The sum of a finite group of weights of a Hopf algebra

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Reference:

A. Khare, *The sum of a finite group of weights of a Hopf algebra*

math.ucr.edu/∼apoorva/job/hopf-weights.pdf

We work over a ground field $\mathbb{F}$.

Definitions.

A *weight* of an $\mathbb{F}$-algebra A is $\lambda \in \mathrm{Hom}_{\mathbb{F}-alg}(A, \mathbb{F})$.

A *left λ-integral* of A is a λ-weight vector Λ_L^λ in A (under left multiplication).

Example: G a finite group, λ a character $: G \to \mathbb{C}^\times$.

A left (and right) λ-integral is $\Lambda^\lambda := \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \lambda(g^{-1})g$.

The *first orthogonality relation* implies: given $\nu \neq \lambda : G \to \mathbb{C}^\times$, $\nu(\Lambda^\lambda) = 0$.

This holds for *any* $\mathbb{F}$-(Hopf) algebra A, with weights $\nu \neq \lambda$ and left λ-integral Λ_L^λ.

Question. Find a Hopf-version of the second orthogonality relation.

We will only use *one-dimensional* characters - i.e., weights.

The second orthogonality relation "changes" to finding the kernel of a sum of weights.

Setup.

H is an $\mathbb{F}$-Hopf algebra.

Γ_H is its *group of weights* $\mathrm{Hom}_{\mathbb{F}-alg}(H, \mathbb{F})$ (under *convolution* $*$).

$\Pi \subset \Gamma_H$ is a finite subgroup.

$\Sigma_\Pi := \sum_{\gamma\in\Pi} \gamma : H \to \mathbb{F}$ is a functional.

Question. Find $\ker \Sigma_\Pi$.

We will focus on examples where H is generated by skew-primitive and grouplike elements.

Example 1. G is any group; $H = \mathbb{F}G$, its group algebra; $\Pi \subset \Gamma_H$.

Proposition 1 (Second orthogonality relation)**.** *$\Sigma_\Pi(g)$ is $|\Pi|$ if g is killed by Π (i.e., $\Pi(g) = \{1\}$), and 0 otherwise.*

Proof. If $\lambda(g) \neq 1$ for $\lambda \in \Pi$, then

$$\Sigma_\Pi(g) = \lambda * \Sigma_\Pi(g) = \lambda(g)\Sigma_\Pi(g)$$

so $(1 - \lambda(g))\Sigma_\Pi(g) = 0$. □

This helps with calculations in a lot of examples, using the following argument.

Proposition 2. *Suppose $H \subset A$ is a Hopf subalgebra of an $\mathbb{F}$-algebra (so A is an $\operatorname{ad} H$-module). Suppose $V \subset A$ satisfies: $V = \bigoplus_{\lambda \neq \varepsilon} V_\lambda$, and $A = H + AVA$. Then*

(1) *For any weight μ of A, $\mu(V) \equiv 0$.*

(2) $\Gamma_A \subset \Gamma_H$.

Example 2 (Fat quantum groups)**.**

- $\mathbb{F}$ is algebraically closed, $\mathrm{char}(\mathbb{F}) \neq 2, 3$.
- $q \in \mathbb{F}^\times$ is not a root of unity.
- A is a *nonsingular* symmetrizable $I \times I$ GCM.
- G is *any* (fixed) abelian group containing $Q^\vee = \mathbb{Z}^I$, the coroot lattice.
- Extend the characters q^{α_i} to $\nu_i : G \to \mathbb{F}^\times$.

(E.g., G can be a lattice "between" the coroot and coweight lattices.)

Now define the *fat quantum group* $U_q(G, \nu)$ to be generated by $\{e_i, f_i : i \in I\}$ and G, modulo the usual relations, such as: $g e_i g^{-1} = \nu_i(g) e_i$.

For example, $U_q(Q^\vee, \{q^{\alpha_i}\}) = U_q(\mathfrak{g})$.

In general, the representation theory of $U_q(G, \nu)$ should be similar to that of $U_q(\mathfrak{g})$. (See Theorem below.)

Theorem 1.

(1) *$U_q(G,\nu)$ is a Hopf algebra, generated by the group-like G and skew-primitive e_i, f_i.*

(2) *It has a* triangular decomposition*:*

$$U_q := U_q(G,\nu) \cong U_q^- \otimes \mathbb{F}G \otimes U_q^+.$$

(3) *If A is of finite type and $[G : Q^\vee] < \infty$, then the BGG Category $\mathcal{O}$ is finite length and has a central character block decomposition.*

(4) *The set of weights is contained in Γ_G.*

The last part follows by setting $H = \mathbb{F}G$ and $V = span(\{e_i, f_i\})$: $U_q = H + U_q V U_q$.

Example 3 (Hopf regular triangular algebras)**.** For example, $\mathfrak{Ug}$ (with $\mathrm{char}(\mathbb{F}) = 0$), where $\mathfrak{g}$ is a *Lie algebra with regular triangular decomposition*, e.g., a

- semisimple Lie algebra.
- symmetrizable Kac-Moody Lie algebra.
- (centerless) Virasoro algebra.
- extended (centerless) Heisenberg algebra.
- (member of a) special class of Borcherds algebras.

Other examples: Fat quantum groups; (quantized) infinitesimal Hecke algebras over $\mathfrak{sl}_2$; some Crawley-Boevey type algebras.

Properties: Each of these algebras A has a triangular decomposition over a commutative cocommutative Hopf algebra: $A \cong B_- \otimes H \otimes B_+$.

$\mathcal{O}$ over some of these algebras, also has a block decomposition into highest weight categories, regardless of the center.

Proposition 3. $\Gamma_A \subset \Gamma_H$.

Moreover, H is generated by its primitive and grouplike elements in all the above examples.

Example 4 (Other quantizations)**.**
The problem similarly reduces to the grouplike elements, for quantizations of

- affine space (i.e., abelian Lie algebras);
- Borel subalgebras of $\mathfrak{U}\mathfrak{g}$;
- the Virasoro algebra;
- GL_n and SL_n.

Example 5 (Finite-dimensional pointed Hopf algebras)**.**

Suppose H is such an algebra over $\mathbb{C}$, and $G(H)$ has size coprime to $2, 3, 5, 7$. By the Theorem of Andruskiewitsch-Schneider,

Proposition 4. $\Gamma_H \subset \Gamma_{G(H)}$.

Conclusion: In a wide variety of examples, to determine $\Sigma_\prod$ on a spanning set, we only need to compute $\Sigma_\prod(g)$ for $g \in G(H)$.

Skew-primitive elements:
Suppose $\Delta(h) = g \otimes h + h \otimes g'$, where $g, g' \in G(H)$. Let $\Pi \subset \Gamma_H$ be a finite subgroup, and $\mathrm{char}(\mathbb{F}) \neq 2$.

Proposition 5. *Let n be the number of elements from $\{g, g'\}$ that are killed by Π.*
(1) *If $n \neq 1$, $\Sigma_\Pi(h) = 0$.*
(2) *If $n = 1$ and $\lambda(gg') \neq 1$ for some $\lambda \in \Pi$,*

$$\Sigma_\Pi(h) = \frac{\lambda(h)|\Pi|}{1 - \lambda(gg')}$$

Products of skew-primitive elements: These and $G(H)$ span H for many H.

Definition. A skew-primitive h is *pseudo-primitive* (with respect to Π) if $g^{-1}g'$ is killed by Π.

Now suppose $h_1, \ldots, h_n \in H$ are pseudo-primitive with respect to Π. Let

- $\mathbf{h} := \prod_i h_i$;
- $\mathbf{g} := \prod_i g_i$;
- $\Phi := \Pi/[\Pi, \Pi]$;
- Given a prime p, let Φ_p be the p-Sylow subgroup of Φ; and
- Φ'_p be a Hall complement of Φ_p in Φ.

Theorem 2.

(1) *If* $\mathrm{char}(\mathbb{F}) \nmid |\Pi|$, *then* $\Sigma_\Pi(\mathbf{h}) = 0$.

(2) *If* $\mathrm{char}(\mathbb{F}) = p > 0$ *divides* $|\Pi|$, *then*

$$\Sigma_\Pi(\mathbf{h}) = |[\Pi, \Pi]| \cdot \Sigma_{\Phi'_p}(\mathbf{g}) \cdot \Sigma_{\Phi_p}(\mathbf{h})$$

(3) *If* $\Sigma_{\Phi_p}(\mathbf{h}) \neq 0$, *then* $(p-1)|n$, *and* $\Phi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$ *for some* $0 \leq k \leq n/(p-1)$. *Moreover, it can assume any value in* $\mathbb{F}$ *(for an abelian Lie algebra).*

Skew-primitive elements: Nothing in general.

Proposition 6. *There exists a Hopf algebra* $H = H(\mathbb{F})$ *with skew-primitive elements* $h_1, \ldots, h_n \in H$*, such that:*

Given any $r \in \mathbb{F}$ *and an abelian group* G *of even order, there exists a subgroup* $\Pi \cong G$ *of* Γ_H*, such that* $\Sigma_\Pi(\mathbf{h}) = r$*.*

This can be done over any field $\mathbb{F}$ such that

- $\mathrm{char}(\mathbb{F}) \gg 0$.
- There exists a primitive $\exp(G)$th root of unity in $\mathbb{F}$.