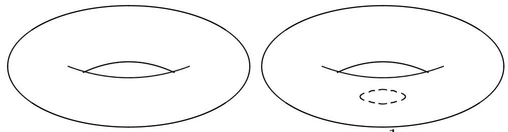
1 Lecture 2

This is the first seminar class given by JB = Julie Bergner. We start with some background on the definitions or general notions of cobordism.

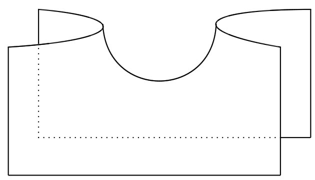
1.1 Cobordisms

Note: All manifolds compact and smooth throughout time, with or without boundary.



(Pictures of torus and torus with S^1 boundary.)

Later, we will see manifolds with corners:

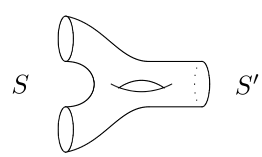


(Picture of manifold with corners.)

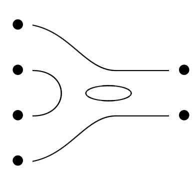
Basic definition of cobordism:

Definition 1. *Two closed $(n-1)$-dimensional manifolds M and N are* **cobordant** *if there exists an n-dimensional manifold W such that $\partial W = M \sqcup N$. W is a* **cobordism**.

Picture of pair of pants with a hole for n = 2.

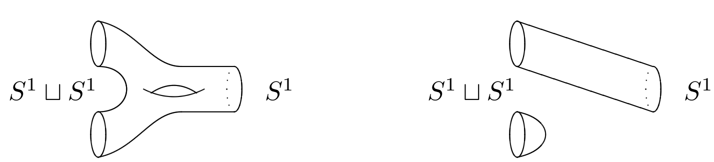


Picture of pair of pants with a hole for n = 1.



Can have different cobordisms between M and N.

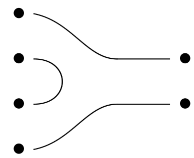
Pictures of cobordisms from $S^1 \sqcup S^1$ to S^1



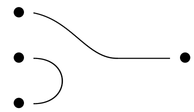
We can define an equivalence relation on $(n-1)$-dimensional closed manifolds by $M \sim N$ if M and N are cobordant.
For $n = 1$, there are 2 cobordism classes:

Picture of "even" and "odd" cobordisms.

The even class:



the odd class:



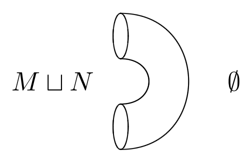
For $n = 2$, there is only 1 cobordism classes:

Picture of cobordism class.

Given $(n-1)$-dimensional M and N we can take $M \sqcup N$. This operation has a unit, the empty manifold.
Fact: After taking cobordism classes this actualy forms a group under $\sqcup$.

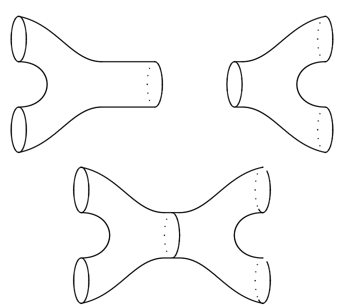
In particular, taking cobordism classes give manifolds as their own inverses.

Picture of an elbow noodle $M \sqcup N \to \emptyset$.



We can compose cobordisms by gluing together copies of S^1 in the 2-dimensional case.

Picture of composition of cobordisms up to diffeomorphism (for purpose of well-defined domain and codomain.)



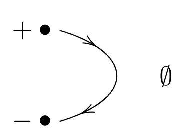
Up to diffeomorphism classes, composition is associative.
What about the composition being smooth?

Technically, this is resolved by adding collars on the boundaries. Can find a way to have the composite cobordism be smooth. We make choices in assigning collars, but up to diffeomorphism they are irrelevant.
We don't want just plain smooth manifolds. We want manifolds with extra structure. In particular, we add orientations, framings and ask to preserve this new structure.

1.2 Oriented manifolds

Definition 2. *Two closed oriented $(n-1)$-dimensional manifolds M and N are* **(oriented) cobordant** *if there exists an n-dimensional manifold W such the $\partial W = \bar{M} \sqcup N$, M with opposite orientation.*

Picture of identity cobordism with orientations.



Definition 3. *Define the category nCob to have:*

- *objects closed oriented* $(n-1)$*-manifolds* M
- *morphisms from* M *to* N *are cobordisms from* M *to* N *under the equivalence relation* $W \sim W'$ *if there is an oreientation-preserving diffeomorphism* $W \to W'$ *compatible with the diffeomorphisms*

$$\partial W \cong \bar{M} \sqcup N \cong \partial W'.$$

- *the identity map* $id\colon M \to M$ *is* $W = M \times [0,1]$.
- *composition of morphisms is composition of cobordisms.*

We are going to look at framed manifolds, which give orientations, but that will come later.