1 Lecture 3

Last time we introduced the category $nCob$ where the objects were closed $(n-1)$-dimensional manifolds and morphisms were diffeomorphism classes of cobordisms between them. In fact, $nCob$ has the additional structure of a symmetric monoidal category.

A quick crash course in symmetric monoidal categories:

Definition 1. *A category $\mathcal{C}$ is* **symmetric monoidal** *if it is equipped with a functor*

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and a unit object $1_{\mathcal{C}}$ such that for objects $C, D, E \in \mathcal{C}$:

$$C \times 1_{\mathcal{C}} \overset{\cong}{\to} C$$

$$C \otimes D \overset{\cong}{\to} D \otimes C$$

$$C \otimes (D \otimes E) \overset{\cong}{\to} (C \otimes D) \otimes E$$

with lots of commuting diagrams. See John Baez's "Some definitions everyone should know".

Example 2. *$nCob$ with $\otimes = \sqcup$ and $1_{nCob} = \emptyset$.*

Example 3. *$Vect(\mathbf{k})$ = category of vector spaces over a field $\mathbf{k}$ with $\otimes$ the usual tensor product of vector spaces and $1_{Vect(\mathbf{k})} = \mathbf{k}$.*

Definition 4. *Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \odot)$ be symmetric monoidal categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is* **symmetric monoidal** *if it is equipped with isomorphisms*

$$F(C \otimes C') \overset{\cong}{\to} F(C) \odot F(C')$$

and

$$F(1_{\mathcal{C}}) \overset{\cong}{\to} 1_{\mathcal{D}}$$

and some diagrams commute:

$$\begin{array}{ccc} F(C) \odot 1_{\mathcal{D}} & \xrightarrow{\cong} & F(C) \\ \uparrow \cong & & \cong \uparrow \\ F(C) \odot F(1_{\mathcal{C}}) & \underset{\cong}{\leftarrow} & F(C \otimes 1_{\mathcal{C}}) \end{array}$$

and some more.

Definition 5. *A* **topological field theory** *(TFT) in dimension n is a symmetric monoidal functor $nCob \to$ $\mathrm{Vect}(\mathbf{k})$.*

What does this mean?

- Every closed $(n-1)$-dimensional manifold M is assigned to a vector space $Z(M)$.
- Every (diffeomorphism class of a) cobordism B from M to N is assigned to a linear map $Z(B) \colon Z(M) \to Z(N)$.
- There are isomorphisms

  $$Z(\emptyset) \overset{\cong}{\to} \mathbf{k}$$

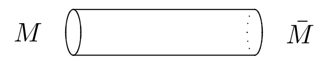
  $$Z(M \sqcup N) \overset{\cong}{\to} Z(M) \otimes Z(N)$$

  and coherence properties.

Suppose B is a closed n-dimensional manifold. So B is a cobordism from $\emptyset$ to $\emptyset$ and B represents a morphism $\emptyset \to \emptyset$ in $nCob$. Given a TFT Z, B correpsonds to a map $Z(B) \colon Z(\emptyset) \to Z(\emptyset)$ essentially, some map $\mathbf{k} \to \mathbf{k}$. But a map $\mathbf{k} \to \mathbf{k}$ is determined by a single vector. So B corresponds to a single vector in $\mathbf{k}$. Suppose B is an n-dimensional manifold with boundary. B can be regarded as a cobordism in several different ways.

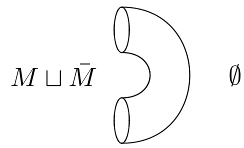
Example 6. *$B = M \times [0,1]$ manifold with boundary $\bar{M} \sqcup M$.*

Picture: cylinder cobordism from M to M, the identity in $nCob$

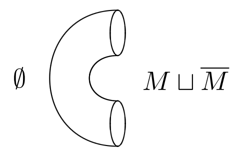


This can be flipped around to get identity on $\bar{M}$.

Picture: cylinder of elbow noodle from $\bar{M} \sqcup M, \bar{M} \sqcup M \to \emptyset$, the evaluation map



Corresponing picture for coevaluation



The idea behind these names:
Apply a TFT Z:

$$Z(\bar{M}) \otimes Z(M) \cong Z(\bar{M} \sqcup M) \overset{Z(ev_M)}{\to} Z(\emptyset) \cong \mathbf{k}$$

corresponds to

$$Z(M)^\vee \otimes Z(M) \to \mathbf{k}$$

the evaluation map.

Proposition 7. *Let Z be a TFT of dimension n. Then for every closed $(n-1)$-dimensional manifold M, the vector space $Z(M)$ is finite-dimensional, and the pairing*

$$Z(\bar{M}) \otimes Z(M) \to \mathbf{k}$$

is perfect: it induces

$$\alpha\colon Z(\bar{M}) \overset{\cong}{\to} Z(M)^\vee$$

Proof. The idea of the proof:

$$Z(\bar{M}) \otimes Z(M) \to \mathbf{k}$$
$$Z(\bar{M}) \overset{\alpha}{\to} Z(M)^\vee$$

We need an inverse to α:
Coevaluation map gives:

$$\mathbf{k}Z(\emptyset) \overset{Z(coev_M)}{\to} Z(M \sqcup \bar{M}) \cong Z(M) \otimes Z(\bar{M})$$

Tensor with $Z(M)^\vee$:

$$\begin{array}{ccc} Z(M)^\vee & \longrightarrow & Z(M)^\vee \otimes Z(M) \otimes Z(\bar{M}) \\ & \searrow^{\beta} & \downarrow ev \\ & & Z(\bar{M}) \end{array}$$

Can show $\beta \cong \alpha^{-1}$. ∎