A Short History of the Interaction Between QFT and Topology

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Today's story should involve lots of people, but the stars are Sir Michael Atiyah and Edward Witten. It begins with a paper Witten wrote in 1982, called "Supersymmetry and Morse Theory" [3]. In this paper, Witten shows how to use 'supersymmetric quantum mechanics' to compute the de Rham cohomology of a compact manifold, M, via Morse theory. This was perhaps the first instance of using quantum theory to find topological invariants.

1 Rudiments of Quantum Theory

Whether we're talking about quantum mechanics or quantum field theories, they all have some common ingredients:

- The **states** of a physical system are nonzero vectors ψ in some complex Hilbert space, $\mathcal{H}$.
- The **observables** of a physical system are the self-adjoint operators $\mathcal{O}$ on $\mathcal{H}$ (modulo some analytic technicalities).

Note: The **adjoint** $\mathcal{O}^*$ of an operator $\mathcal{O}$ is the unique operator such that

$$\langle \mathcal{O}^* \psi, \phi \rangle = \langle \psi, \mathcal{O} \phi \rangle$$

for all $\psi, \phi \in \mathcal{H}$. An operator $\mathcal{O}$ is called **self-adjoint** if $\mathcal{O}^* = \mathcal{O}$.

Physically, $\mathcal{O}$'s eigenvalues are the possible values of $\mathcal{O}$ upon measurement, and the eigenstates are states which, when we measure their $\mathcal{O}$, yield the corresponding eigenvalue.

Typically, the observables satisfy certain algebraic relations. So, we're actually looking for representations $\mathcal{H}$ of some "algebra of observables", A. This algebra could be a Lie algebra, a super-Lie algebra, or some other gadget.

2 Supersymmetric Quantum Theory

The adjective "super" means that everything in sight in $\mathbb{Z}_2$-graded:

- The Hilbert space is $\mathbb{Z}_2$-graded:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

- In turn, this gives a grading on the operators:
  - $\mathcal{O}$ is **even** if it preserves the grade on $\mathcal{H}$.
  - $\mathcal{O}$ is **odd** if it reverses the grade on $\mathcal{H}$.

In addition, we have distinguished operators on $\mathcal{H}$ satisfying certain relations. We could take these relations to define a low-dimensional super-Lie algebra, with:

- An even generator, H, called the **Hamiltonian**.
- Two odd generators, Q_1 and Q_2, called the **supersymmetry generators**.
- Satisfying:

$$H = \frac{1}{2}[Q_1, Q_1] = \frac{1}{2}[Q_2, Q_2]$$
$$[Q_1, Q_2] = 0$$

- All other brackets vanish.

In the jargon of physics, these relations define an $N = 2$ Poincaré super-Lie algebra in $0 + 1$ dimensions. This jargon has a specific meaning:

- $N = 2$ means there are two supersymmetry generators, Q_1 and Q_2.
- $0 + 1$ means spacetime with 0-dimensions of space, so really just time.

We think of H as the generator of time translations, which is the only possible direction of translation in $0 + 1$ dimensions. We think of the Q_i as generating supertranslations in certain infinitesimal superdirections of "superspacetime".

As the defining relations of a super-Lie algebra, the above bracket $[-, -]$ is abstract. But when we represent H, Q_1, and Q_2 as operators on $\mathcal{H}$, $[-, -]$ stands for the supercommutator on $\mathfrak{gl}(\mathcal{H})$, which is defined to be

$$[x, y] = xy - (-1)^{|x||y|} yx$$

for homogeneous elements $x, y \in \mathfrak{gl}(\mathcal{H})$. (Remember, operators on $\mathcal{H}$ are graded!) So our relations become:

$$H = Q_1^2 = Q_2^2$$

$$Q_1 Q_2 + Q_2 Q_1 = 0$$

We now give two representations of this algebra, and the connection to topology arises here.

3 Supersymmetric Quantum Mechanics and Morse Theory

Fix M a compact oriented Riemannian manifold. Let $\mathcal{H} = \mathbb{C} \otimes \Omega(M)$, the complexified differential forms on M. This has an inner product induced from the Riemannian structure, g: we get a fiberwise inner product on 1-forms, which we then integrate over M to get an inner product. We can then extend this to get an inner product on all of $\Omega(M)$, and complexify it to a Hermitian inner product on $\mathbb{C} \otimes \Omega(M)$. (We may also need to complete this space to make it into a Hilbert space, but once again we adopt the policy of ignoring analytical technicalities.)

$\mathcal{H}$ is $\mathbb{Z}_2$-graded into the forms of even and odd degree. It has two odd operators on it: d, the differential, and d^*, the codifferential (which is just the adjoint of d). From these, we construct our representation:

- The Hamiltonian is simply the Laplacian: $H = d^*d + dd^*$.
- The supersymmetry generators are $Q_1 = d + d^*$ and $Q_2 = -i(d - d^*)$.

It's easy to check these operators satisfy $H = Q_i^2$, $Q_1Q_2 + Q_2Q_1 = 0$, and are all self-adjoint.

We have a second, very similar representation on $\mathcal{H}$. Fix h a smooth real-valued function on M, and let t be a real number. Define a new differential and codifferential:

$$d_t = e^{-th} d e^{th}, \quad d_t^* = e^{th} d e^{-th}.$$

Now use these in place of d and d^* to define our representation:

- The Hamiltonian is $H_t = d_t^* d_t + d_t d_t^*$.
- The supersymmetry generators are $Q_{1t} = d_t + d_t^*$ and $Q_{2t} = -i(d_t - d_t^*)$.

Note that we recover our original representation when $t = 0$.

In fact, d_t gives the same cohomology as d, and for a p-form ψ, the map

$$\psi \mapsto e^{th}\psi$$

is the cochain map relating d and d_t.

Witten was interested in these representations as toy models for supersymmetry breaking. In supersymmetric physics, the Hamiltonian is always positive-definite, since it is the sum of squares of self-adjoint operators. For example:

$$H = \frac{1}{2}(Q_1^2 + Q_2^2)$$

This implies the Hamiltonian has nonnegative eigenvalues: the energy is always nonnegative! The states with the smallest such eigenvalues are termed the **ground states**.

Now, if $H_t\psi = 0$ has a solution, not only is ψ a ground state, but the supersymmetry is **unbroken**: the state ψ is invariant under Q_{it}, as a simple calculation suffices to show. On the flip side, if this has no solution, there are no ground states invariant under Q_{it}—the supersymmetry is **broken**.

Remarkably, if we apply the typical techniques used in physics to find the ground states and solve

$$H_t\psi = 0$$

for this given Hilbert space and Hamiltonian, we can also find the cohomology of M, bringing in Morse theory with h as our Morse function.

To see how this works, we need to compute H_t at a point x of M. To do this right, we need a little machinery: choose an orthonormal basis e^k of tangent vectors at x. We can consider the operator a^{k*} to the exterior algebra at x: the operation of wedging with e_k, the 1-form dual to e^k. That is,

$$a^{k*}\psi(x) = e_k \wedge \psi(x)$$

for any p-form ψ. We can also consider a^k, the adjoint operator to a^{k*}. It just takes the interior product with e^k:

$$a^k\psi(x) = i(e^k)\psi(x)$$

Finally, we can also form the covariant Hessian of h, a tensor of rank 2. In keeping with Witten's notation, we write the ijth component of this tensor (in our orthonormal basis) as $\dfrac{D^2h}{D\phi^i D\phi^j}$.

With these tools in hand, we can expand H_t:

$$H_t = d^*d + dd^* + t^2(dk)^2 + \sum_{i,j} t\frac{D^2h}{D\phi^i D\phi^j}[a^{i*}, a^j]$$

For large t, the eigenstates should be concentrated near the critical points of h. That is, points x which satisfy $dh(x) = 0$. This is physical intuition at play: we think of the term $(dh)^2$ has being like a potential, and t a coupling constant which tells us the strength of this potential. When t is large and the potential is very high, the system tends to get trapped in a potential well around critical points.

This is how Morse theory gets in the game. Morse theory is all about studying the topology of M via the critical points of h, which is called a **Morse function** if its critical points are nonsingular.

When h is a Morse function, we define a critical point x to have type p if the Hessian $\frac{D^2h}{D\phi^i D\phi^j}$ has p negative eigenvalues. Witten was able to show:

- For each critical point x of type p, H_t has exactly one ground state ψ_x, which is a p-form that is concentrated near x.
- Define the cochain groups

$$X_p = \mathrm{sp}\{\psi_x : x \text{ is a critical point of type } p\}$$

and define the coboundary operator

$$d_t \colon X_p \to X_{p+1}.$$

Remarkably, d_t of ψ_x lies in X_{p+1}. Even more remarkably, the cohomology of this cochain complex is the de Rham cohomology of M.

4 Topological Quantum Field Theories

We just outlined how Witten took supersymmetric quantum mechanics and got the de Rham cohomology of M via Morse theory. It's quantum *mechanics*, and not quantum *field theory*, because M is finite-dimensional. Roughly, physicists think of a system with a finite-dimensional configuration space like M as being some sort of mechanics. A field theory is one with an infinite-dimensional configuration space.

While Witten was working this out, mathematicians were cooking up some new invariants of 3- and 4-dimensional manifolds. Yet, examined closely, these new invariants bore a strong resemblance to Witten's approach to de Rham cohomology.

- Floer homology, a kind of homology theory done on the infinite-dimensional space $\mathcal{C}$ of flat SU(2) connections on Y, via a Morse function.
- Donaldson invariants, which are invariants of the smooth structure of 4-dimensional manifolds, and also possible to view as a homology theory on an infinite-dimensional manifold.

Atiyah noticed these similarities [1], and proposed that just as Witten was able to invent a supersymmetric quantum mechanics to do de Rham theory on a finite-dimensional manifold, there was an analogous construction in which one could realize Floer homology and Donaldson theory via supersymmetric quantum field theories. He further speculated the Jones polynomial could also be realized in this way.

Atiyah urged Witten to devise these quantum field theories. And Witten eventually did so [4, 5, 6]! Peter Woit, who was becoming active in this area of mathematics just as these events took place, recalls [7]:

> Spring 1987: Atiyah conjectures that Andreas Floers new homology groups (inspired by Witten's supersymmetry and Morse theory paper) are the Hilbert space of a QFT. There are two cases where Floer theory works: $1+1$ dimensions where the observables of the QFT would count curves (later to be known as Gromov-Witten invariants), and $3+1$ dimensions where the observables count instantons (Donaldson invariants). Atiyah conjectures the existence of two corresponding QFTs, and also notes that the new knot polynomials of Vaughan Jones might correspond to a QFT in $2+1$ d. He talks to Witten about this and gives an amazing lecture at a conference at

Duke explaining these ideas. Witten tries to find a supersymmetric QFT that will do what Atiyah wants, but initially doesn't succeed.

Late 1987: Atiyah visits Witten again at the IAS and keeps after him about the TQFT idea. Witten finally realizes that things work if he uses a "twisted" version of $N = 2$ supersymmetry.

Atiyah then attempted to axiomatize Witten's methods by defining a *topological quantum field theory* [2]. While this was only a partial axiomatization of Witten's work, it nonetheless opened up vast new territory in mathematics, some of which we shall explore in the weeks and months ahead.

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