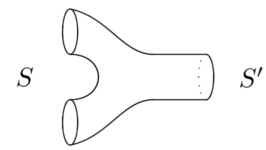
1 Lecture 1

Theorem 1. *(The cobordism hypothesis)*
$nCob$ is the free stable (∞, n)-category on a fully dualizable object.

$nCob$ began life as a category where

- objects are framed (compact smooth) $(n-1)$-dimensional manifolds and
- morphisms are framed (compact smooth) n-dimensional cobordisms.

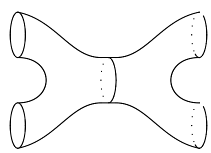
A picture:



(Picture of an oriented cobordism M going from $S = S^1 \sqcup S^1$ to $S' = S^1$, where orientation of M should match orientations of S' and opposite to orientation of S.)

Roughly, M is an n-dimensional manifold with boundary $\partial M = S \cup S'$.
A "framing" is a structure on a manifold which gives (among other things) an orientation.

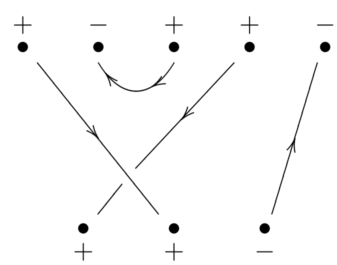
We compose morphisms in $nCob$ roughly as follows:



We need to carefully define "cobordism" to make composition well-defined, associative and have identities. The challenge: to find a purely algebraic description of $nCob$.
For

- $n = 1$, this is easy
- $n = 2$, still easy
- $n = 3$, a real mess...
- $n = 4$, nobody knows...here it becomes good to think of $nCob$ as an n-category or an (∞, n)-category

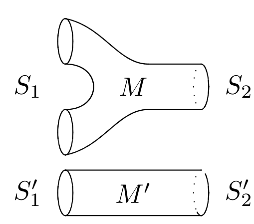
Consider $n = 1$,



(Picture of 1-tangle in 2-dimensions with orientations on 0 and 1-dimensional manifolds.)

This is a typical morphism in $1Cob$. Here we do not have to worry about closed links since we are not embedded in 3-dimensional space.

It turns out that $nCob$ is better than just a category; it is a symmetric monoidal (or stable) category: *roughly* a category with a "tensor product":

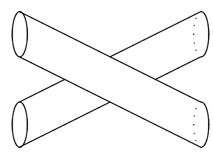


(Picture of cobordisms M and M' with $n = 2$ set side by side representing $M \otimes M'$.)

This captures the idea of monoidal. The idea of symmetric is a further structure which says that we have a "symmetry" isomorphism

$$S \otimes S' \xrightarrow{\cong} S' \otimes S.$$

(Picture of crossing cobordisms representing symmetry isomorphism.)



So $1Cob$ is a symmetric monoidal category, but also there is a special object:

$$x = \bullet_+$$

the positively oriented point, which has a "dual":

$$x^* = \bullet_-.$$

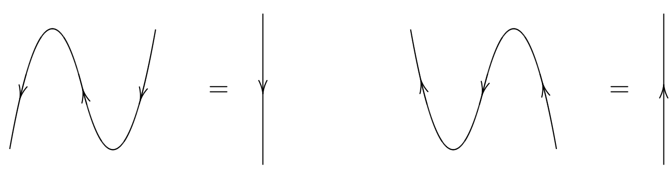
If x is an object in a symmetric monoidal category, we say a **dual** for it is an object x^* with morphisms:

$$e_x \colon x^* \otimes x \to I(\text{the counit}),$$

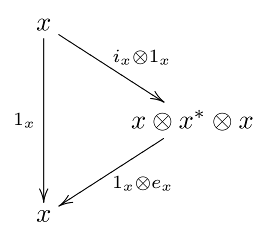
where I is the identity for $\otimes$ and

$$i_x \colon I \to x \otimes x^*(\text{the unit})$$

satisfying zig-zag identities: *(Picture of zig-zag identities)*



This comes from part of the definition of duals which says the following diagram commutes:



and a similar diagram for the second zig-zag equation.

Now, we have an *almost theorem*:

Theorem 2. *$1Cob$ is the free symmetric monoidal category on one object x with a dual.*

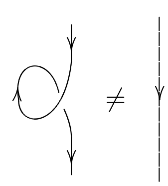
Typical morphisms $x \otimes x^* \otimes x^* \otimes x \otimes x \xrightarrow{f} x$ are very easy to describe using the rules of orientation in the diagram pictures.

Then using the zig-zag equations we see that we can wiggle this picture around without changing the morphism.

So this is now starting to sound like the cobordism hypothesis. But we need to see what "fully dualizable" means.

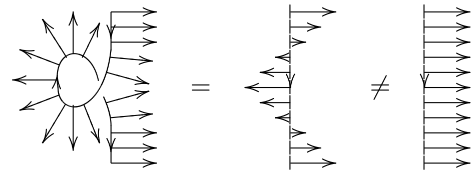
There is a problem: in the free stable category on one object with a dual:

(Picture of nonequality of first Reidermeister move.)



but in 1Cob they seem equal. We need to change our definition of 1Cob so that these are not equal. This is where the notion of "framings" comes into play. In this case, framings are a choice of smooth field of normal vectors along a 1-morphism.

(Picture of nonequality of first Reidermeister move with framings.)



Passing to the case of $n = 2$ it becomes harder to see how we will find a purely algebraic description. We will need to think of cobordisms as living in a 2-category with very elementary building blocks with managable collections of critical points.

So the original Baez-Dolan idea was to generalize this almost theorem to say:

nCob is the free symmetric monoidal n-category on one object x with a dual.