1 Lecture 6

Recall from Chris Carlson's talk: explicit descriptions of n-dimensional TFT's for $n = 1, 2$. What is the value of the TFT on a closed $(n-1)$-dimensional manifold?

$n = 1$: determined by the image of a point.

$n = 2$: determined by the image of S^1.

We had an equivalence of categories

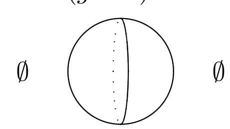
$$(2-\text{dimensional TFT's}) \leftrightarrow (\text{commutative Frobenius algebras})$$

Recall from Julie's last talk: the image of a closed n-dimensional manifold specifies a vector in $\mathbf{k}$.

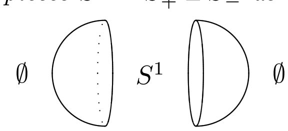
Lurie points out that in our category $nCob$, our morphisms are diffeomorphism classes of cobordisms. (n-dimensional manifolds) So this choice of vector is then a diffeomorphism invariant of closed n-dimensional manifolds. We can view the rest of the structure of a TFT as allowing us to find $Z(M)$ by breaking M up into smaller pieces. We notice in these low-dimensional cases that our TFT's are completely determined by the value we assign to the point. This will be a guiding principle later on.

Example 1. *Let A be a commutative Frobenius algebra. Then A determines a 2-dimensional TFT, Z. Z assigns a vector in $\mathbf{k}$ to every closed 2-manifold Σ. But these are completely classified by their genus g. So, $\forall g \geq 0$, we get a vector $Z(\Sigma_g) \in \mathbf{k}$. How can we find it?*

picture of sphere $S^2 (g = 0)$ as a cobordism $\emptyset \to \emptyset$



picture of sphere cut into two pieces $S^2 = S^2_+ \sqcup S^2_-$ as cobordisms $\emptyset \to S^1$ and $S^1 \to \emptyset$



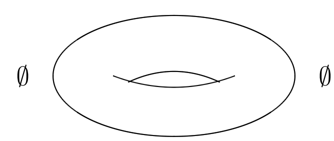
Applying Z, we get:

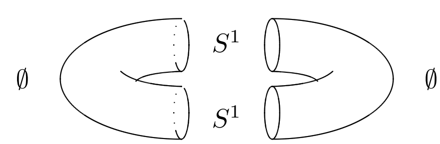
$$\mathbf{k} \cong Z(\emptyset) \xrightarrow{Z(S^2_-)} Z(S^1) = A \xrightarrow{Z(S^2_+)} Z(\emptyset) \cong \mathbf{k}$$

where the first map is the inclusion of the identity and the second map is the trace map tr. So, the map $\mathbf{k} \to \mathbf{k}$ is given by $tr(1) \in \mathbf{k}$.

Example 2.

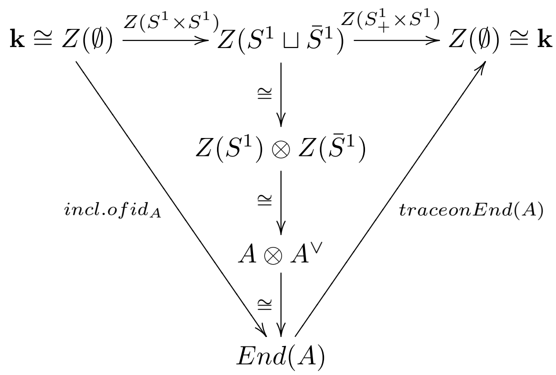
Picture of torus $S^1 \times S^1 (g = 1)$ and picture of torus split into two two-dimensional elbow noodles $S^1_- \times S^1$ and $S^1_+ \times S^1$





these are maps $\emptyset \to \{\pm 1\} \times S^1$ and $\{\pm 1\} \times S^1 \to \emptyset$

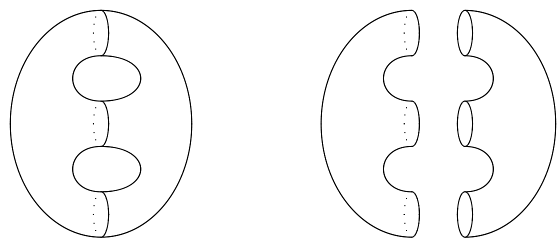
the boundary needs $\pm$ choices



This implies $Z(\Sigma_1) = trace(id_A) = dimA$.

We continue for higher genus:

Pictures of choppings of 2-dimensional surfaces with various genus



Idea: Cut an n-dimensional manifold M along closed $(n-1)$-dimensional manifolds N to break M up into pieces.

This gets hard: for higher dimensions we can't require N to be closed. What if N itself has boundary? Think of the case of triangulations of M.

Suppose N may not be closed, but ∂N is a closed $(n-2)$-dimensional manifold.

Definition 3. *Sketch*
A **2-extended TFT** Z *consists of:*

a) $\forall$ *closed* n*-dimensional manifolds* M*, an element* $Z(M) \in \mathbf{k}$.

b) $\forall$ *closed* $(n-1)$*-dimensional manifold* N*, a* $\mathbf{k}$*-vector space* $Z(N)$.

c) $\forall$ n*-dimensional manifold* M*, an element of* $Z(\partial M)$*. (when* M *is closed, coincides with a).)*

d) $\forall$ *closed* $(n-2)$*-dimensional* P *a* **k-linear category** $Z(P)$ *category enriched over vector spaces. For each* x, y, $\hom(x,y)$ *is a* $\mathbf{k}$*-vector space and the composition map is bilinear.*

If $P = \emptyset$*, then* $Z(P) \cong Vect(\mathbf{k})$.

e) $\forall$ $(n-1)$*-dimensional manifolds* N*, an object of the* $\mathbf{k}$*-linear category* $Z(\partial N)$*. If* N *is closed, then* $Z(N)$ *should be the vector space (i.e., object of* $Vect(\mathbf{k})$*) specified by b), using*

$$Z(\partial N) = Z(\emptyset) \cong Vect(\mathbf{k})$$

We have to specify how this all fits together. How do we glue the pieces together?
For ordinary TFT's: used symmetric monoidal structure. Here, we no longer have a symmetric monoidal category.
Before $nCob$ had:

- closed $(n-1)$-dimensional manifolds as objects
- diffeomorphism clases of cobordisms as morphisms
- plus extra structure.

Now:

- closed $(n-2)$-manifolds are objects
- cobordisms between them: $(n-1)$-manifolds are morphisms
- cobordisms between these: n-dimensional manifolds "morphisms between morphisms"

So we basically have a higher category – a 2-category or a bicategory or a symmetric monoidal bicategory! What's that?

2 Lecture 7

Last week we hinted at the need for a 2-extended theory, and that this does not form a category anymore. Today, we will talk about strict 2-categories, "weak" 2-categories, and what will and will not work for our purposes.

Definition 4. *A* **strict 2-category** $\mathcal{C}$ *consists of:*

- *a collection of objects;*
- *for every pair X, Y of objects, a category $Map_{\mathcal{C}}(X,Y)$ consisting of:*
  - *objects of $Map_{\mathcal{C}}(X,Y)$ are 1-morphisms of $\mathcal{C}$*
  - *morphisms of $Map_{\mathcal{C}}(X,Y)$ are 2-morphisms of $\mathcal{C}$*
- $$\textit{Picture of 2-morphism with } f, g \in \mathrm{Ob}(Map_{\mathcal{C}}(X,Y)), \alpha \in \hom_{Map_{\mathcal{C}}(X,Y)}(f,g)$$
- *for every object X of $\mathcal{C}$, an object id_X in $Map_{\mathcal{C}}(X,X)$;*
- *composition functors*
  $$Map_{\mathcal{C}}(X,Y) \times Map_{\mathcal{C}}(Y,Z) \to Map_{\mathcal{C}}(X,Z)$$
  such that identities are units and composition is associative.

Example 5. *Consider* $\mathrm{Vect}_2(\mathbf{k})$ *consisting of:*

- *objects are (finitely) co-complete $\mathbf{k}$-linear categories (meaning closed under $\oplus$ and has cokernels)*

  A $\mathbf{k}$-linear category: $\hom(X,Y)$ *has structure of a $\mathbf{k}$-vector space*
- $Map_{\mathrm{Vect}_2(\mathbf{k})}(\mathcal{C},\mathcal{D})$ *is the category of cocontinuous (preserves $\oplus$, cokernels) $\mathbf{k}$-linear functors $F\colon \mathcal{C} \to \mathcal{D}$ and for any objects $x, y \in \mathcal{C}$*
  $$\hom_{\mathcal{C}}(x,y) \to \hom_{\mathcal{D}}(Fx, Fy)$$
  is $\mathbf{k}$-linear.

I.e.,

- *1-morphisms of* $\mathrm{Vect}_2(\mathbf{k})$ *are $\mathbf{k}$-linear functors*
- *2-morphisms are $\mathbf{k}$-linear natural transformations.*

We want: a 2-category $Cob_2(n)$ as described last time consisting of:

- objects: closed $(n-2)$-dimensional manifolds
- $Map_{Cob_2(n)}(M,N)$ given by:
  - objects: cobordisms from M to N, that is, $(n-1)$-dimensional manifolds B with $\partial B \cong \bar{M} \sqcup N$;
  - $\hom_{Map_{Cob_{2(n)}}(M,N)}(B,B') =$ diffeomorphism classes of cobordisms X from B to B', which are trivial along $\partial B \cong \bar{M} \sqcup N \cong \partial B$.

Problem: Composition

- When gluing cobordisms, we make choices about smooth structure.
- If we take diffeomorphism classes, it does not matter.
- But, here we need B, B', not their diffeomorphism classes.
- Otherwise, we can not take cobordisms between them.

Associativity also only holds up to diffeomorphism.
A way to fix this: work with weak 2-categories or bicategories

- Here associativity up to isomorphism, not on the nose and coherence laws.

Julie passes out Tom Leinster's "Basic Bicategories". Next week John Baez will talk about structures that we can see using symmetric monoidal bicategories.

We are going to work with $(\infty, 2)$-categories instead. So we return to the question of why we need $Cob_2(n)$ rather than $Cob(n)$. This was because we could not necessarily cut n-dimensional manifolds into nice pieces with closed boundary.

In general, we need to extend more than 2-dimensions and this is where we begin to need higher categories. We can define higher categories inductively. We are going to define a strict n-category to have a strict $(n-1)$-category of morphisms.

Definition 6. *Let $n \geq 0$ be an integer*

- *a* **strict** 0**-category** *is a set;*
- *for $n > 0$, then a* **strict** n**-category** $\mathcal{C}$ *is a category enriched over strict $(n-1)$-categories:*
  - *objects*
  - *for objects X, Y, an $(n-1)$-category $Map_{\mathcal{C}}(X, Y)$*
  - *identity objects of $Map_{\mathcal{C}}(X, Y)$, composition, unit, associativity*

Definition 7. *A* **strict** n**-functor**

$$F : \mathcal{C} \to \mathcal{D}$$

sends objects to objects and has a $(n-1)$-functor (defined inductively along with the product for the composition).

I.e., We have:

- objects
- 1-morphisms between objects
- 2-morphisms between 1-morphisms
- 3-morphisms between 2-morphisms
- $\vdots$
- n-morphisms between $(n-1)$-morphisms

and at every stage you have all the structure of a category of the nose.
However, examples are rarely strict – need weak n-categories. There are many definitions for this and people are trying to understand how to compare these. But we will not dwell on this since we are interested in ∞-categories.
We want, for $k \leq n$, $Cob_k(n)$, a weak k-category:

- objects: closed oriented $(n-k)$-dimensional manifolds
- 1-morphisms: cobordisms between them: $(n+k-1)$-manifolds with boundary
- 2-morphisms are cobordisms between cobordisms

- $\vdots$
- k-morphisms: diffeomorphism classes of cobordisms between $(n-1)$-dimensional manifolds (with corners)

and composition is defined by gluing.

- $k = 0$: diffeomorphism classes of n-manifolds $Cob_0(n)$.
- $k = 1$: We get $Cob(n)$
- $k = 2$: $Cob_2(n)$

Definition 8. *Let $\mathcal{C}$ be a symmetric monoidal weak n-category. An* **extended $\mathcal{C}$-valued TFT** *of dimension n is a symmetric monoidal functor*

$$Z\colon Cob_n(n) \to \mathcal{C}.$$