times X) - \mathrm{mod} &= \mathrm{Func}(\bullet) - \mathrm{mod} \\ &= \mathrm{VBun}(\bullet) \end{aligned}$$

More generally, consider a surjective map $X \to Y$ of finite sets.

$$\begin{aligned} \mathrm{End}_{\mathrm{Func}(Y)}(\mathrm{Func}(X)) &= \mathrm{Func}(X \times_Y X) \\ &= X \times X \text{ block matrices with blocks} \\ &\quad\ \text{labeled by elements of } Y \end{aligned}$$

This algebra of endomorphisms is Morita equivalent to Func($Y$):

$$\begin{aligned} \mathrm{Func}(X \times_Y X) - \mathrm{mod} &= \mathrm{Func}(Y) - \mathrm{mod} \\ &= \mathrm{VBun}(Y) \end{aligned}$$

Back in the world of algebraic stacks, a result due to Ben-Zvi, Francis and Nadler states that the category of endo-functors $\mathcal{H}^{\mathcal{Q}}_{G,K}$ is Morita equivalent to $\mathcal{Q}(G)$:

$$\mathcal{Q}(\bullet/K \times_{\bullet/G} \bullet/K) - \mathrm{mod} \quad = \quad \mathcal{Q}(\bullet/G) - \mathrm{mod}$$

for any subgroup $K \subset G$. In particular, we can define $\mathcal{Z}^{\mathcal{Q}}_G$ by assigning to the point the 2-category of $\mathcal{H}^{\mathcal{Q}}_{G,K}$–modules. This no longer holds for $\mathcal{Z}^{\mathcal{D}}_G$ as we will see shortly.

## 1.6 Smooth $G$-categories.

Examples of smooth $G$-categories arise in a similar fashion. We restrict attention to the case where $B \subset G$ is a Borel subgroup of $G$ and consider the flag variety $G/B$ of $G$. It can be decomposed into (contractible) $B$-orbits with respect to multiplication on the left. These are the *Schubert cells* parametrized by elements of the Weyl group $W$ of $G$.

For example, let $G$ be $GL_n(\mathbb{C})$, $B$ be the subgroup of upper triangular matrices and $W \cong S_n$. The flag variety is the manifold of full flags in $\mathbb{C}^n$. Decomposition into Schubert cells is given by $\coprod_{\sigma \in S_n} BP_\sigma B$ where $P_\sigma$ is a permutation matrix.

As discussed previously, the Hecke category is the category of endo-functors

$$\begin{aligned} \mathcal{H}^{\mathcal{D}}_{G,B} &:= \mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}(G/B)) \\ &= \mathcal{D}(B \backslash G/B) \end{aligned}$$

By the above, $K_0(\mathcal{H}) \cong \mathcal{Z}W$ and there exist natural "bases" of $\mathcal{H}$ given by the different ways of extending a trivial flat bundle $\mathbb{C}_w$ on each orbit labeled $w \in W$.

One such basis is the "standard" basis $T_w := (i_w)_* \mathbb{C}_w \in \mathcal{H}$. For simple reflections, $s_i \in W$, $1 \leq i \leq n-1$, we have

$$\begin{aligned} T_{s_i} * T_{s_j} * T_{s_i} &\cong T_{s_j} * T_{s_i} * T_{s_j} \quad && |i-j| = 1 \\ T_{s_i} * T_{s_j} &\cong T_{s_j} * T_{s_i} && |i-j| \geq 2 \end{aligned}$$

As a result, an action of $\mathcal{H}$ on a category $\mathcal{C}$ makes it into a $B_n$–category[2], not a $W$-category!

[2] $B_n$ denotes the *Braid group* on $n$ strands.

A well known theorem due to Beilinson and Berstein states an equivalence of categories (via global sections)

$$\Gamma : \mathcal{D}(G/B) \xrightarrow{\sim} \mathfrak{g} - \mathrm{mod}_0$$

where the subscript on the right hand side means a trivial action of $Z(U\mathfrak{g})$[3]. This is an equivalence of $G$-categories where $\mathcal{D}(G)$ acts on $\mathcal{D}(G/B)$ and $G$ acts on $\mathfrak{g} - \mathrm{mod}_0$ via the Adjoint action.

The category $\mathcal{D}(G/B)$, and hence $\mathfrak{g} - \mathrm{mod}_0$, comes equipped with two commuting actions

$$\mathcal{D}(G) \circlearrowright \mathcal{D}(G/B) \circlearrowleft \mathcal{H}^{\mathcal{D}}_{G,B}$$

The action of $\mathcal{H}^{\mathcal{D}}_{G,B}$ on $\mathfrak{g} - \mathrm{mod}_0$ is given geometrically by convolution with the Hecke algebra, or algebraically by intertwining operators on the category of representations (generalizing the classical intertwiners on representations).

The Hecke category $\mathcal{H}^{\mathcal{D}}_{G,B}$ has the advantage that the assignment

$$\mathcal{Z}^{\mathcal{H}}_G : \bullet \rightsquigarrow \mathcal{H}^{\mathcal{D}}_{G,B} - \mathrm{mod}$$

extends to a 2-dimensional TFT by a theorem of Ben-Zvi and Nadler which is not the case for $\mathcal{Z}^{\mathcal{D}}_G$.

Examples of $\mathcal{H}^{\mathcal{D}}_{G,B}$-modules arise as "subs" of $\mathcal{D}(G/B)$, namely, $\mathcal{D}(K\backslash G/B)$ for $K \subset G$. By a result of Beilinson and Berstein,

$$\mathcal{D}(K\backslash G/B) = \text{Harish-Chandra } (\mathfrak{g}, K)\text{−modules}$$

When $K \subset G$ is a subgroup of fixed points of an involution of $G$, $\mathcal{D}(K\backslash G/B)$ is the category of (possibly infinite dimensional) representations of a real form of $G$.

### 1.7 Center and Dim.

The first half of the second lecture was devoted to a dicussion of the character and action maps. Given $\mathcal{Z}$ a fully extended 2-dimensional TFT, the action and character maps are what $\mathcal{Z}$ assigns to the bordism in fig. 1 and its mirror image. This bordism relates the open and closed sectors of the theory. For every object $V \in \mathcal{Z}(\bullet)$ we have two dual maps

$$\begin{array}{lcl} \text{action} & : & \mathcal{Z}(S^1) \to \mathrm{End}_{\mathcal{Z}(\bullet)}(V) \\ \text{character} & : & \mathrm{End}_{\mathcal{Z}(\bullet)}(V) \to \mathcal{Z}(S^1) \end{array}$$

Note that $\mathcal{Z}(S^1)$ plays a double role. On the one hand, it acts on every object $V \in \mathcal{Z}(\bullet)$, and on the other, it carries its characters. We can define the action map, when $\mathcal{Z}(S^1)$ is identified with the "center" or Hochschild cohomology of $\mathcal{Z}(\bullet)$. When we dualize, $\mathcal{Z}(S^1)$ is identified with the "dimension" or Hochschild homology of $\mathcal{Z}(\bullet)$.

[3]This statement generalizes to generic parameters $\lambda \in \mathfrak{h}^*/W$.

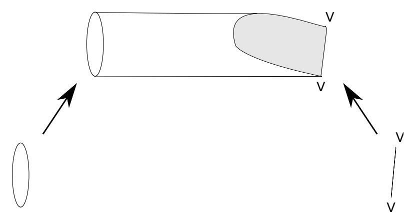

Figure 1: Bordism relating open and closed sectors.

**Example I.** Consider $\mathcal{Z}_\Gamma$ the extended 2-dimensional TFT arising from gauge theory with finite symmetry group $\Gamma$. In the first lecture we argued that

$$\mathcal{Z}_\Gamma(\bullet) = \mathbb{C}[\Gamma] - \mathrm{mod} \ , \ \ \mathcal{Z}_\Gamma(S^1) = \mathbb{C}[\Gamma]^\Gamma$$

We can also compute $\mathcal{Z}_\Gamma(S^1)$ using a decomposition of $S^1$ into bordisms as in fig. 2. $\mathcal{Z}_\Gamma(S^1)$ is given as the image of the unit object $\mathbb{1} \in \mathrm{Vect}_\mathbb{C}$ under the associated map.

$$\mathbb{1} \longmapsto \sum_{\substack{V \\ \text{irrep}}} (V, V) \longmapsto \bigoplus_{\substack{V \\ \text{irrep}}} \mathrm{Hom}_{\mathcal{Z}_\Gamma(\bullet)}(V, V)$$

We can identify $\mathcal{Z}_\Gamma(\bullet)^{\mathrm{op}}$ with $\mathrm{mod} - \mathbb{C}[\Gamma]$ and equivalently calculate $\mathcal{Z}_\Gamma(S^1)$ as the image of $\mathbb{1}$ under

$$\mathbb{1} \longmapsto \sum_{\substack{V \\ \text{irrep}}} (V^\vee, V) \longmapsto \bigoplus_{\substack{V \\ \text{irrep}}} V^\vee \otimes_{\mathbb{C}[\Gamma]} V$$

This re-establishes well known identifications

$$\mathbb{C}[\Gamma]^\Gamma = \bigoplus_{\substack{V \\ \text{irrep}}} \mathrm{End}_\Gamma(V) = \bigoplus_{\substack{V \\ \text{irrep}}} V^\vee \otimes_{\mathbb{C}[\Gamma]} V$$

The center of $\mathcal{Z}_\Gamma(\bullet)$[4] is given by the categorical trace of the identity functor:

$$\mathrm{Center}(\mathcal{Z}_\Gamma(\bullet)) = \mathrm{End}(\mathrm{id}_{\mathcal{Z}_\Gamma(\bullet)}) = \bigoplus_{\substack{V \\ \text{irrep}}} \mathrm{Hom}_{\mathcal{Z}_\Gamma(\bullet)}(V, V)$$

with the later equality due to semi-simplicity of the category $\mathcal{Z}_\Gamma(\bullet)$. The center of $\mathbb{C}[\Gamma] - \mathrm{mod}$ also equals the Hochschild cohomology of $\mathbb{C}[\Gamma]$.

$$\mathrm{Center}(\mathcal{Z}_\Gamma(\bullet)) = HH^0(\mathbb{C}[\Gamma]) = Z(\mathbb{C}[\Gamma]) = \mathbb{C}[\Gamma]^\Gamma$$

With this identification in mind, we can define the action map by the restriction of $\eta \in \mathrm{End}(\mathrm{id}_{\mathcal{Z}_\Gamma(\bullet)})$ to each object $V \in \mathcal{Z}_\Gamma(\bullet)$.

[4] Also known as the *Bernstein center*.

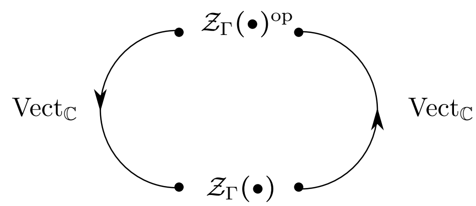


Figure 2: $S^1$ as a composition of two bordisms.

To get the character map we can take the dual of the action map. Explicitly we have for every $\varphi \in \mathrm{End}_{\mathcal{Z}(\bullet)}(V)$

$$\text{character} : \varphi \longmapsto \{f_\varphi : \gamma \in \Gamma \mapsto \mathrm{tr}(\gamma\varphi) \in \mathbb{C}\}$$

with $f_\varphi$ being a class function. The dimension of $\mathcal{Z}_\Gamma(\bullet)$ is given by the Hochschild homology of $\mathbb{C}[\Gamma]$.

$$\mathrm{Dim}(\mathcal{Z}_\Gamma(\bullet)) = HH_0(\mathbb{C}[\Gamma]) = \mathrm{Ab}(\mathbb{C}[\Gamma]) = \mathbb{C}[\Gamma]/[\mathbb{C}\Gamma, \mathbb{C}\Gamma]$$

where $\mathrm{Ab}(\mathbb{C}[\Gamma])$ stands for the abelianization of $\mathbb{C}[\Gamma]$.

**Example II.** $\mathcal{Z}_\Gamma$ is the 3-dimensional TFT associated to $\Gamma$ (see section 1.2). To the point, $\mathcal{Z}_\Gamma$ assigns the 2-category of $\mathrm{Vect}_{\mathbb{C}}\Gamma-$modules. The center of $\mathcal{Z}_\Gamma(\bullet)$ is the *Drinfeld center* of the monoidal category $\mathrm{Vect}_{\mathbb{C}}\Gamma$ of vector bundles over $\Gamma$. Explicitly, the Drinfeld center is a category whose objects are pairs consisting of $V \in \mathrm{Vect}_{\mathbb{C}}\Gamma$ and a natural isomorphism $V \otimes - \xrightarrow{\sim} - \otimes V$. In this particular case, the Drinfeld center of $\mathrm{Vect}_{\mathbb{C}}\Gamma$ is given by the category of $\Gamma-$equivariant vector bundles over $\Gamma$

$$Z(\mathrm{Vect}_{\mathbb{C}}\Gamma) = \mathrm{Vect}\frac{\Gamma}{\Gamma} = \prod_{[\gamma]} \mathrm{Rep}\, Z_\Gamma(\gamma)$$

where $[\gamma]$ denotes a conjugacy class in $\Gamma$.

**Example III.** The following theorem establishes the assignment $\mathcal{Z}_G^{\mathcal{Q}}(S^1) = \mathcal{Q}(\frac{G}{G})$.

**Theorem:** [Ben-Zvi - Francis - Nadler] The category $\mathcal{Q}(\frac{G}{G})$ is the Center and Dim of $\mathcal{Q}(G)$ and of $\mathcal{Q}(K\backslash G/K)$ for all $K \subset G$.