The Hecke Bicategory

Alexander E. Hoffnung
Department of Mathematics, University of California
Riverside, CA 92521 USA

April 29, 2010

Abstract

We present an application of the program of groupoidification leading up to a sketch of the Fundamental Theorem of Hecke Operators. In stating this theorem we construct the Hecke bicategory — a categorification of the permutation representations of a finite group. As an immediate corollary, we obtain a categorification of the Hecke algebra, which leads to solutions of the Zamolodchikov tetrahedron equation. This paper is expository in style and is meant as a companion to *Higher Dimensional Algebra VII: Groupoidification* and the work in progress, *Higher Dimensional Algebra VIII: The Hecke Bicategory*, which introduces the Hecke bicategory in detail.

Contents

1 Introduction

Categorification is the attempt to shed new light on familiar mathematical notions by replacing a set-theoretic interpretation with a category-theoretic analogue. Loosely speaking, categorification replaces sets with categories and functions with functors. This often brings a new layer of structure previously hidden from view — the natural transformations. While categorification is not a systematic process — in other words, finding this new layer of structure may require a certain amount of creativity — the reverse process of *decategorification* should be a systematic way of recovering the original set-theoretic structure or concept.

In *Higher Dimensional Algebra VII: Groupoidification* [4], we introduced a program called *groupoidification* aimed at categorifying various notions from representation theory and mathematical physics. The very simple idea was that one could replace vector spaces by groupoids, i.e., categories with only isomorphisms, and replace linear operators by spans of groupoids. In fact, what we really did was define a systematic process called *degroupoidification*:

$$\text{groupoids} \to \text{vector spaces}$$

$$\text{spans of groupoids} \to \text{matrices}$$

We then suggested some applications of groupoidification to Hall algebras, Hecke algebras, and Feynman diagrams, so that other researchers could begin to categorify *their* favorite notions from representation theory.

In this paper, we give an expository account of *categorified intertwining operators* for representations of a very basic type: the *permutation representations* of a finite group. There is an easy way to categorify the theory of intertwining operators between such representations. Unfortunately, the construction is, in some sense, unnatural. Groupoidification offers a more natural construction. Much of this paper is devoted to explaining how to recover the permutation representations from this categorified structure, and further, to point out how categorified Hecke algebras follow directly from such a construction.

We make use of the techniques of groupoidification along with the machinery of enriched bicategories and some very basic topos theory. Thus, this paper is intended to give an introduction to some concepts which should play a significant role as the subject of categorified representation theory continues to develop. We plan to give a more detailed account of the necessary structures along with proofs in a paper in progress with John Baez — *Higher Dimensional Algebra VIII: The Hecke Bicategory* [5].

Section 2 recalls the basic notions which are studied in this paper. In Section 2.1, we recall definitions and facts about spans of finite sets, and we discuss the relationship between spans and matrices. In Section 2.2, we discuss permutation representations of a finite group and the intertwining operators between these representations. In Section 2.3, we describe the bicategory of spans of finite G-sets, for any finite group G. In Section 2.4, we describe a decategorification process in the spirit of the Grothendieck group construction. In Section 2.5, we give a first definition of the Hecke bicategory and state a corresponding version of the *Fundamental Theorem of Hecke Operators*. The theorem states that decategorifying the Hecke bicategory yields the category of permutation representations.

Section 3 recalls the definition of degroupoidification and introduces enriched bicategories. In Section 3.1, we describe the *action groupoid* construction and introduce *groupoid cardinality*. In Section 3.2, we recall the construction of the degroupoidification functor. In particular, we highlight the role of groupoid cardinality in linearizing spans of groupoids. This functor takes groupoids to vector spaces and spans of groupoids to linear operators. In Section 3.3, we present a definition of enriched bicategories along with a *change of base* theorem.

Section 4 accomplishes the main tasks of the paper. In Section 4.1, we give a more natural description of the Hecke bicategory using enriched bicategories. In Section 4.2, we state the Fundamental Theorem of Hecke Operators in the language of enriched bicategories and degroupoidification.

Section 5 describes the monoidal bicategories and functors needed to reconcile the two descriptions of the Hecke bicategory. In Section 5.1, we introduce the monoidal bicategory of 'nice topoi'. In Section 5.2, we exploit some ideas from topos theory to explain the relationship between these presheaf categories and groupoids, and highlight their roles in categorified linear algebra. In particular, we sketch a monoidal functor between bicategories, which is analogous to constructing the free 'categorified vector space' on a 'categorified basis'. In Section 5.3, we examine the structure of the hom-categories of the Nice-enriched Hecke bicategory more closely and reinterpret them as spans of finite G-sets. In Section 5.4, we describe an endofunctor on Nice, which passes from presheaves on an action groupoid to the presheaves on the ordinary G-set quotient. In Section 5.5, we again encounter the Hecke bicategory of spans of G-sets — this time as a change of base of the Span-enriched bicategory Hecke(G) of Section 4.1. We see that change of base allows a statement of the Fundamental Theorem of Hecke Operators in terms of the Span-enriched bicategory Hecke(G), and either decategorification or degroupoidification.

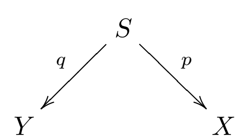
Section 6 discusses the first application of the Fundamental Theorem of Hecke Operators — a categorification of the Iwahori-Hecke algebra — as well as future directions in low-dimensional topology and higher-category theory. In Section 6.1, we recall the notion of Hecke algebras associated to Dynkin diagrams and prime powers. We describe how a categorification of the Hecke algebra naturally arises from the Fundamental Theorem of Hecke Operators. Finally, in Section 6.2, we describe a concrete example of the categorified Hecke algebra, while hinting at its relationship to knot theory. In particular, we describe solutions to the Zamolodchikov tetrahedron equation.

2 Spans, Matrices, and Decategorification

2.1 Spans as Matrices

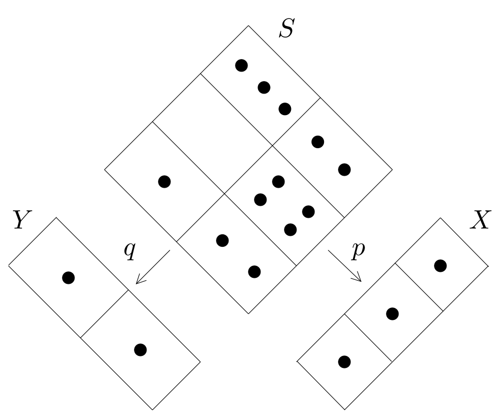
The first tool of representation theory is linear algebra. Vector spaces and linear operators have nice properties, which allow representation theorists to extract a great deal of information about algebraic gadgets ranging from finite groups to Lie groups to Lie algebras and their various relatives and generalizations. We start at the beginning, considering the representation theory of finite groups. Noting the utility of linear algebra in representation theory, this paper is fundamentally based on the idea that the heavy dependence of linear algebra on fields, very often the complex numbers, obscures the combinatorial skeleton of the subject. Then, we hope that by peeling back the soft tissue of the continuum, we may expose and examine the bones, revealing new truths by working directly with the underlying combinatorics. In this section, we consider the notion of spans of sets, a very simple idea, which is at the heart of categorified representation theory.

A **span of sets** from X to Y is a pair of functions with a common domain like so:



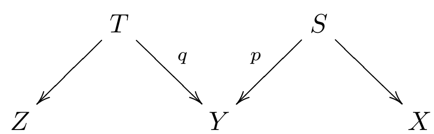
We will often denote a span by its apex, when no confusion is likely to arise.

A span of sets can be viewed as a a matrix of sets:

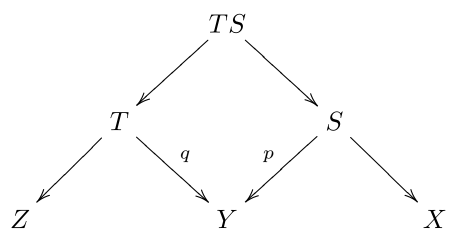


For each pair (x, y), we have a set $S_{x,y} = p^{-1}(x) \cap q^{-1}(y)$. In particular, if all the sets $S_{x,y}$ are finite, this can be 'decategorified' to a matrix of natural numbers $|S_{x,y}|$ — a very familiar object in linear algebra. In this sense, a span is a 'categorification' of a matrix. We will consider only *finite* sets throughout this paper.

Even better than spans giving rise to matrices, composition of spans gives rise to matrix multiplication. Given a pair of composable spans:



we define the composite to be the **pullback** of the pair of functions $p\colon S \to Y$ and $q\colon T \to Y$, which is a new span:



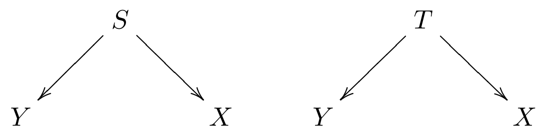
where TS is the subset of $T \times S$:

$$\{(t, s) \subseteq T \times S \mid p(s) = q(t)\},$$

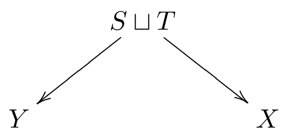
with the obvious projections to S and T. It is straightforward to check that this process agrees with matrix multiplication after decategorifying.

While we have not defined the notions of categorification and decategorification explicitly, we have been hinting at their role in the relationship between spans of finite sets and matrices of natural numbers. The reason for skirting the definitions is that the notion of 'categorification' is simply a heuristic tool allowing us to 'undo' the process of decategorification. Thus, in the above example, we turn spans of finite sets into matrices of natural numbers simply by counting the number of elements in each set $S_{x,y}$. We note that there is a standard basis of the vector space of linear maps, and each basis element can be realized as a span of finite sets. Thus, we can recover the entire vector space of linear maps by constructing the free vector space on this basis. Further, we can turn a set X into a vector space by constructing the free vector space with basis X. Checking that composition of spans and matrix multiplication agree after taking the cardinality of the set-valued entries is the main step in showing that our decategorification process — spans of sets to linear operators — is *functorial.*

Since we are interested in the relationship between spans and matrices, we expect that a good decategorification process should be 'additive' in some suitable manner. In particular, we should say how to add 'categorified linear operators', or spans. To define *addition of spans*, we consider a pair of spans from a set X to a set Y:

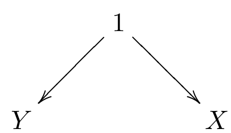


We can define the **sum** of S and T as the span:



where the legs of $S \sqcup T$ are induced by the universal property of the coproduct.

Using this notion of addition, we can write down a 'categorified basis' of spans of finite sets from X to Y — that is, a set of spans whose corresponding matrices span the vector space of linear operators from the free vector space with basis X to the free vector space with basis Y. These *categorified basis vectors* are the spans:



where there is one span corresponding to each element $(x, y) \in X \times Y$. These are the **irreducible spans** — those that cannot be written as the sum of two 'non-trivial spans'. A **non-trivial span** is a span whose apex is the empty set. Colloquially we say that *spans of finite sets categorify linear operators between finite dimensional vector spaces.*

2.2 Permutation Representations

Again we start with a very simple idea. We want to study the actions of a finite group G on finite sets — *finite G-sets.* These extend to *permutation representations of G.* We fix the field of complex numbers and consider only complex representations throughout this paper.

Definition 1. *A* **permutation representation** *of a finite group G is a finite-dimensional representation of G together with a chosen basis such that the action of G maps basis vectors to basis vectors.*

Thus, finite G-sets can be linearized to obtain permutation representations of G. In fact, we have described a relationship between the objects of the *category* of finite G-sets and the objects of the *category* of permutation representations of G. Given a finite group G, the category of finite G-sets has:

- finite G-sets as objects,

- G-equivariant functions as morphisms,

and the category of permutation representations PermRep(G) has:

- permutation representations of G as objects,
- intertwining operators as morphisms.

The main goal of this paper is to categorify the very special algebras of Hecke operators called the *Iwahori-Hecke algebras* [10]. Of course, an algebra is a Vect-enriched category with exactly one object, and the Hecke algebras are isomorphic to certain one-object subcategories of the Vect-enriched category of permutation representations. Thus, we consider the morphisms of the category PermRep(G) to be *Hecke operators* and refer to the category as the *Hecke algebroid* — a many-object generalization of the Hecke algebra. We will construct a bicategory — or more precisely, an *enriched bicategory* — called the *Hecke bicategory* that categorifies the Hecke algebroid for any finite group G.

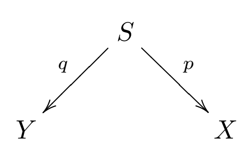
There is a functor from finite G-sets to permutation representations of G. As stated above, the maps between G-sets are G-equivariant functions — that is, functions between G-sets X and Y that respect the actions of G. Such a function $f\colon X \to Y$ gives rise to a G-equivariant linear map (or intertwining operator) $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$. However, there are many more intertwining operators from $\widetilde{X}$ to $\widetilde{Y}$ than there are G-equivariant maps from X to Y. In particular, the former is a complex vector space, while the latter is a finite set. For example, an intertwining operator $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ may take a basis vector $x \in \widetilde{X}$ to any $\mathbb{C}$-linear combination of basis vectors in $\widetilde{Y}$, whereas a map of G-sets does not have the freedom of scaling or adding basis elements.

So, in the language of category theory the process of linearizing finite G-sets to obtain permutation representations is a faithful, essentially surjective functor, which is not at all full.

2.3 Spans of G-Sets

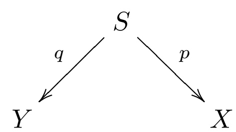
In the previous section, we discussed the relationship between finite G-sets and permutation representations. In Section 2.1, we saw a basis for the vector space of linear operators between the free vector spaces on a pair of finite sets X and Y coming from spans between X and Y. Thinking of finite sets as G-sets with a trivial action of G suggests that we can similarly obtain a basis for the vector space of intertwining operators between permutation representations from *spans of finite G-sets.*

A **span of finite G-sets** from a finite G-set X to a finite G-set Y is a pair of maps with a common domain like so:



where S is a finite G-set, and p and q are G-equivariant maps.

Since the category of finite G-sets and G-equivariant maps has a coproduct, we can define addition of spans of finite G-sets as we did for finite sets in Section 2.1. Further, we obtain a basis of intertwining operators from $\widetilde{X}$ to $\widetilde{Y}$ as spans of G-sets from X to Y, These *categorified basis vectors* are spans:



where $S \subseteq X \times Y$ is the orbit of some point in $X \times Y$, and p and q are the obvious projections.

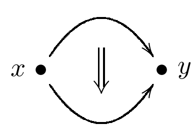
At this point, we see the first hints of the existence of a categorified Hecke algebroid. Having found a basis coming from spans of finite G-sets is promising because spans of sets — and, similarly G-sets — naturally form a bicategory.

The development of bicategories by Benabou [8] is an early example of categorification. A (small) category consists of a *set of objects* and a *set of morphisms.* A bicategory is a categorification of this concept, so there is a new layer of structure [26]. In particular, a (small) bicategory $\mathcal{B}$ consists of:

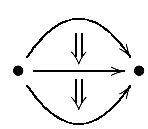
- a set of objects $x, y, z \ldots$,
- for each pair of objects a set of morphisms,
- for each pair of morphisms a set of 2-morphisms,

and given any pair of objects x, y, there is a hom-category $\hom(x, y)$ which has:

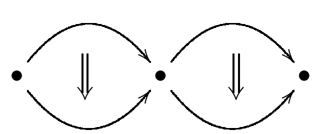
- 1-morphisms $x \to y$ of $\mathcal{B}$ as objects,
- 2-morphisms:



of $\mathcal{B}$ as morphisms. There is a *vertical composition* of 2-morphisms,



as well as a *horizontal composition*,

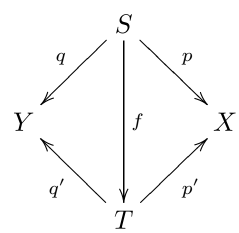


and these are required to satisfy certain coherence axioms, which make these operations simultaneously well-defined.

Benabou's definition followed from several important examples of bicategories, which he presented in [8], and which are very familiar in categorified and geometric representation theory. The first example is the bicategory of spans of sets, which has:

- sets as objects,
- spans of sets as morphisms,
- maps of spans of sets as 2-morphisms.

We defined spans of sets in Section 2.1. A **map of spans of sets** from a span S to a span T is a function $f\colon S \to T$ such that the following diagram commutes:



For each finite group G, there is a closely related bicategory Span(GSet), which has:

- finite G-sets as objects,
- spans of finite G-sets as morphisms,
- maps of spans of finite G-sets as 2-morphisms.

The definitions are the same as in the bicategory of spans of sets, except for the obvious finiteness condition and that every arrow should be made G-equivariant.

This bicategory seems to be a good candidate for a categorification of PermRep(G). In the next section, we define a process of decategorification, and see that we do *not* recover the Hecke algebroid. Of course, that is not the end of the story. In Section 2.5, we alter the construction of Span(GSet) slightly and, as a result, obtain the Hecke algebroid, as desired.

2.4 Decategorification

In this section, we describe a functor from the bicategory of spans of G-sets to the category of permutation representations of G.

Consider the bicategory of spans of finite G-sets from the previous section. We have seen that such spans can be interpreted as categorified intertwining operators between permutation representations, i.e., matrices whose entries are sets. However, while counting the number of elements of these sets produces a matrix with natural number entries, we have not specified a decategorification process, which takes the bicategory of spans of finite G-sets to the category $\mathrm{PermRep}(G)$. Our goal is to obtain the entire category of permutation representations.

Let us propose such a process and see what goes wrong, when we apply it to $\mathrm{Span}(G\mathrm{Set})$. Since the bicategory of spans has a coproduct it is natural to apply a functor, which one might call an 'additive Grothendieck construction' in analogy with the usual split Grothendieck group construction on abelian categories. This is a functor:

$$\mathcal{L}\colon \mathrm{Span}(G\mathrm{Set}) \to \mathrm{PermRep}(G)$$

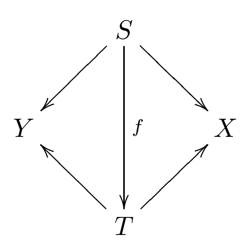
We note that the objects of the bicategory are finite G-sets. Thus, the functor only needs to linearize these to obtain permutation representations of G as described in Section 2.2:

$$X \mapsto \widetilde{X}$$

The interesting part of this process is turning the hom-categories consisting of spans of finite G-sets and maps between these spans into vector spaces. To do this, we take the free vector space on the set of matrices corresponding to isomorphism classes of *irreducible spans* — those spans which cannot be written as a coproduct of two non-trivial spans.

However, there are many more isomorphism classes of irreducible spans of G-sets from X to Y than needed to span the space of intertwining operators between the permutation representations $\widetilde{X}$ and $\widetilde{Y}$. This is most obvious if we take $X = Y = 1$ and take G to be non-trivial. Then the space of intertwining operators is 1-dimensional, but there are many more isomorphism classes of irreducible spans of G-sets from X to Y. Such a span has an apex that is a G-set with just a single orbit.

Fortunately, the problem is clear. Given spans of finite G-sets S and T from X to Y, we say S and T are the same as matrices if they are the same in each matrix entry, i.e., in each fiber over a pair $(x, y) \in X \times Y$. This is true precisely when there is a bijection from $f\colon S \to T$ making the following diagram commute:



The problem arises when we consider only the G-equivariant maps from S to T. In the next section, we see that relaxing this requirement solves this problem.

2.5 The Hecke Bicategory — Take One

This section introduces the Hecke bicategory as a bicategory of spans of finite G-sets. We alter the bicategory Span(GSet) by considering a larger class of 2-morphisms. We then show that extending the decategorification functor $\mathcal{L}$ introduced in the last section to this new bicategory, we obtain precisely the Hecke algebroid as its image. Unfortunately, this leaves us with a less than desirable solution, which we now describe.

We consider the bicategory $\mathrm{Span}^*(G\mathrm{Set})$ consisting of:

- finite G-sets as objects,
- spans of finite G-sets as morphisms,
- *not necessarily G-equivariant* maps of spans as 2-morphisms.

The raised asterisk is there to remind us of the new description of the 2-morphisms. Of course, our decategorification functor $\mathcal{L}$ can be applied equally well to any bicategory of spans of finite sets. Thus, we have, for each finite group G, the functor:

$$\mathcal{L}\colon \mathrm{Span}^*(G\mathrm{Set}) \to \mathrm{PermRep}(G)$$

The bicategory $\mathrm{Span}^*(G\mathrm{Set})$ categorifies the Hecke algebroid PermRep(G), or more precisely:

Claim 2. *Given a finite group G,*

$$\mathcal{L}\left(\mathrm{Span}^*(G\mathrm{Set})\right) \simeq \mathrm{PermRep}(G)$$

as Vect*-enriched categories.*

Unfortunately, the use of *not-necessarily equivariant* maps of G-sets makes this construction appear artificial. One goal of this paper is to solve this problem by giving a more natural description of the categorified Hecke algebroid. We do this in Section 4. In Section 5, we show how our two descriptions of the Hecke bicategory are related.

3 Groupoidification and Enriched Bicategories

The following sections introduce the necessary machinery to present a more natural description of the Hecke bicategory. Enriched bicategories are developed for use in Section 4 to construct the Hecke bicategory and state the Fundamental Theorem of Hecke Operators, and in Section 5 to make an explicit connection with the first construction of the Hecke bicategory in Section 2.5.

3.1 Action Groupoids and Groupoid Cardinality

In this section, we draw a connection between G-sets and groupoids via the 'action groupoid' construction. We then introduce *groupoid cardinality*, which makes this connection explicit.

For any G-set, there exists a corresponding groupoid, called the *action groupoid* or *weak quotient*:

Definition 3. *Given a group G and a G-set X, the* **action groupoid** $X//G$ *is the category which has:*

- *elements of X as objects,*
- *pairs $(g, x) \in G \times X$ as morphisms $(g, x)\colon x \to x'$, where $g \cdot x = x'$.*

Composition of morphisms is defined by the product on G.

Of course, associativity follows from associativity in G and the construction defines a groupoid since any morphism $(g, x)\colon x \to x'$ has an inverse $(g^{-1}, x')\colon x' \to x$.

So every finite G-set defines a groupoid, and we will see in Section 4.1 that the weak quotient of G-sets plays an important role in understanding categorified permutation representations.

Next, we recall the definition of groupoid cardinality [3]:

Definition 4. *Given a (small) groupoid $\mathcal{G}$, its* **groupoid cardinality** *is defined as:*

$$|\mathcal{G}| = \sum_{\text{isomorphism classes of objects [x]}} \frac{1}{|\mathrm{Aut}(x)|}$$

If this sum diverges, we say $|\mathcal{G}| = \infty$.

In this paper, we will only consider *finite groupoids* — groupoids with a finite set of objects and finite set of morphisms. In general, we could allow groupoids with infinitely many isomorphism classes of objects, and the cardinality of a groupoid would take values in the non-negative real numbers when the sum converges. Generalized cardinalities have been studied by a number of authors [14, 25, 27, 34].

Groupoid cardinality makes explicit the relationship between a G-set and the corresponding action groupoid. In particular, we have the following equation:

$$|X//G| = |X|/|G|$$

whenever G is a finite group acting on a finite set X.

Weakening the quotient $(X \times Y)/G$, we obtain the action groupoid $(X \times Y)//G$, which will be central in categorifying the permutation representation category of a finite group G. In the next section, we define degroupoidification using the notion of groupoid cardinality.

3.2 Degroupoidification

In this section, we recall some of the main ideas of groupoidification. Of course, in practice this means we will discuss the corresponding process of decategorification — the degroupoidification functor.

To define degroupoidification in [4], we considered a functor from the category of spans of groupoids to the category of linear operators between vector spaces. In the present setting, we will need to extend degroupoidification to a functor between bicategories.

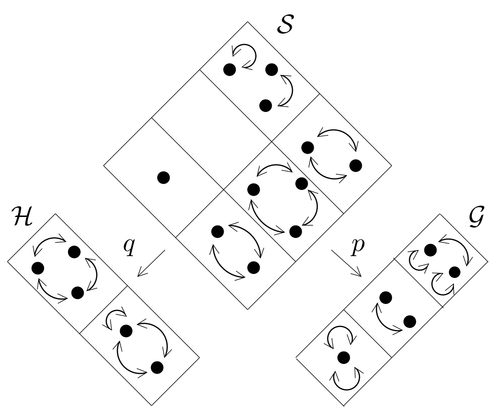
We extend the functor to a bicategory Span, which has:

- (finite) groupoids as objects;
- spans of (finite) groupoids as 1-morphisms;
- 'isomorphism classes of maps' of spans of (finite) groupoids as 2-morphisms.

Since all groupoids that show up in this paper arise from the action groupoid construction on finite G-sets, there is no problem restricting our attention to finite groupoids.

Arbitrary spans of groupoids form a tricategory, which has not only *maps of spans* as 2-morphisms, but also *maps of maps of spans* as 3-morphisms. Thus, it takes some work to restrict this structure to a bicategory. While there are more sophisticated ways of obtaining such a bicategory, we do so by taking *isomorphism classes of maps of spans* as 2-morphisms.

Spans of finite groupoids are categorified matrices of non-negative rational numbers in the same way that spans of finite sets are categorified matrices of natural numbers. A *span of groupoids* is a pair of functors with common domain, and we can picture one of these roughly as follows:



Whereas one uses set cardinality to realize spans of sets as matrices, we can use groupoid cardinality to obtain a matrix from a span of groupoids.

We have seen evidence that a span of groupoids is a categorified matrix, so a groupoid must be a categorified vector space. To make these notions precise, we define the degroupoidification functor:

$$\mathcal{D}\colon \text{Span} \to \text{Vect},$$

as follows. Given a groupoid $\mathcal{G}$, we obtain a vector space $\mathcal{D}(\mathcal{G})$, called the **degroupoidification** of $\mathcal{G}$, by taking the free vector space on the set of isomorphism classes of objects of $\mathcal{G}$.

We say a groupoid $\mathcal{V}$ *over* a groupoid $\mathcal{G}$:

$$\begin{array}{c} \mathcal{V} \\ \big\downarrow{\scriptstyle p} \\ \mathcal{G} \end{array}$$

is a **groupoidified vector**. In particular, from the functor p we can produce a vector in $\mathcal{D}(\mathcal{G})$ in the following way.

The **full inverse image** of an object x in $\mathcal{G}$ is the groupoid $p^{-1}(x)$, which has:

- objects v of $\mathcal{V}$, such that $p(v) \cong x$, as objects,
- morphisms $v \to v'$ in $\mathcal{V}$ as morphisms.

We note that this construction depends only on the isomorphism class of x. Since the set of isomorphism classes of $\mathcal{G}$ determine a basis of the corresponding vector space, the vector determined by p can be defined as:

$$\sum_{\text{isomorphism classes of objects [x]}} |p^{-1}(x)|[x],$$

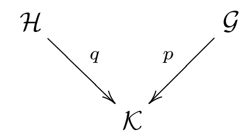
where $|p^{-1}(x)|$ is the groupoid cardinality of $p^{-1}(x)$. We note that a 'groupoidified basis' can be obtained in this way as a set of functors from the terminal groupoid **1** to a representative object of each isomorphism class of $\mathcal{G}$. A **groupoidified basis** of $\mathcal{G}$ is a set of groupoids $\mathcal{V} \to \mathcal{G}$ over $\mathcal{G}$ such that the corresponding vectors give a basis of the vector space $\mathcal{D}(\mathcal{G})$.

Given a span of groupoids,

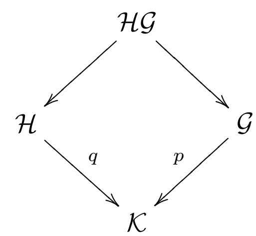
$$\mathcal{H} \xleftarrow{\;q\;} \mathcal{S} \xrightarrow{\;p\;} \mathcal{G}$$

we want to produce a linear map $\mathcal{D}(\mathcal{S})\colon \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{H})$. The details are checked in [4]. Here we show only that given a basis vector of $\mathcal{D}(\mathcal{G})$, the span $\mathcal{S}$ determines a vector in $\mathcal{D}(\mathcal{H})$. To do this, we need the notion of the weak pullback of groupoids — a categorified version of the pullback of sets.

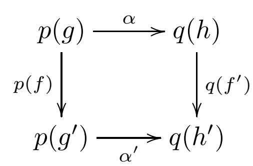
Given a diagram of groupoids:



the **weak pullback** of $p\colon \mathcal{G} \to \mathcal{K}$ and $q\colon \mathcal{H} \to \mathcal{K}$ is the diagram:

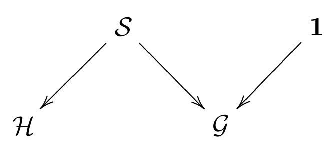


where $\mathcal{HG}$ is a groupoid whose objects are triples (h, g, α) consisting of an object $h \in \mathcal{H}$, an object $g \in \mathcal{G}$, and an isomorphism $\alpha\colon p(g) \to q(h)$ in $\mathcal{K}$. A morphism in $\mathcal{HG}$ from (h, g, α) to (h', g', α') consists of a morphism $f\colon g \to g'$ in $\mathcal{G}$ and a morphism $f'\colon h \to h'$ in $\mathcal{H}$ such that the following square commutes:

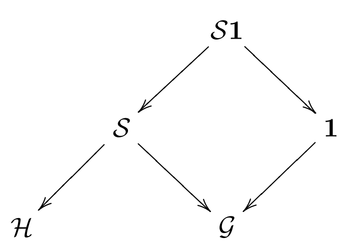


As in the case of the pullback of sets, the maps out of $\mathcal{HG}$ are the obvious projections. Further, this construction should satisfy a certain universal property.

Now, given our span and a chosen *groupoidified basis vector*:



we obtain a groupoid over $\mathcal{H}$ by constructing the weak pullback:



Now, $\mathcal{S}\mathbf{1}$ is a groupoid over $\mathcal{H}$, and we can compute the resulting vector. In general, to guarantee that this process defines a linear operator, we need to restrict to the so-called 'tame' spans defined in [4]. However, spans of finite groupoids are automatically tame, so we can safely ignore this issue.

One can check that this process defines a linear operator from a span of groupoids, and, further, that this process is functorial. This is the degroupoidification functor. It is straightforward to extend this to our bicategory of spans of groupoids by adding identity 2-morphisms to the category of vector spaces and sending all 2-morphisms between spans of groupoids to the corresponding identity 2-morphism.

In the next section, we define a notion of *enriched bicategories*. We will see that constructing an enriched bicategory depends heavily on having a monoidal bicategory in hand. The bicategory Span defined above is, in fact, a monoidal bicategory — that is, Span has a tensor product, which is a functor

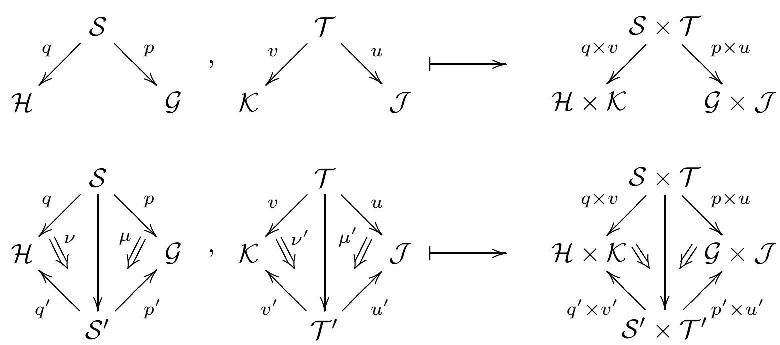
$$\otimes \colon \mathrm{Span} \times \mathrm{Span} \to \mathrm{Span},$$

along with further structure and satisfying some coherence relations.

We describe the main components of the tensor product on Span. Given a pair of groupoids $\mathcal{G}, \mathcal{H}$, the tensor product $\mathcal{G} \times \mathcal{H}$ is the product in Cat. Further, for each pair of pairs of groupoids $(\mathcal{G}, \mathcal{H}), (\mathcal{J}, \mathcal{K})$ there is a functor:

$$\otimes \colon \mathrm{Span}(\mathcal{G}, \mathcal{H}) \times \mathrm{Span}(\mathcal{J}, \mathcal{K}) \to \mathrm{Span}(\mathcal{G} \times \mathcal{J}, \mathcal{H} \times \mathcal{K}),$$

defined as follows:



3.3 Enriched Bicategories

A monoidal structure such as the tensor product on Span discussed in the previous section is the crucial ingredient for defining *enriched bicategories*. In particular, given a monoidal bicategory $\mathcal{V}$ with the tensor product $\otimes$, a $\mathcal{V}$-enriched bicategory has for each pair of objects x, y, an object $\hom(x, y)$ of $\mathcal{V}$. Further, composition involves the tensor product in $\mathcal{V}$:

$$\circ \colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

Monoidal bicategories were defined by Gordon, Powers, and Street [16] in 1994 as a special case of tricategories [12, 17].

In this section, we give a definition of enriched bicategories followed by a *change of base* theorem, which says which sort of map $f\colon \mathcal{V} \to \mathcal{V}'$ lets us turn a $\mathcal{V}$-enriched bicategory into a $\mathcal{V}'$-enriched bicategory.

Remember that for each finite group G there is a category of permutation representations PermRep(G). Enriched bicategories allow us to define a Span-enriched bicategory called the Hecke bicategory, and denoted Hecke(G), which categorifies PermRep(G) in two ways. The change of base theorem is the main tool employed in proving the Fundamental Theorem of Hecke Operators via degroupoidification. Using some basic topos theory, the change of base theorem also provides an alternative view of categorified intertwining operators as spans of G-sets as discussed in Section 2.2.

Before giving the definition of an *enriched bicategory*, we recall the definition of an enriched category — that is, a category enriched over a monoidal category $\mathcal{V}$ [24]. An *enriched category* consists of:

- a set of objects $x, y, z \ldots$,
- for each pair of objects x, y, an object $\hom(x, y) \in \mathcal{V}$,
- composition and identity-assigning maps that are morphisms in $\mathcal{V}$.

For example, PermRep(G) is a category enriched over the monoidal category of vector spaces.

We now define enriched bicategories, which are simply a categorified version of enriched categories.

Definition 5. *Let $\mathcal{V}$ be a monoidal bicategory. A* **$\mathcal{V}$-bicategory** *$\mathcal{B}$ consists of the following data subject to the following axioms:*
Data:

- *a collection of objects $x, y, z, \ldots$,*
- *for every pair of objects x, y, a* **hom-object** *$\hom(x, y) \in \mathcal{V}$, which we will often denote (x, y),*
- *a morphism called* **composition**

$$\circ\colon \hom(x, y) \otimes \hom(y, z) \to \hom(x, z)$$

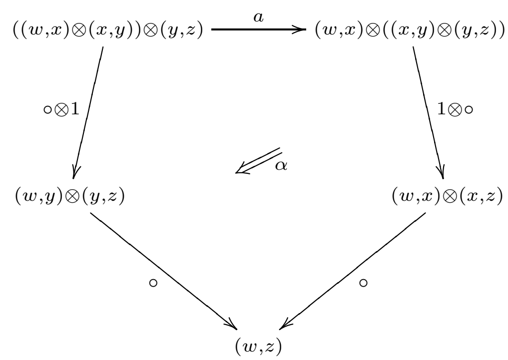
  for each triple f objects $x, y, z \in \mathcal{B}$,

- *an* **identity-assigning** *morphism*

$$i_x\colon I \to \hom(x, x)$$

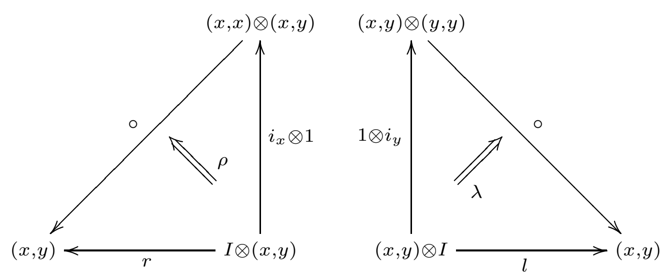
  for each object $x \in \mathcal{B}$,

- *an invertible* 2*-morphism called the* **associator**



for each quadruple of objects $w, x, y, z \in \mathcal{B}$*,*

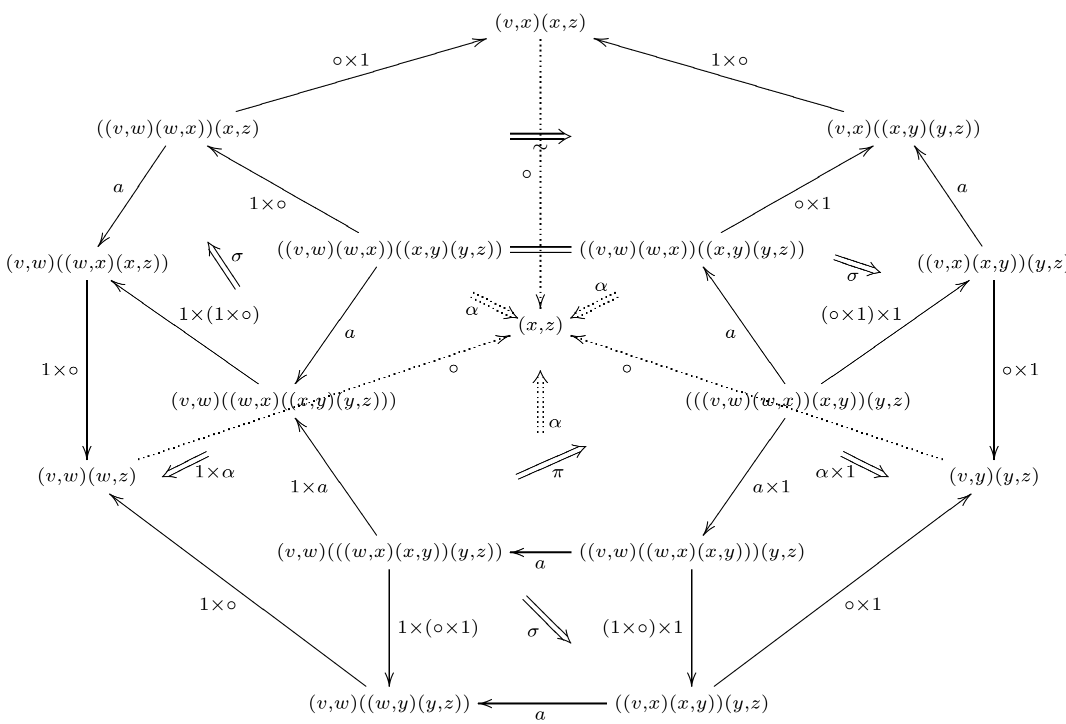
- *and invertible* 2*-morphisms called the* **right unitor** *and* **left unitor**

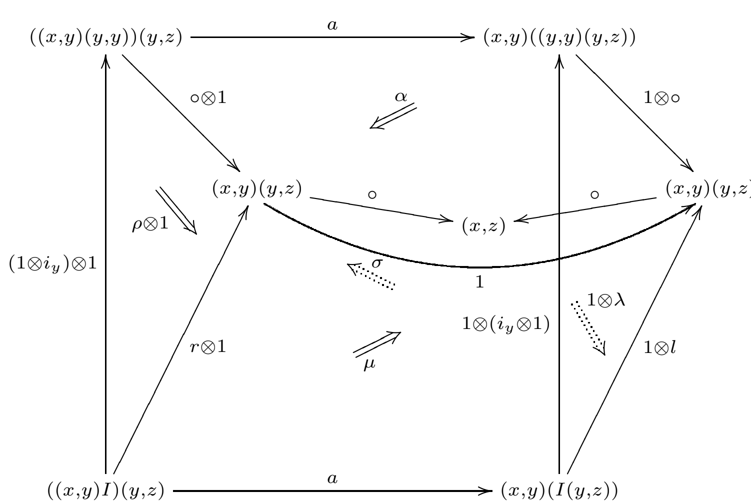


for every pair of objects $x, y \in \mathcal{B}$*.*

In the following diagrams we omit the tensor product in $\mathcal{V}$*.*

Axioms:





σ is the component 2-morphism expressing the pseudo naturality of the associator for the tensor product in the monoidal bicategory $\mathcal{V}$, and the arrow marked $\sim$ is just the associator natural isomorphism in the underlying bicategory of $\mathcal{V}$.

We note that diagrams similar to our axioms of enriched bicategories appeared in the work of Aguilar and Mahajan [1].

Given a monoidal bicategory $\mathcal{V}$, which has only identity 2-morphisms, then every $\mathcal{V}$-bicategory is a $\mathcal{V}$-category in the obvious way, and every $\mathcal{V}$-enriched category can be trivially extended to a $\mathcal{V}$-bicategory. This flexibility will allow us to think of PermRep(G) as either a Vect-enriched category or as a Vect-enriched bicategory.

Now we state a *change of base* construction which allows us to change a $\mathcal{V}$-enriched bicategory to a $\mathcal{V}'$-enriched bicategory.

Claim 6. *Given a lax-monoidal homomorphism of monoidal bicategories $f \colon \mathcal{V} \to \mathcal{V}'$ and a $\mathcal{V}$-bicategory $\mathcal{B}_{\mathcal{V}}$, then there is a $\mathcal{V}'$-bicategory*

$$\bar{f}(\mathcal{B}_{\mathcal{V}}).$$

A monoidal homomorphism is just the obvious sort of map between monoidal bicategories [16, 17]. A *lax*-monoidal homomorphism f is a bit more general: it need not preserve the tensor product up to isomorphism. Instead, it preserves the tensor product only up to a *morphism*:

$$f(x) \otimes' f(y) \to f(x \otimes y).$$

The data of the enriched bicategory $\bar{f}(\mathcal{B}_{\mathcal{V}})$ is straightforward to write down and the proof of the claim is a trivial, yet tedious surface diagram chase. Here we just point out the most important idea. The new enriched bicategory $\bar{f}(\mathcal{B}_{\mathcal{V}})$ has the same objects as $\mathcal{B}_{\mathcal{V}}$, and for each pair of objects x, y, the hom-category of $\bar{f}(\mathcal{B}_{\mathcal{V}})$ is:

$$\mathrm{hom}_{\bar{f}(\mathcal{B}_{\mathcal{V}})}(x, y) := f(\mathrm{hom}_{\mathcal{B}_{\mathcal{V}}}(x, y))$$

This theorem will allow us to pass from the more natural definition of the Hecke bicategory, which we define in the next section, to our original definition of the Hecke bicategory as the bicategory of spans of finite G-sets Span$^*(G\mathrm{Set})$.

4 Fundamental Theorem of Hecke Operators

The following sections are devoted to the proposal of a different construction categorifying the Hecke algebroid. This construction will enable a natural description of the Hecke bicategory, while still retaining a close relationship to the bicategory Span$^*(G\mathrm{Set})$ described in Section 2.5. We will show how to again obtain the category of permutation representations by decategorification, except this time we will use the process of *degroupoidification*. Using the theory of enriched bicategories described in Section 3.3, we show that both degroupoidification and the Grothendieck-style decategorification described in Section 2.4 yield the same result, if we begin with the nicer version of the Hecke bicategory described in the next section.

4.1 The Hecke Bicategory — Take Two

We are now in a position to present a more satisfactory categorification of the intertwining operators between permutation representations of a finite group G. This is the Span-enriched category Hecke(G) — the *Hecke bicategory*.

Claim 7. *Given a finite group G, there is a* Span*-enriched bicategory* Hecke(G) *which has:*

- *finite G-sets $X, Y, Z \ldots$ as objects,*
- *for each pair of finite G-sets X, Y, an object of* Span*, the action groupoid:*

$$\mathrm{hom}(X,Y) = (X \times Y)/\!/G,$$

- *composition*

$$\circ\colon (X \times Y)/\!/G \times (Y \times Z)/\!/G \to (X \times Z)/\!/G$$

  is the span of groupoids,

$$(X \times Z)/\!/G \xleftarrow{\;p_X \times p_Z\;} (X \times Y \times Z)/\!/G \xrightarrow{\;(p_X \times p_Y) \times (p_Y \times p_Z)\;} (X \times Y)/\!/G \times (Y \times Z)/\!/G$$

- *for each finite G-set X, an identity assigning span from the terminal groupoid 1 to $(X \times X)/\!/G$,*
- *invertible 2-morphisms in* Span *assuming the role of the associator and left and right unitors.*

Given this structure one needs to check that the axioms of an enriched bicategory are satisfied; however, we will not prove this here. Combining the degroupoidification functor of Section 3.2, the change of base theorem of Section 3.3, and the enriched bicategory Hecke(G) described above, we can now state the Fundamental Theorem of Hecke Operators. This is the content of the next section.

4.2 The Fundamental Theorem of Hecke Operators

In this section, we make the relationship between the *Hecke algebroid* PermRep(G) of permutation representations of a finite group G and the Hecke bicategory Hecke(G) precise. The idea is that for each finite group G, the Hecke bicategory Hecke(G) categorifies PermRep(G).

We recall the functor *degroupoidification*:

$$\mathcal{D}\colon \mathrm{Span} \to \mathrm{Vect}$$

which replaces groupoids with vector spaces and spans of groupoids with linear operators. With this functor in hand, we can apply the change of base theorem to the Span-enriched bicategory Hecke(G). In other words, for each finite group G there is a Vect-enriched bicategory:

$$\bar{\mathcal{D}}\left(\mathrm{Hecke}(G)\right),$$

which has

- permutation representations $X, Y, Z, \ldots$ of G as objects,
- for each pair of permutation representations X, Y, the vector space

$$\mathrm{hom}(X,Y) = \mathcal{D}\left((X \times Y)//G\right)$$

with G-orbits of $X \times Y$ as basis. Of course, a Vect-enriched bicategory is also a Vect-enriched category. The following is the statement of the Fundamental Theorem of Hecke Operators, an equivalence of Vect-enriched categories.

Claim 8. *Given a finite group G,*

$$\bar{\mathcal{D}}\left(\mathrm{Hecke}(G)\right) \simeq \mathrm{PermRep}(G)$$

as Vect*-enriched categories.*

More explicitly, this says that given two permutation representations X and Y, the vector space of intertwining operators between them can be constructed as the degroupoidification of the groupoid $(X \times Y)//G$.

An important corollary of the Fundamental Theorem of Hecke Operators is that for certain G-sets, which are the flag varieties X associated to Dynkin diagrams, the hom-groupoid Hecke(X, X) categorifies the associated Hecke algebra. We will describe these Hecke algebras in Section 6.1 and make the relationship to the Hecke bicategory and some of its applications explicit in Section 6.2.

In the next section we describe a monoidal bicategory which we use to connect the Hecke bicategory by change of base to the bicategory Span*(GSet) of spans of finite G-sets defined in Section 2.5.

5 A Different Point of View

In the following sections, we recall our original construction of the Hecke bicategory from spans of G-sets. Given the unsatisfactory nature of this construction, we defined the Hecke bicategory as a Span-enriched bicategory and proved the Fundamental Theorem of Hecke Operators using degroupoidification and change of base in Section 4,. Now we use change of base to see that we can obtain the Hecke bicategory Span*(GSet) from the Span-enriched Hecke bicategory Hecke(G). For this we will need to introduce the monoidal bicategory of 'nice topoi'.

5.1 The Monoidal Bicategory of Nice Topoi

In Section 3.2, we began promoting the idea that groupoids are categorified vector spaces. In particular, the isomorphism classes of objects assume the role of a basis of the corresponding free vector space. A slightly different point of view, which we discussed at length in [4], assigns to a groupoid the vector space of functions on the set of isomorphism classes of that groupoid. Thus, promoting functions to functors, we can think of a categorified vector space as the presheaf category on a groupoid. The objects of the monoidal bicategory described in this section will be defined to be equivalent to such presheaf categories. Although presheaves traditionally take values in the category of sets, in the present work we restrict our definition to take values in the category of finite sets, which we denote FinSet.

Definition 9. *A* **presheaf (of finite sets) on a groupoid $\mathcal{G}$ is a contravariant functor from $\mathcal{G}$ to** FinSet**.**

Given a groupoid $\mathcal{G}$, its category of presheaves $\widehat{\mathcal{G}}$ has:

- presheaves on $\mathcal{G}$ as objects,
- natural transformations as morphisms.

Definition 10. *A* **nice topos** *is a category equivalent to the category of presheaves on a (small) groupoid $\mathcal{G}$.*

By the above definition, there is a nice topos $\widehat{\mathcal{G}}$ of presheaves corresponding to any groupoid $\mathcal{G}$. However, mapping groupoids to these very special presheaf categories suggests that nice topoi should have an intrinsic characterization. To give such a characterization of these topoi, especially when we generalize to presheaves of not-necessarily finite sets, we should look to the generalization to topoi of Grothendieck's Galois theory of schemes. See the survey article of [30] and references therein. Giving this intrinsic characterization liberates the nice topos from its dependence on a particular groupoid. In particular, this supports the point of view that nice topoi are the objects of a truly basis independent theory of categorified vector spaces. We saw in Section 3.2 that a groupoid does indeed correspond to a categorified vector space with a chosen basis.

Following this line of reasoning, the maps between nice topoi are thought of as categorified linear operators. Thus, they should preserve sums, or more accurately, they should preserve a categorified and generalized notion of 'sums' — colimits.

Definition 11. *A functor is said to be* **cocontinuous** *if it preserves all (small) colimits.*

Thus, cocontinuous functors seem to be the right choice for categorified linear operators.

In the next section, we will see further support for the analogy: *nice topos is to groupoid* as *abstract vector space is to vector space with chosen basis* and *cocontinuous functor is to span of groupoids* as *linear operator is to matrix*.

The monoidal bicategory of nice topoi consists of:

- nice topoi $\mathcal{D}, \mathcal{E}, \mathcal{F}$ as objects,
- cocontinuous functors as 1-morphisms,
- natural transformations as 2-morphisms.

Objects of Nice are categories and the morphisms between them are functors. Thus, there is a faithful functor from Nice to Cat. The monoidal structure on Nice is then just a restriction of the monoidal structure on Cat. In particular, the tensor product of nice topoi $\mathcal{E} \simeq \widehat{\mathcal{G}}$ and $\mathcal{F} \simeq \widehat{\mathcal{H}}$ is the cartesian product $\mathcal{E} \times \mathcal{F}$. This product is particularly easy to work with since:

$$\widehat{\mathcal{G}} \times \widehat{\mathcal{H}} \cong \widehat{\mathcal{G} \times \mathcal{H}}$$

That is, the product of a presheaf on $\mathcal{G}$ and a presheaf on $\mathcal{H}$ is a presheaf on $\mathcal{G} \times \mathcal{H}$.

In the next section, we will describe the relationship between spans of groupoids and cocontinuous functors between nice topoi. Further, we will describe some basic notions of topos theory.

5.2 From Spans of Groupoids to Nice Topoi

The change of base construction for enriched bicategories offers a new interpretation of the Hecke bicategory. We have described two closely related monoidal bicategories. The relationship between groupoids and nice topoi is made manifest as a map of monoidal bicategories.

Claim 12. *There is a homomorphism of monoidal bicategories*

$$\mathcal{K}\colon \mathrm{Span} \to \mathrm{Nice}.$$

It is clear from the definition of nice topoi that $\mathcal{K}$ should take a groupoid $\mathcal{G}$ to its corresponding presheaf topos $\widehat{G}$. Continuing our analogy with abstract vector spaces and vector spaces with a chosen basis, we explain how a span of groupoids induces a cocontinuous functor between the corresponding presheaf categories. In fact, a groupoid can be recovered up to equivalence from its presheaf category, just as a basis can be recovered up to isomorphism from its vector space, but in each case the equivalence or isomorphism is non-canonical.

First, we give some basic definitions from topos theory. A topos is a category which resembles the category of sets. Categories of presheaves (of sets, not necessarily finite) are examples of *Grothendieck topoi*. In general, a Grothendieck topos is a category of sheaves on a site. A site is just a category with a notion of a *covering* of objects called a *Grothendieck topology* [19, 28]. A familiar example with a particularly simple Grothendieck topology is the category of presheaves on a topological space. In this paper we consider only presheaves of *finite* sets. However, a category of such presheaves is also a topos.

So if a topos is just a special type of category, then how does topos theory differ from category theory? One answer is that while the morphisms between categories are functors, the morphisms between topoi must satisfy extra properties. Such a morphism is called a *geometric morphism* [28]. We define the morphisms between nice topoi, although the definition is exactly the same in the more general setting of topoi.

Definition 13. *A* **geometric morphism** $e\colon \mathcal{E} \to \mathcal{F}$ *between nice topoi is a pair of functors* $e^*\colon \mathcal{F} \to \mathcal{E}$ *and* $e_*\colon \mathcal{E} \to \mathcal{F}$ *such that* e^* *is left adjoint to* e_* *and* e^* *is left exact, i.e., preserves finite limits. A geometric morphism* $e\colon \mathcal{E} \to \mathcal{F}$ *is said to be* **essential** *if there exists a functor* $e_!\colon \mathcal{E} \to \mathcal{F}$ *which is left adjoint to* e^*.

We note a relationship to functors between finite groupoids, which allows us to define cocontinuous functors from spans. Any functor $f\colon \mathcal{G} \to \mathcal{H}$ defines a geometric morphism between the corresponding presheaf categories:

$$\widehat{f}\colon \widehat{\mathcal{G}} \to \widehat{\mathcal{H}}.$$

More specifically, the geometric morphism $\widehat{f}$ consists of the functor:

$$f^*\colon \widehat{\mathcal{H}} \to \widehat{\mathcal{G}},$$

which pulls presheaves backwards from $\mathcal{H}$ to $\mathcal{G}$, together with

$$f_*\colon \widehat{\mathcal{G}} \to \widehat{\mathcal{H}},$$

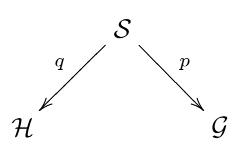
which pushes presheaves forward from $\mathcal{G}$ to $\mathcal{H}$. The particularly important fact is that a geometric morphism induced by a functor between groupoids will always be essential — that is, there exists a *left* adjoint to f^*:

$$f_!\colon \widehat{\mathcal{G}} \to \widehat{\mathcal{H}}.$$

Definition 14. *A* **map of geometric morphisms** $\alpha\colon e \Rightarrow f$ *is a natural transformation:*

$$\alpha\colon e^* \Rightarrow f^*.$$

Using the fact that a functor between groupoids induces an essential geometric morphism, we see that from a span of groupoids:



we can define a functor:

$$q_!p^*\colon \widehat{\mathcal{G}} \to \widehat{\mathcal{H}},$$

which is a composite of left adjoint functors, and thus, cocontinuous. Although we do not go into detail at present, using this construction on spans it is not difficult to define a natural transformation between cocontinuous functors from a map of spans. After checking details, it is evident that we have defined a homomorphism of monoidal bicategories

$$\mathcal{K}\colon \text{Span} \to \text{Nice}$$

as claimed.

Using the map $\mathcal{K}$, we can apply the change of base to the Hecke bicategory to obtain a new interpretation of the Hecke bicategory as a Nice-enriched bicategory. We will discuss the details and benefits of this interpretation in the next section.

5.3 Spans of G-Sets as Nice Topoi

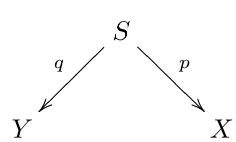
In this section we take a closer look at the structure of the Nice-enriched bicategory $\bar{\mathcal{K}}(\text{Hecke}(G))$. Our goal is to rephrase the construction as a monoidal bicategory of spans of finite G-sets. However, recalling the unnatural change we were forced to make in Section 2.3 to obtain a categorification of the Hecke algebroid, we will have to perform one more change of base to obtain a version of the Hecke bicategory, which reconciles these structures.

It turns out that the hom-categories of the Nice-enriched bicategory $\bar{\mathcal{K}}(\text{Hecke}(G))$ have a very simple description as spans of finite G-sets and maps of spans. Given finite G-sets X and Y, the product $X \times Y$ is a finite G-set, so we can construct the action groupoid $(X \times Y)//G$. Presheaves on this groupoid will be a nice topos.

Lemma 15. *Given a pair of finite G-sets X, Y, the category of spans of finite G-sets and maps of spans of finite G-sets is equivalent to the nice topos $\widehat{(X \times Y)//G}$.*

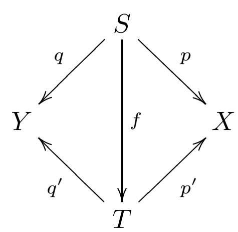
This lemma is usually called the *Grothendieck construction.* It says that given a pair of finite G-sets X and Y, presheaves on $(X \times Y)//G$ are spans from X to Y and natural transformations are maps of spans. We sketch the proof of this lemma now.

Proof. (Sketch) Given a span of finite G-sets:



there is a presheaf on $(X \times Y)//G$, which we can think of approximately as a categorified matrix of natural numbers, i.e., a matrix of sets. Each object (x, y) determines an entry in the matrix, and the entries are the sets $S_{x,y} = p^{-1}(x) \cap q^{-1}(y)$, defined in Section 2.1. For each morphism $(g,(x,y))\colon (x,y) \to (x',y')$, we define a function from $S_{x',y'}$ to $S_{x,y}$ by the action of g^{-1} on the G-set $S_{x',y'}$. Thus, we get a presheaf from the span S.

Now from a map of spans of finite G-sets:



we construct a natural transformation between the presheaves corresponding to S and T. For each object (x, y) of $(X \times Y)//G$, the component of the natural transformation takes an element $s \in S_{x,y}$ to $f(s) \in T_{x,y}$. Since f is G-equivariant, the naturality squares commute.

It is not difficult to check that this process defines an equivalence of categories. □

We have seen that the hom-categories of the Nice-enriched version of Hecke(G) are actually categories whose objects are spans of finite G-sets and whose morphisms are maps between such spans. In particular, this bicategory is equivalent to Span(GSet), our original attempt at categorifying the Hecke algebroid in Section 2.3. Unfortunately, we were forced to weaken the requirements on the 2-morphisms in that bicategory — we considered *not necessarily* G-equivariant maps of spans of finite G-sets in place of the usual equivariant maps between these spans. In the next section, we will describe a functor from Nice to itself, which corrects for this change in the 2-morphisms, and using the functor $\mathcal{L}$ from Section 2.5, allows us to recover the Hecke algebroid PermRep(G) once again.

5.4 A Useful Change of Base

In Section 2.5, we considered a slight modification of the proposed categorification of the Hecke algebroid. This was an attempt to rectify the close relationship between spans of finite G-sets and intertwining operators between permutation representations. In particular, there were too many isomorphism classes of irreducible spans. We remedied this by allowing more maps of spans in the bicategory. This, in effect, reduced the number of isomorphism classes of irreducible spans. In this section, we compensate for this change by constructing a monoidal functor from Nice to itself and apply change of base to the Nice-enriched Hecke bicategory.

We now describe the monoidal functor:

$$\mathcal{J} \colon \text{Nice} \to \text{Nice}.$$

We assign to each object $\mathcal{E} \simeq \widehat{\mathcal{G}}$ in Nice the presheaf on the set of orbits of the groupoid $\mathcal{G}$:

$$\mathcal{J} \colon \widehat{\mathcal{G}} \mapsto \underline{\widehat{\mathcal{G}}},$$

where $\underline{\mathcal{G}}$ is the set of orbits, or connected components, of $\mathcal{G}$. Of course, the set of orbits is a discrete groupoid — that is, a groupoid with only identity morphisms — and, thus, an object of Nice.

Given a cocontinuous functor $f\colon \widehat{\mathcal{G}} \to \widehat{\mathcal{H}}$, we send it to the cocontinuous functor:

$$\mathcal{J}(f)\colon \widehat{\underline{\mathcal{G}}} \to \widehat{\underline{\mathcal{H}}},$$

which is straightforward to define by abstract nonsense.

Now, to apply change of base to the Hecke bicategory, we need this functor to preserve the monoidal structure laxly. In other words, there should be an arrow:

$$\mathcal{J}(\widehat{\mathcal{G}}) \otimes \mathcal{J}(\widehat{\mathcal{H}}) \to \mathcal{J}(\widehat{\mathcal{G}} \otimes \widehat{\mathcal{H}}).$$

We will just discuss the monoidal character of the function on objects here, as it is not difficult to fill in the details for 1- and 2-morphisms.

We have:

$$\mathcal{J}(\widehat{\mathcal{G}}) \otimes \mathcal{J}(\widehat{\mathcal{H}}) = \widehat{\underline{\mathcal{G}}} \otimes \widehat{\underline{\mathcal{H}}} = \widehat{\underline{\mathcal{G}} \times \underline{\mathcal{H}}}$$

and

$$\mathcal{J}(\widehat{\mathcal{G}} \otimes \widehat{\mathcal{H}}) = \widehat{\underline{\mathcal{G} \times \mathcal{H}}}.$$

Obtaining this map is simple once we note that there is a morphism:

$$\underline{\mathcal{G} \times \mathcal{H}} \to \underline{\mathcal{G}} \times \underline{\mathcal{H}},$$

and that the functor taking a groupoid to its category of presheaves is contravariant. We should note that the map from $\underline{G \times H}$ to $\underline{G} \times \underline{H}$ may not have an inverse, and therefore the monoidal structure is only laxly preserved. However, one can see from the change of base axioms for enriched bicategories that a lax monoidal functor is sufficient to obtain an enriched bicategory by base change.

5.5 Revisiting the Hecke Bicategory

The purpose of constructing the monoidal functor $\mathcal{J}$ in the last section was to recover the Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$ from this new Nice-enriched bicategory.

In Section 5.3, we showed that presheaf categories on action groupoids obtained from products of finite G-sets, were equivalent to categories of spans of finite G-sets and G-equivariant maps between these spans. However, in Section 2.5, we saw that to obtain the Hecke algebroid from a bicategory of finite G-sets, spans of finite G-sets, and maps of spans, we had to relax the equivariance condition on the maps of spans. We now show that using the monoidal functors:

$$\mathcal{K}\colon \mathrm{Span} \to \mathrm{Nice}$$

from Section 5.2 and:

$$\mathcal{J}\colon \mathrm{Nice} \to \mathrm{Nice}$$

from Section 5.4, we can pass from the Span-enriched Hecke bicategory to the Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$. Thus, our two versions of the Hecke bicategory each admit a statement of the Fundamental Theorem of Hecke Operators. This means we can decategorify the Span-enriched Hecke bicategory $\mathrm{Hecke}(G)$ via degroupoidification and change of base:

$$\bar{\mathcal{D}}\colon \mathrm{Hecke}(G) \to \mathrm{PermRep}(G),$$

or the functor:

$$\mathcal{L}\colon \mathrm{Span}^*(G\mathrm{Set}) \to \mathrm{PermRep}(G),$$

each process yielding the Hecke algebroid $\mathrm{PermRep}(G)$.

We recall that from the Span-enriched bicategory $\mathrm{Hecke}(G)$, we have obtained a Nice-enriched bicategory:

$$\bar{\mathcal{J}}\bar{\mathcal{K}}\,(\mathrm{Hecke}(G))$$

via our change of base theorem. It is useful to note that this version of the Hecke bicategory is not only an enriched bicategory, but is an actual bicategory consisting of:

- finite G-sets X, Y, Z,... as objects,
- for each pair of finite G-sets X, Y, a hom-category $(\widehat{X \times Y})/G$ consisting of:
  - presheaves on the discrete groupoid $(X \times Y)/G$ as 1-morphisms,
  - natural transformations as 2-morphisms.

Starting from the Span-enriched Hecke bicategory, we have used change of base to recover a bicategory very similar to our original Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$. In fact, they are equivalent.

Of course, the objects are the same, so we will sketch the equivalence by noting the functor between hom-categories. Given finite G-sets X and Y, we need to reinterpret the hom-category:

$$(\widehat{X \times Y})/G$$

as a category of spans of G-sets and maps of spans. This is quite simple.

First, given any presheaf S, we precompose with the function from $X \times Y$:

$$X \times Y \to (X \times Y)/G \xrightarrow{S} \mathrm{FinSet},$$

and we call the composite $\bar{S}$. Recall that we are treating sets as discrete groupoids, so the above arrows are, in fact, functors. Such a functor:

$$\bar{S}\colon X \times Y \to \mathrm{FinSet}$$

is equivalent to the span:

$$Y \longleftarrow \coprod_{(x,y)} \bar{S}(x,y) \longrightarrow X$$

Natural transformations in $(\widehat{X \times Y})/G$ are just families of functions indexed by $X \times Y$, and, therefore, induce maps between the corresponding spans.

Claim 16. *Given a finite group G,*

$$\bar{\mathcal{J}}\bar{\mathcal{K}}\left(\mathrm{Hecke}(G)\right) \simeq \mathrm{Span}^*(G\mathrm{Set})$$

as bicategories.

Passing through this equivalence, we can apply our decategorification functor $\mathcal{L}$ to $\bar{\mathcal{J}}\bar{\mathcal{K}}\left(\mathrm{Hecke}(G)\right)$. Thus, we have now have two statements of the Fundamental Theorem of Hecke Operators coming from $\mathrm{Hecke}(G)$:

Claim 17. *Given a finite group G,*

$$\mathcal{L}(\bar{\mathcal{J}}\bar{\mathcal{K}}\left(\mathrm{Hecke}(G)\right)) \simeq \mathrm{PermRep}(G)$$

as Vect*-enriched categories.*

and,

Claim 18. *Given a finite group G,*

$$\mathcal{D}\left(\mathrm{Hecke}(G)\right) \simeq \mathrm{PermRep}(G)$$

as Vect*-enriched categories.*

6 The Hecke Algebra and Applications

The main theorem of this paper was the Fundamental Theorem of Hecke Operators. This is, in fact, a statement about categorified Hecke algebras. There is a nice collection of literature on categorified Hecke algebras as they have played a central role in the development of categorified and geometric representation theory.

In Section 2, we motivated the notion of a categorification of permutation representations and intertwining operators via connections with spans of sets — especially finite G-sets. This leads us to an awkward first construction of the Hecke bicategory as $\mathrm{Span}^*(G\mathrm{Set})$. We spent the rest of the paper building a more natural construction of the Hecke bicategory, and a way to relate said construction $\mathrm{Hecke}(G)$ to $\mathrm{Span}^*(G\mathrm{Set})$.

In applying the categorified Hecke algebra to knot theory, the more concrete description of the Hecke bicategory $\mathrm{Span}^*(G)$ as spans of finite G-sets, allows a hands-on approach to the 2-morphisms, i.e., the not-necessarily G-equivariant maps of spans. We show that in certain cases these are Yang-Baxter operators that satisfy the Zamolodchikov tetrahedron equation.

6.1 Hecke Algebras

In this section, we recall some descriptions of the Hecke algebra and note a categorification of these algebras in the context of the Fundamental Theorem of Hecke Operators. Categorified Hecke algebras have been studied by a number of authors in various contexts including Soergel bimodules citeSoe, a recent diagrammatic interpretation of the work of Soergel by Elias and Khovanov [13], and a geometric interpretation by Webster and Williamson [33]. Further, Hecke categories have been studied in the context of the Kazhdan-Lusztig conjectures [23]. Hecke algebras are close relatives of quantum groups, which have provided the major thrust in research in categorified representation theory. See [15, 21, 31], for example.

Hecke algebras are constructed from a Dynkin diagram and a prime power. Moreover, they are algebras of Hecke operators. The term 'Hecke operator' is largely confined to the realm of number theory and modular forms, but it makes sense to say that the Hecke algebras with which we are concerned at present consist of Hecke operators. That is, the notion of Hecke operator can be interpreted quite broadly as the Hecke algebra is just a special example of a vector space of intertwining operators between permutation representations.

There are several well-known equivalent descriptions of the Hecke algebra $\mathcal{H}(\Gamma, q)$ obtained from a Dynkin diagram Γ and a prime power q. One kind of Hecke algebra, commonly referred to as the *Iwahori-Hecke algebra*, is a q-deformation of the group algebra of the Coxeter group of Γ. A standard example of a Coxeter group associated to a Dynkin diagram is the symmetric group on n letters S_n, which is the Coxeter group of the A_{n-1} Dynkin diagram. We will return to this definition in Section 6.2 and see that it lends itself to combinatorial applications of the Hecke algebra. This combinatorial aspect comes from the close link between the Coxeter group and its associated Coxeter complex, a finite simplicial complex which plays an essential role in the theory of buildings [9].

Hecke algebras have an alternative definition as algebras of intertwining operators between certain coinduced representations [10]. Given a Dynkin diagram Γ and prime power q, there is an associated simple algebraic group $G = G(\Gamma, q)$. Choosing a Borel subgroup $B \subset G$, i.e., a maximal solvable subgroup, we can construct the corresponding flag variety $X = G/B$, a transitive G-set.

Now, for a finite group G and a representation V of a subgroup $H \subset G$, the *coinduced representation* of G from H is defined as the V-valued functions on G, which commute with the action of H:

$$\mathrm{CoInd}_H^G = \{f \colon G \to V \mid h \cdot f(g) = f(hg)\}$$

The action of $g \in G$ is defined on a function $f \colon G \to V$ as $g \cdot f(g') = f(g'g^{-1})$. A standard fact about finite groups says that the representation coinduced from the trivial representation of any subgroup is the permutation representation on the cosets of that subgroup.

Thus, from the trivial representation of a Borel subgroup B, we obtain the permutation representation on the cosets of B, i.e., the flag variety X. Then the Hecke algebra is defined as the algebra of intertwining operators from $\widetilde{X}$ to itself:

$$\mathrm{PermRep}(G)(\widetilde{X}, \widetilde{X}) := \mathcal{H}(\Gamma, q),$$

where $G = G(\Gamma, q)$ and we use the notation $\mathcal{C}(A, B)$ to denote $\hom(A, B)$ in the category $\mathcal{C}$.

Given this definition of the Hecke algebra, we have an immediate corollary to Claim 8:

Claim 19. *Given a finite group $G = G(\Gamma, q)$, the* hom-*category* $\mathrm{Hecke}(G)(\widetilde{X}, \widetilde{X})$ *categorifies $\mathcal{H}(\Gamma, q)$.*

6.2 The Categorified Hecke Algebra and 2-Tangles

Now that we have developed the machinery of the Fundamental Theorem of Hecke Operators, and we have seen a categorification of Hecke algebras abstractly as a corollary, we can look at a concrete example. The categorified Hecke algebra is particularly easy to understand from our original definition of the Hecke bicategory $\mathrm{Span}^*(G\mathrm{Set})$ as the bicategory of finite G-sets, spans of G-sets, and not-necessarily equivariant maps of spans. Further, in this categorified picture we can see relationships with 2-tangles in 4-dimensional space.

While we found it useful in considering the Fundamental Theorem of Hecke Operators to view Hecke algebras as algebras of intertwining operators, viewing the Hecke algebra as a q-deformation of a Coxeter group [18] is helpful in examples.

Any Dynkin diagram gives rise to a simple Lie group, and the Weyl group of this simple Lie group is a Coxeter group. Let Γ be a Dynkin diagram. We write $d \in \Gamma$ to mean that d is a dot in this diagram. Associated to each unordered pair of dots $d, d' \in \Gamma$ is a number $m_{dd'} \in \{2, 3, 4, 6\}$. In the usual Dynkin diagram conventions:

- $m_{dd'} = 2$ is drawn as no edge at all,
- $m_{dd'} = 3$ is drawn as a single edge,
- $m_{dd'} = 4$ is drawn as a double edge,
- $m_{dd'} = 6$ is drawn as a triple edge.

For any prime power q, our Dynkin diagram Γ gives a Hecke algebra. The *Hecke algebra* $\mathcal{H}(\Gamma, q)$ corresponding to this data is the associative $\mathbb{C}$-algebra with one generator σ_d for each $d \in \Gamma$, and relations:

$$\sigma_d^2 = (q - 1)\sigma_d + q$$

for all $d \in \Gamma$, and

$$\sigma_d \sigma_{d'} \sigma_d \cdots = \sigma_{d'} \sigma_d \sigma_{d'} \cdots$$

for all $d, d' \in \Gamma$, where each side has $m_{dd'}$ factors.

When $q = 1$, this Hecke algebra is simply the group algebra of the Coxeter group associated to Γ: that is, the group with one generator s_d for each dot $d \in \Gamma$, and relations

$$s_d^2 = 1, \qquad (s_d s_{d'})^{m_{dd'}} = 1.$$

So, the Hecke algebra can be thought of as a q-deformation of this Coxeter group.

We recall the flag variety $X = G/B$ from Section 6.1. This set is a smooth algebraic variety, but we only need the fact that it is a finite set equipped with a transitive action of the finite group G. Starting from just this G-set X, we can see an explicit picture of the categorified Hecke algebra of spans of G-sets from X to X.

The key is that for each dot $d \in \Gamma$ there is a special span of G-sets that corresponds to the generator $\sigma_d \in \mathcal{H}(\Gamma, q)$. To illustrate these ideas, let us consider the simplest nontrivial example, the Dynkin diagram A_2:

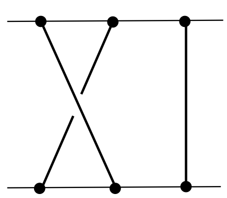
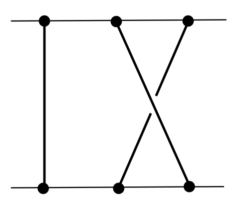
The Hecke algebra associated to A_2 has two generators, which we call P and L, for reasons soon to be revealed:

$$P = \sigma_1, \qquad L = \sigma_2.$$

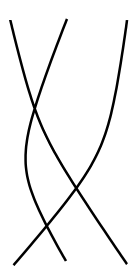
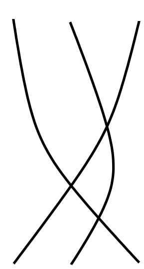
The relations are

$$P^2 = (q-1)P + q, \qquad L^2 = (q-1)P + q, \qquad PLP = LPL.$$

It follows that this Hecke algebra is a quotient of the group algebra of the 3-strand braid group, which has two generators P and L, which we can draw as tangles in 3-dimensional space:

$P =$  $L =$ 

and one relation $PLP = LPL$,:

 $=$ 

called the *Yang–Baxter equation* or *third Reidemeister move*. This is why Jones could use traces on the A_n Hecke algebras to construct invariants of knots [20]. This connection to knot theory makes it especially interesting to categorify Hecke algebras.

So, let us see what the categorified Hecke algebra looks like, and where the Yang–Baxter equation comes from. The algebraic group corresponding to the A_2 Dynkin diagram and the prime power q is $G = \mathrm{SL}(3, \mathbb{F}_q)$, and we can choose the Borel subgroup B to consist of upper triangular matrices in $\mathrm{SL}(3, \mathbb{F}_q)$. Recall that a complete flag in the vector space $\mathbb{F}_q^3$ is a pair of subspaces

$$0 \subset V_1 \subset V_2 \subset \mathbb{F}_q^3.$$

The subspace V_1 must have dimension 1, while V_2 must have dimension 2. Since G acts transitively on the set of complete flags, while B is the subgroup stabilizing a chosen flag, the flag variety $X = G/B$ in this example is just the set of complete flags in $\mathbb{F}_q^3$—hence its name.

We can think of $V_1 \subset \mathbb{F}_q^3$ as a point in the projective plane $\mathbb{F}_q\mathrm{P}^2$, and $V_2 \subset \mathbb{F}_q^3$ as a line in this projective plane. From this viewpoint, a complete flag is a chosen point lying on a chosen line in $\mathbb{F}_q\mathrm{P}^2$. This viewpoint is natural in the theory of 'buildings', where each Dynkin diagram corresponds to a type of geometry [9]. Each dot in the Dynkin diagram then stands for a 'type of geometrical figure', while each edge stands for an 'incidence relation'. The A_2 Dynkin diagram corresponds to projective plane geometry. The dots in this diagram stand for the figures 'point' and 'line':

$$\text{point} \bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \bullet \text{ line}$$

The edge in this diagram stands for the incidence relation 'the point p lies on the line ℓ'.

We can think of P and L as special elements of the A_2 Hecke algebra, as already described. But when we categorify the Hecke algebra, P and L correspond to irreducible spans of G-sets – that is, not a coproduct of two non-trivial spans of G-sets. Let us describe these spans and explain how the Hecke algebra relations arise in this categorified setting.

The objects P and L can be defined by giving irreducible spans of G-sets:

$$X \leftarrow P \rightarrow X \qquad\qquad X \leftarrow L \rightarrow X$$

In general, any span of G-sets

$$X \xleftarrow{q} S \xrightarrow{p} X$$

such that $q \times p \colon S \to X \times X$ is injective can be thought of as G-invariant binary relation between elements of X. Irreducible G-invariant spans are always injective in this sense. So, such spans can also be thought of as G-invariant relations between flags. In these terms, we define P to be the relation that says two flags have the same line, but different points:

$$P = \{((p,\ell),(p',\ell)) \in X \times X \mid p \neq p'\}$$

Similarly, we think of L as a relation saying two flags have different lines, but the same point:

$$L = \{((p,\ell),(p,\ell')) \in X \times X \mid \ell \neq \ell'\}.$$

Given this, we can check that

$$P^2 \cong (q-1) \times P + q \times 1, \qquad L^2 \cong (q-1) \times L + q \times 1, \qquad PLP \cong LPL.$$

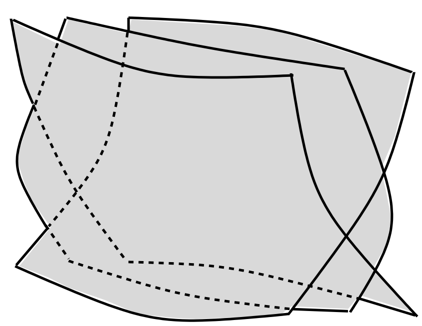
Here both sides refer to spans of G-sets. Addition of spans is defined using coproduct, while 1 denotes the identity span from X to X. We use 'q' to stand for a fixed q-element set, and similarly for '$q-1$'. We compose spans of G-sets using the ordinary pullback.

To check the existence of the first two isomorphisms above, we just need to count. In $\mathbb{F}_q\mathrm{P}^2$, the are $q+1$ points on any line. So, given a flag we can change the point in q different ways. To change it again, we have a choice: we can either send it back to the original point, or change it to one of the $q-1$ other points. So, $P^2 \cong (q-1) \times P + q \times 1$. Since there are also $q+1$ lines through any point, similar reasoning shows that $L^2 \cong (q-1) \times L + q \times 1$.

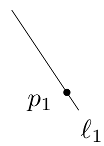
The Yang–Baxter isomorphism

$$PLP \cong LPL$$

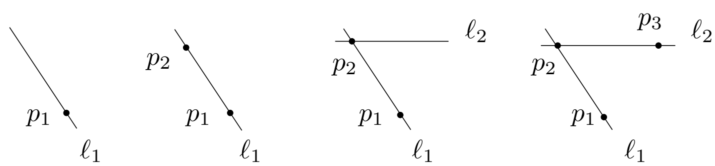
is more interesting. For this isomorphism we will draw the corresponding 2-tangle in 4-dimensional space [6]:



We construct it as follows. First consider the left-hand side, PLP. So, start with a complete flag called (p_1, ℓ_1):

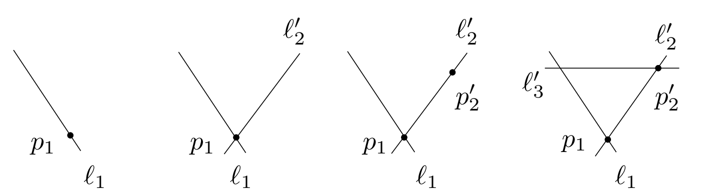


Then, change the point to obtain a flag (p_2, ℓ_1). Next, change the line to obtain a flag (p_2, ℓ_2). Finally, change the point once more, which gives us the flag (p_3, ℓ_2):



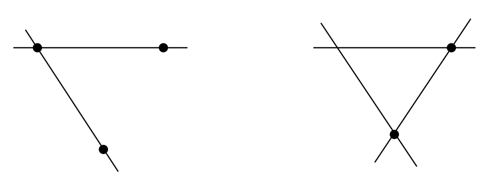
The figure on the far right is a typical element of PLP.

On the other hand, consider LPL. So, start with the same flag as before, but now change the line, obtaining (p_1, ℓ'_2). Next change the point, obtaining the flag (p'_2, ℓ'_2). Finally, change the line once more, obtaining the flag (p'_2, ℓ'_3):



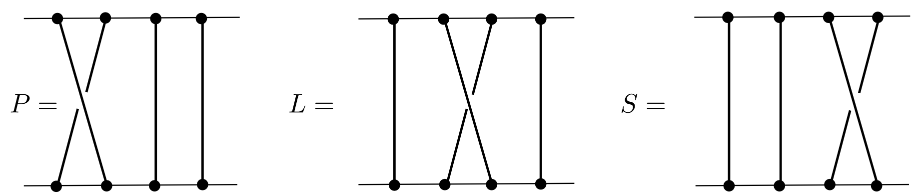
The figure on the far right is a typical element of LPL.

Now, the axioms of projective plane geometry say that any two distinct points lie on a unique line, and any two distinct lines intersect in a unique point. So, any figure of the sort shown on the left below determines a unique figure of the sort shown on the right, and vice versa:

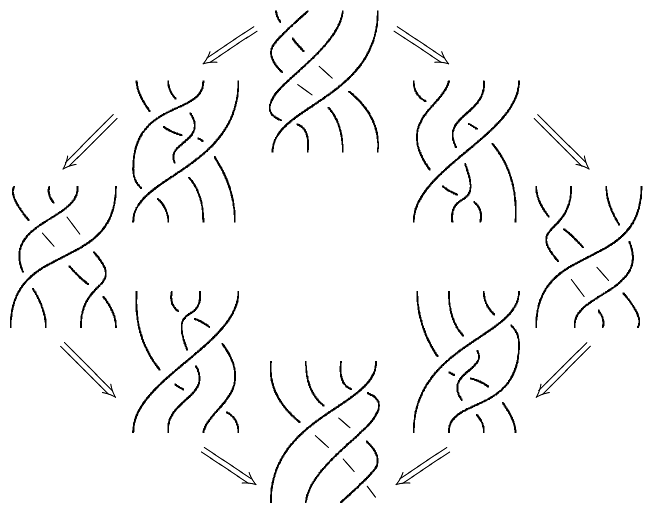


Comparing this with the pictures above, we see this bijection induces an isomorphism of spans $PLP \cong LPL$. So, we have derived the Yang–Baxter isomorphism from the axioms of projective plane geometry!

While the Yang-Baxter *equation* is present in the generators and relations description of the Hecke algebra, we have seen that the categorified setting allows us to view these equations as *isomorphisms* of spans of G-sets. As such, these *Yang-Baxter operators* satisfy an equation of their own – the *Zamolodchikov tetrahedron equation* [22]. However, this equation only appears in the categorified A_n Hecke algebra, for $n \geq 3$. We can assign braids on four strands to the generators of the A_3 Hecke algebra:



where composition of spans, or multiplication in the Hecke algebra, corresponds to stacking of braid diagrams. Then we can express the Zamolodchikov equation – as an equation in the categorified Hecke algebra – in the form of a commutative diagram of braids citeBaezCrans,CarterSaito:



This is just the beginning of a wonderful story involving Dynkin diagrams of more general types, incidence geometries, logic, braided monoidal 2-categories [7, 29], knot invariants, topological quantum field theories, geometric representation theory, and more!

Acknowledgements

A tremendous debt is owed to John Baez, James Dolan and Todd Trimble for initiating an interesting and creative long-term endeavor, as well as for generously sharing their time and ideas. Also to the contributors to the n-Category Café, especially John Baez, Bruce Bartlett, Denis-Charles Cisinski, Tom Leinster, Mike Shulman, Todd Trimble, Urs Schreiber and Simon Willerton for taking an interest and sharing and clarifying so many ideas. Finally, I thank Nick Gurski for patiently teaching me about tricategories, David Ben-Zvi, Anthony Licata, Urs Schreiber and the members of the *Journal Club* for introducing me to the larger world of geometric function theories, Mikhail Khovanov and Aaron Lauda for generously hosting me at Columbia University while many of these ideas were developed, Jim Stasheff for encouragement on a very early draft, and Julia Bergner and Christopher Walker for patiently listening and contributing to many useful conversations. The author was supported by the National Science Foundation under Grant No. 0653646.

References

[1] M. Aguiar and S. Mahajan, Monoidal functors, species and Hopf algebras. Available at http://www.math.tamu.edu/∼maguiar/a.pdf

[2] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004), 492528. Also available as arXiv:math/0307263.

[3] J. Baez and J. Dolan, From finite sets to Feynman diagrams, in *Mathematics Unlimited—2001 and Beyond*, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29–50. Also available as arXiv:math/0004133.

[4] J. Baez, A. E. Hoffnung, and C. Walker, Higher Dimensional Algebra VII: Groupoidification, in *Theory and Applications of Categories*, Available at arXiv:0908.4305.

[5] J. Baez and A. E. Hoffnung, Higher Dimensional Algebra VIII: The Hecke Bicategory. Available at http://math.ucr.edu/home/baez/hecke.pdf.

[6] J. Baez and L. Langford, Higher-dimensional algebra IV: 2-tangles, in *Adv. Math.* **180** (2003), 705764. Also available as arXiv:q-alg/9703033.

[7] J. Baez and M. Neuchl, Higher-dimensional algebra I: braided monoidal categories, in *Adv. Math.* **121** (1996), 196–244. Also available as arXiv:q-alg/9511013.

[8] J. Bénabou, Introduction to Bicategories, in *Reports of the Midwest Category Seminar*, *Lecture Notes in Math.*, Springer-Verlag, Berlin, (1967), 1–77.

[9] K. Brown, *Buildings*, Springer, Berlin, 1989.

[10] D. Bump, *Lie groups*, Springer, New York, 2004.

[11] J. S. Carter and M. Saito, *Knotted Surfaces and Their Diagrams*, American Mathematical Society, Providence, 1998.

[12] B. Day and R. Street, Monoidal bicategories and Hopf algebroids, in *Adv. Math.* **129** (1997), 99157.

[13] B. Elias and M. Khovanov, Diagrammatics for Soergel Categories, 2009. Available as arXiv:0902.4700.

[14] M. Fiore and T. Leinster, Objects of categories as complex numbers, *Adv. Math.* **190** (2005), 264–277. Also available as arXiv:math/0212377.

[15] I. B. Frenkel, M. Khovanov, and C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum sl(2) and their tensor products, *Selecta Math.* **12** (2006), 379–431. Also available as arXiv:math/0511467.

[16] R. Gordon, A. J. Powers, and R. Street, Coherence for Tricategories, *Mem. Amer. Math. Soc.* **117**, Providence, 1995.

[17] N. Gurski, An algebraic theory of tricategories, PhD thesis, University of Chicago, June 2006. Available as http://www.math.yale.edu/∼mg622/tricats.pdf

[18] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge U. Press, Cambridge, 1992.

[19] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Oxford U. Press, Oxford, 2002.

[20] V. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* **126** (1987), 335–388.

[21] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347. Also available as arXiv:0803.4121.

[22] M. Kapranov and V. Voevodsky, 2-Categories and Zamolodchikov tetrahedra equations, in *Proc. Symp. Pure Math.* **56** Part 2 (1994), AMS, Providence, 177-260.

[23] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53**, no.2 (1979), 165–184.

[24] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, Cambridge University Press, Cambridge, 1982.

[25] M. Kim, A Lefschetz trace formula for equivariant cohomology, *Ann. Sci. École Norm. Sup.* (4) **28** (1995), no. 6, 669688.

[26] T. Leinster, Basic Bicategories. Available as arXiv:math/9810017.

[27] T. Leinster, The Euler characteristic of a category, *Doc. Math.* **13** (2008), 21-49. Also available as arXiv:math/0610260.

[28] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, Berlin, 1992.

[29] P. McCrudden, Balanced coalgebroids, in *Theory and Applications of Categories* **7**(6), 71-147, 2000.

[30] I. Moerdijk, Toposes and groupoids, in *Categorical algebra and its applications* (Louvain-La-Neuve, 1987), 280–298, Lecture Notes in Math., 1348, Springer, Berlin, 1988.

[31] R. Rouquier, Categorification of the braid groups. Available as arXiv:math/0409593.

[32] W. Soergel, The combinatorics of Harish-Chandra bimodules, *Journal Reine Angew. Math.* **429**, (1992) 49–74.

[33] B. Webster and G. Williamson, A geometric model for Hochschild homology of Soergel bimodules, *Geometry and Topology* **12** (2008), 1243–1263. Also available as arXiv:0707.2003.

[34] A. Weinstein, The Volume of a Differentiable Stack, *proceedings of Poisson 2008 (Lausanne, July 2008)*. Also available as arXiv:0809.2130.