A Categorification of the Hecke Algebra

Alexander E. Hoffnung

Joint with John Baez and James Dolan
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Department of Mathematics, University of California, Riverside

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Outline

1. The idea
2. Enriched bicategory theory
3. The theorem
4. Hecke algebras
5. The categorified Hecke algebra and incidence geometries
The category of permutation representations

Given a finite group $G$ there is a category $\text{PermRep}(G)$ consisting of:

- objects: permutation representations of finite groups – in other words, actions of $G$ on finite sets;
- morphisms: intertwining operators.

Our goal is to categorify $\text{PermRep}(G)$. 
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Degroupoidification and monoidal bicategories

Thinking back half an hour or so, recall the functorial process, *degroupoidification*:

- Groupoids $\mapsto$ Vector spaces
- Spans of groupoids $\mapsto$ Linear maps
- Maps of spans $\mapsto$ Identity 2-morphisms.

And a span (or correspondence) is just a pair of maps with common domain:

\[ \begin{array}{c} S \\ X \\
\end{array} \quad \begin{array}{c} \Downarrow \\ Y \end{array} \]
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Some of the structure

Let $\mathcal{V}$ be a monoidal bicategory. A $\mathcal{V}$-bicategory $\mathcal{B}$ consists of:

- A set of objects $a, b, c, \ldots$;
- for every pair of objects $a, b$, a hom-object $\text{hom}(a, b) \in \mathcal{V}$;
- for each triple of objects $a, b, c \in \mathcal{B}$, a morphism in $\mathcal{V}$ called composition

$$c = c_{abc} : \text{hom}(b, c) \otimes \text{hom}(a, b) \rightarrow \text{hom}(a, c)$$

- and an identity-assigning morphism in $\mathcal{V}$;
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- and an identity-assigning morphism in $\mathcal{V}$;
an invertible 2-morphism in \( \mathcal{V} \) called the **associator**

\[
((c,d) \otimes (b,c)) \otimes (a,b) \xrightarrow{a} (c,d) \otimes ((b,c) \otimes (a,b))
\]

\[
\begin{array}{ccc}
(c \otimes 1) & & (1 \otimes c) \\
\downarrow & \alpha_{abcd} & \downarrow \\
(b,d) \otimes (a,b) & & (c,d) \otimes (a,c) \\
\downarrow c & & \downarrow c \\
(a,d) & & (c,d) \otimes (a,c)
\end{array}
\]

for each quadruple of objects \( a, b, c, d \in \mathcal{B} \) and some **unitor** 2-morphisms;
One of the axioms

\[
\begin{align*}
(d,e)((c,d)((b,c)(a,b))) &= ((d,e)(c,d))(a,c) \\
((d,e)(c,d))((b,c)(a,b)) &= ((d,e)(c,d))(a,c) \\
(c,e)((b,c)(a,b)) &= 1 \\
(c,d) &= 1 \\
(b,c) &= 1 \\
(a,b) &= 1
\end{align*}
\]
Theorem

Given a monoidal functor \( f : \mathcal{V} \to \mathcal{V}' \) and a \( \mathcal{V} \)-enriched bicategory \( \mathcal{B} \), there exists a \( \mathcal{V}' \)-enriched bicategory \( \bar{f}(\mathcal{B}) \).

We have such a monoidal functor:

\[
\mathbf{D} : \text{FinSpan} \to \text{FinVec},
\]

where \( \text{FinSpan} \) is the monoidal bicategory of spans of finite groupoids.
The idea

Enriched bicategory theory

The theorem

Hecke algebras

The categorified Hecke algebra and incidence geometries

Change of base

Theorem

Given a monoidal functor \( f : \mathcal{V} \rightarrow \mathcal{V}' \) and a \( \mathcal{V} \)-enriched bicategory \( \mathcal{B} \), there exists a \( \mathcal{V}' \)-enriched bicategory \( \tilde{f}(\mathcal{B}) \).

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The action groupoid

Given a $G$-set $S$, i.e. a set with an action of $G$, we can form the **action groupoid** $S//G$ with:

- Objects: elements $s \in S$;
- Morphisms: $(g, s) : s \to gs$. 
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The Hecke bicategory

Given a finite group $G$, there is an enriched bicategory $\text{Hecke}(G)$ consisting of:

- objects: finite $G$-sets;
- for each pair of finite $G$-sets $A, B$, a groupoid of morphisms
  \[ \text{hom}(A, B) = (A \times B) \bowtie G \]
- a composition map which is a span of groupoids;
- etc.,

**Lemma**

That is, $\text{Hecke}(G)$ is enriched over the monoidal bicategory of spans of groupoids, $\text{FinSpan}$.
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**Lemma**

That is, $\text{Hecke}(G)$ is enriched over the monoidal bicategory of spans of groupoids, $\text{FinSpan}$. 
From our change of base theorem and the degroupoidification functor, we have a \( \text{Vect} \)-enriched bicategory (really just a category)

\[
\bar{D}(\text{Hecke}(G))
\]

\[\bar{D}(\text{Hecke}(G)) \cong \text{PermRep}(G)\]

(as enriched bicategories)
The fundamental theorem

From our change of base theorem and the degroupoidification functor, we have a $\text{Vect}$-enriched bicategory (really just a category)

$$\tilde{D}(\text{Hecke}(G))$$

**Theorem**

$$\tilde{D}(\text{Hecke}(G)) \cong \text{PermRep}(G)$$

(as enriched bicategories)
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Hecke algebras are permutation representations

- This fundamental theorem is in effect a theorem about categorified Hecke algebras.
- This is because the Hecke algebra is the endomorphism ring of the representation $V$ of a simple algebraic group $G = \text{SL}(n, \mathbb{F}_q)$ induced from the trivial representation of the Borel subgroup of upper triangular matrices.
Hecke algebras: generators and relations

Let $S$ be the set of vertices of a Dynkin diagram $D$. Denote an edge between $s$ and $t$ in $S$ by $st$ and the label on $st$ by $m_{st}$.

**Definition**

Let $D$ be a Dynkin diagram and $q$ a nonzero complex number. The **Hecke algebra** corresponding to this data is the associative $\mathbb{Z}[q, q^{-1}]$-algebra with generators $\sigma_s$, for each $s \in S$, and relations:

$$\sigma_s\sigma_t\sigma_s \ldots = \sigma_t\sigma_s\sigma_t \ldots$$

where each side has $m_{st}$ factors, and

$$\sigma_s^2 = (q - 1)\sigma_s + q$$

for all $s \in S$. 
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Consider the Dynkin diagram $A_2$:

$$\bullet \quad \longrightarrow \quad \bullet$$

We fix a prime power $q$. We have $G = SL(3, \mathbb{F}_q)$ and $B$ is the upper triangular matrices. $X = G/B$ is the set of complete flags in $\mathbb{F}_q^3$, i.e.,

$$\{ V_1 \subset V_2 \}$$

and

$$\text{dim} V_i = i.$$
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Consider the Dynkin diagram $A_2$:

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) [fill] {}; 
\node (B) at (1,0) [fill] {}; 
\node (C) at (2,0) [fill] {}; 
\node (D) at (3,0) [fill] {}; 
\draw (A) -- (B); 
\end{tikzpicture}
\end{center}

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\[ \{ V_1 \subseteq V_2 \} \]

and

\[ \dim V_i = i. \]
Projective perspective

In the projective space $\mathbb{F}_q P^2$, the flags are just a chosen point lying on a chosen line. The vertices of our Dynkin diagram represent “figures” and the edges represent “incidence relations”.

These figures should be thought of as generators of the categorified Hecke algebra.
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point \quad ---- \quad line

These figures should be thought of as generators of the categorified Hecke algebra.
The Coxeter group

The Coxeter group $S_3$ controls the structure of $(X \times X)//G$. For $A_2$ there are two generators:

![Diagram](image)

The elements of $S_3$ correspond to the possible incidence relations between pairs of flags. The multiplication in the categorified Hecke algebra is a deformed version of this multiplication.

$$P^2 = (q - 1)P + q1$$
$$L^2 = (q - 1)L + q1$$
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The flag variety:

The Coxeter group – the apartments – shows up in four ways.
The idea
Enriched bicategory theory
The theorem
Hecke algebras

The categorified Hecke algebra and incidence geometries
The Bowtie

A Categorification of the Hecke Algebra
The idea
Enriched bicategory theory
The theorem
Hecke algebras

The categorified Hecke algebra and incidence geometries

The Cow
English nursery rhymes
The fourth apartment – the untold story
The idea
Enriched bicategory theory
The theorem
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The Coffin
For each pair of flags ...
For each pair of flags there is a unique apartment, i.e. a unique element of $S_3$. 
This type of thinking leads one to find Yang-Baxter isomorphisms (these are isomorphisms of spans of sets over the flag variety) which are solutions to the Zamolodchikov tetrahedron equation.