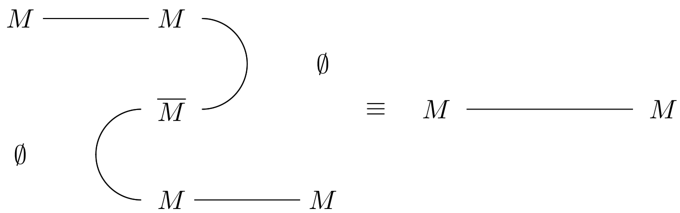
Lecture 4 by Chris Carlson

We first discuss non-degeneracy (called perfect pairing by Lurie).
The Snake Lemma can be viewed in terms of the diagram:



This picture gives us that the identity morphism:

$$V \xrightarrow{1} V$$

is the same as the composition:

$$V \cong V \otimes \mathbb{F} \xrightarrow{1\otimes\gamma} V \otimes V^* \otimes V \xrightarrow{\beta\otimes 1} \mathbb{F} \otimes V \cong V$$

where $\beta\colon V \otimes V^* \to \mathbb{F}$ is the evaluation map, and $\gamma\colon \mathbb{F} \to V^* \otimes V$ is the coevaluation map. What this gives is the following. If we fix a vector $x \in V$ we have an isomorphism

$$\beta_{left}\colon V \to V^{**} \cong V$$

or as a commutative diagram:

$$\begin{array}{ccc} V \otimes V^* & \xrightarrow{\beta} & \mathbb{F} \\ {\scriptstyle \beta_{left}\otimes 1}\downarrow & \nearrow_{eval} & \\ V \otimes V^* & & \end{array}$$

We can do the same for the coevaluation map γ.

$$\gamma\colon \mathbb{F} \to V \otimes V^* \cong End(V)$$

$$x \mapsto x \sum v_i \otimes v^j = x1_V$$

Example - Consider Cob(1)
Objects - 0-dimensional manifolds. If $P = +$ and $Q = -$, then any manifold can be written as

$$M = M_+ \coprod M_- = \left(\coprod P\right) \coprod \left(\coprod Q\right).$$

If Z is a TFT:

$$Z\colon Cob(1) \to \mathrm{Vect}_{\mathbb{F}}$$

then it is defined on objects by

$$Z(\emptyset) = \mathbb{F}$$

$$Z(P) = V$$

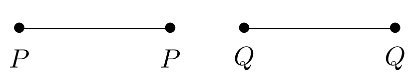
then we get

$$Z(Q) = V^*.$$

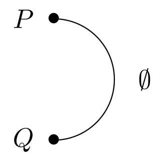
So for any manifold M:

$$Z(M) = Z(M_+) \otimes Z(M_-) = V^{\otimes n} \otimes (V^*)^{\otimes m}$$

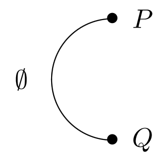
On the morphisms (cobordisms) Z is defined by:



$$V \xrightarrow{1} V \qquad V^* \xrightarrow{1} V^*$$

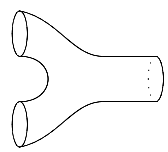


$\Longrightarrow \quad \beta\colon V \otimes V^* \to \mathbb{F}$

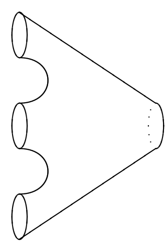


$\Longrightarrow \quad \gamma\colon \mathbb{F} \to V \otimes V^*$

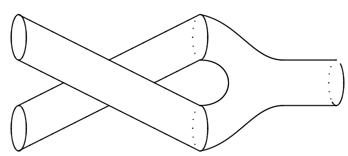
Example - in Cob(2) $P = Q$ so we only need to define a TFT Z on the circle S^1, i.e. let $Z(S^1) = V$.
Multiplication comes from "pants":



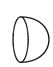
associativity is given by the three legged pants:



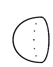
and commutativity is given by:



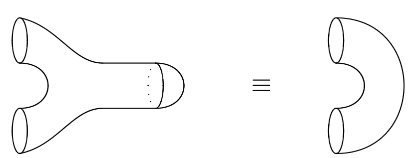
Also, we have the counit map:



and the unit map:



Using these we can create other interesting identities, like the following:



$$V \otimes V \xrightarrow{m} V \xrightarrow{tr} \mathbb{F} \qquad\qquad V \otimes V \xrightarrow{ev} \mathbb{F}$$

Now since the evaluation map is non-degenerate by the snake lemma, then $tr(ab)$ is non-degenerate.

Definition 1. *A* **commutative Frobenius algebra** *over $\mathbb{F}$ is a finite dimensional $\mathbb{F}$-algebra A with a linear map $tr\colon A \to \mathbb{F}$ such that*

$$(a, b) \mapsto tr(ab)$$

is non-degenerate.

$Z(Cob(2))$ with tr as defined above satisfies this definition.

Proposition 2. *$Z(Cob(2))$ is a CFA over $\mathbb{F}$*