1 Lecture 3

Last time we introduced the category \( n\text{Cob} \) where the objects were closed \((n-1)\)-dimensional manifolds and morphisms were diffeomorphism classes of cobordisms between them. In fact, \( n\text{Cob} \) has the additional structure of a symmetric monoidal category.

A quick crash course in symmetric monoidal categories:

**Definition 1.** A category \( C \) is **symmetric monoidal** if it is equipped with a functor 
\[
\otimes : C \times C \to C
\]
and a unit object \( 1_C \) such that for objects \( C, D, E \in C \):
\[
C \times 1_C \xrightarrow{\sim} C
\]
\[
C \otimes D \xrightarrow{\sim} D \otimes C
\]
\[
C \otimes (D \otimes E) \xrightarrow{\sim} (C \otimes D) \otimes E
\]
with lots of commuting diagrams. See John Baez’s “[Some definitions everyone should know”.

**Example 2.** \( n\text{Cob} \) with \( \otimes = \sqcup \) and \( 1_{n\text{Cob}} = \emptyset \).

**Example 3.** \( \text{Vect}(k) \) = category of vector spaces over a field \( k \) with \( \otimes \) the usual tensor product of vector spaces and \( 1_{\text{Vect}(k)} = k \).

**Definition 4.** Let \( (C, \otimes) \) and \( (D, \odot) \) be symmetric monoidal categories. A functor \( F : C \to D \) is **symmetric monoidal** if it is equipped with isomorphisms

\[
F(C \otimes C') \xrightarrow{\sim} F(C) \odot F(C')
\]
and

\[
F(1_C) \xrightarrow{\sim} 1_D
\]

and some diagrams commute:

\[
\begin{array}{ccc}
F(C) \odot 1_D & \xrightarrow{\sim} & F(C) \\
\uparrow_{\sim} & & \uparrow_{\sim} \\
F(C) \odot F(1_C) & \xrightarrow{\sim} & F(C \otimes 1_C)
\end{array}
\]

and some more.

**Definition 5.** A **topological field theory** (TFT) in dimension \( n \) is a symmetric monoidal functor \( n\text{Cob} \to \text{Vect}(k) \).

What does this mean?

- Every closed \((n-1)\)-dimensional manifold \( M \) is assigned to a vector space \( Z(M) \).
- Every (diffeomorphism class of a) cobordism \( B \) from \( M \) to \( N \) is assigned to a linear map \( Z(B) : Z(M) \to Z(N) \).
- There are isomorphisms
  \[
  Z(\emptyset) \xrightarrow{\sim} k
  \]
  \[
  Z(M \sqcup N) \xrightarrow{\sim} Z(M) \otimes Z(N)
  \]
  and coherence properties.

Suppose \( B \) is a closed \( n \)-dimensional manifold. So \( B \) is a cobordism from \( \emptyset \) to \( \emptyset \) and \( B \) represents a morphism \( \emptyset \to \emptyset \) in \( n\text{Cob} \). Given a TFT \( Z \), \( B \) corresponds to a map \( Z(B) : Z(\emptyset) \to Z(\emptyset) \) essentially, some map \( k \to k \). But a map \( k \to k \) is determined by a single vector. So \( B \) corresponds to a single vector in \( k \). Suppose \( B \) is an \( n \)-dimensional manifold with boundary. \( B \) can be regarded as a cobordism in several different ways.
Example 6. $B = M \times [0,1]$ manifold with boundary $\bar{M} \sqcup M$.

Picture: cylinder cobordism from $M$ to $M$, the identity in $nCob$

\[
\begin{array}{c}
\includegraphics{cylinder_1} \\
M \quad \bar{M}
\end{array}
\]

This can be flipped around to get identity on $\bar{M}$.

Picture: cylinder of elbow noodle from $\bar{M} \sqcup M$, $\bar{M} \sqcup M \to \emptyset$, the evaluation map

\[
\begin{array}{c}
\includegraphics{cylinder_2} \\
M \sqcup \bar{M}
\end{array}
\]

Corresponding picture for coevaluation

\[
\begin{array}{c}
\includegraphics{cylinder_3} \\
\emptyset \quad M \sqcup \bar{M}
\end{array}
\]

The idea behind these names:

Apply a TFT $Z$:

\[
Z(\bar{M}) \otimes Z(M) \cong Z(\bar{M} \sqcup M)^{Z(ev_M)} \to Z(\emptyset) \cong k
\]

corresponds to

\[
Z(M)^\vee \otimes Z(M) \to k
\]

the evaluation map.

Proposition 7. Let $Z$ be a TFT of dimension $n$. Then for every closed $(n-1)$-dimensional manifold $M$, the vector space $Z(M)$ is finite-dimensional, and the pairing

\[
Z(\bar{M}) \otimes Z(M) \to k
\]

is perfect: it induces

\[
\alpha : Z(\bar{M}) \cong Z(M)^\vee
\]

Proof. The idea of the proof:

\[
Z(\bar{M}) \otimes Z(M) \to k
\]

\[
Z(M) \cong Z(M)^\vee
\]

We need an inverse to $\alpha$:

Coevaluation map gives:

\[
kZ(\emptyset) \xrightarrow{Z(coev_M)} Z(M \sqcup \bar{M}) \cong Z(M) \otimes Z(\bar{M})
\]

Tensor with $Z(M)^\vee$:

\[
Z(M)^\vee \xrightarrow{\beta} Z(M)^\vee \otimes Z(M) \otimes Z(\bar{M})
\]

Can show $\beta \cong \alpha^{-1}$. □