How to Measure a Groupoid

First Steps towards Topological Groupoidification

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Counterrevolutionaries

Degroupoidification

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	$\dashrightarrow$	
discrete groupoid		*vector space*

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discrete groupoid $\dashrightarrow$ *vector space*

where $\underline{X} = X^{(0)}/X$ is the orbit space of X.

$$X \xleftarrow{q} S \xrightarrow{p} Y \quad \dashrightarrow \quad \widetilde{S} : \mathbb{R}^{\underline{Y}} \longrightarrow \mathbb{R}^{\underline{X}}$$

span of groupoids $\dashrightarrow$ *linear operator*

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1. Start with a specific algebraic structure.
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** *"Higher-Dimensional Algebra VII: Groupoidification"* by John Baez, Alex Hoffnung and Christopher Walker. arXiv:0908.4305v1

Goal

Extend degroupoidification from the realm of discrete groupoids to the topological and measure theoretic setting (with groupoidification of structures like operator algebras in mind).

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inverse: $(i,t,j)^{-1} = (j,t,i)$
range: $r(i,t,j) = (i,t,i)$
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Suggests richness when considering topology: as a groupoid $\underline{G}$ is always cotrivial, but as a topological space it can be any $\mathcal{T}$ above.

Groupoid Cardinality

A key notion in the discrete setting.

Definition (Baez and Dolan, 2001)

$$||X|| = \sum_{[u]\in \underline{X}} \frac{1}{|Aut(u)|}$$

where $Aut(u) = X_u^u$ is the automorphism (or isotropy) group of a unit $u \in X^{(0)}$, which is assumed to be finite for every u.
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Example:
Let Γ be a finite group acting on a finite set S, and let $X = S \times \Gamma$ denote the corresponding transformation groupoid.

$$||X|| = \frac{|S|}{|\Gamma|}$$

Motivating example for Baez and Dolan.

Example:

Let E be the groupoid of bijections of finite sets.

$$||E|| = \sum_{n \in \mathbb{N}} \frac{1}{|S_n|} = \sum_{n \in \mathbb{N}} \frac{1}{n!} = e$$

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Remark:

Equivalent formula for groupoid cardinality:

$$||X|| = \sum_{u \in X^{(0)}} \frac{1}{|X^u|}$$

where $X^u = r^{-1}(u)$ is assumed to be finite for every u.

Groupoid Measure

Let G be a topological groupoid that admits a continuous left Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$, and let μ be a (Radon) measure on $G^{(0)}$. Assume that each λ^u is finite and that $u \mapsto (\int_{G^u} 1 d\lambda^u)^{-1}$ is μ-measurable.

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* Alex, please don't ask what is a stack.

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Let Γ be a locally compact group acting on a locally compact Hausdorff space X, and let $G = X \times \Gamma$ be the corresponding transformation groupoid. Note that $G^{(0)} \cong X$. Let μ be a (Radon) measure on X and let λ_Γ be the left Haar measure of Γ.

λ_Γ induces a left Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$ on G given by $\lambda^u = \lambda^{(x,e)} = \delta_x \times \lambda_\Gamma$, where δ_x is the point mass at $x \in X$.

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Proposition

If Γ is compact and μ is finite then:

$$|| \, G \, ||_{\lambda,\mu} = \frac{\mu(X)}{\lambda_\Gamma(\Gamma)}$$

Example

Recall the open cover groupoid $\mathcal{G}$. Take μ to be the Lebesgue measure on $\mathcal{T}$. μ induces a measure μ^0 on $\mathcal{G}^{(0)} = \coprod_{i=1}^{n} U_i$, defined by

$$\mu^0(S) = \sum_{i=1}^{n} \mu(S \cap U_i), \quad S \subseteq \mathcal{G}^{(0)}.$$

Let $\lambda = \{\lambda^u\}$ be the counting Haar system on the étale groupoid $\mathcal{G}$. Observe that for any $u \in \mathcal{G}^{(0)}$, the set $\mathcal{G}^u$ is nonempty and finite, and moreover

$$\left(\int_{\mathcal{G}^u} 1 d\lambda^u\right)^{-1} = \frac{1}{|\mathcal{G}^u|}.$$

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$$||\,\mathcal{G}\,||_{\lambda,\,\mu^0} = \mu(\mathcal{T}).$$

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Idea of proof - by example:

$\mathcal{T} = [0,1] \times [0,1]$

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$$\begin{aligned} ||\,\mathcal{G}\,||_{\lambda,\,\mu^0} &= \int_{G^{(0)}} \left(\int_{G^u} 1 d\lambda^u \right)^{-1} d\mu \\ &= \int_{G^{(0)}} \frac{1}{|\mathcal{G}^u|} d\mu \\ &= 4\left((1-\alpha)^2 \cdot \frac{1}{1} + 2(\alpha-(1-\alpha))\cdot(1-\alpha)\cdot\frac{1}{2} + (\alpha-(1-\alpha))^2\cdot\frac{1}{4} \right) \\ &= 1 \end{aligned}$$

The space $\mathbb{R}^{\underline{X}}$

Recall that in the discrete case:

$\underline{X}$	$\dashrightarrow$	$\mathbb{R}^{\underline{X}}$
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where $\widetilde{\Psi}([u]) := ||p^{-1}([u])||$

The space $M(\underline{G})$

In the topological case:

$$\begin{array}{ccc} G & & M(\underline{G}) \\ & \dashrightarrow & \\ \textit{topological groupoid} & & \textit{space of measures} \end{array}$$

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What is $\tilde{p}$?

Let $p : H \longrightarrow G$ be a continuous groupoid homomorphism. Assume H admits a Haar system η and a measure ν on $H^{(0)}$.
Denote by $\pi : G \longrightarrow \underline{G}$ the continuous map $x \longmapsto [r(x)]$. For every Borel subset S of $\underline{G}$, $\hat{S} = p^{-1}(\pi^{-1}(S))$ is a subgroupoid of H.

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Example:

For the open cover groupoid $\mathcal{G}$, we recover the original measure μ on $\underline{G} = \mathcal{T}$ from the identity homomorphism.

$$id : \mathcal{G} \longrightarrow \mathcal{G} \quad \dashrightarrow \quad \widetilde{id}_{\mathcal{G}} = \mu$$

Spans

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In progress.

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But let me tell you a bit about our progress.

Recall that:

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X & & Y \end{array} \quad \dashrightarrow \quad \widetilde{S} : \mathbb{R}^{\underline{Y}} \longrightarrow \mathbb{R}^{\underline{X}}$$

span of groupoids — *linear operator*

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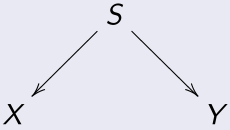
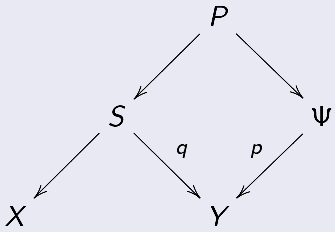


span of groupoids $\dashrightarrow$ $\widetilde{S} : \mathbb{R}^{\underline{Y}} \longrightarrow \mathbb{R}^{\underline{X}}$ *linear operator*

There is an explicit formula for $\widetilde{S}$.

In particular, the operator $\widetilde{S}$ is applied to the vector in $\mathbb{R}^{\underline{Y}}$ corresponding to $\Psi \to Y$ via a certain "weak pullback" construction, as in the following diagram:



Where $P = \{(s, y, \psi) \mid r(q(s)) = r(y) \;\; and \;\; r(p(\psi)) = d(y)\}$.

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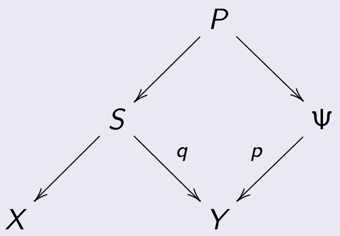

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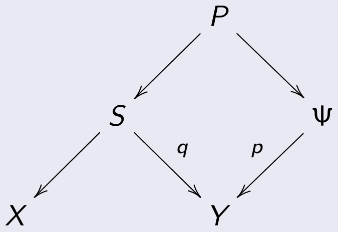

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Composition of spans is also achieved via weak pullbacks.

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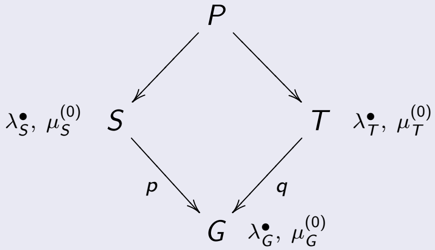

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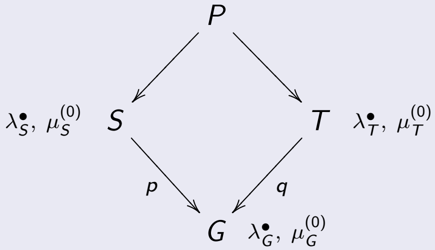

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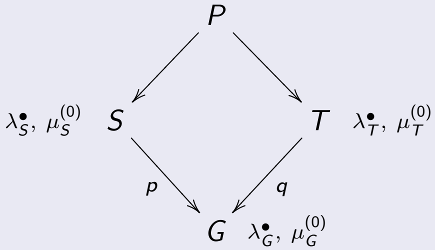

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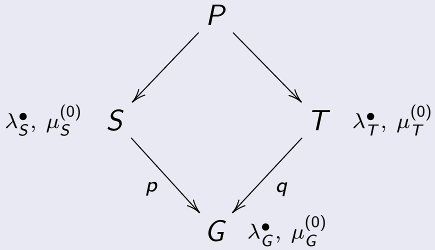

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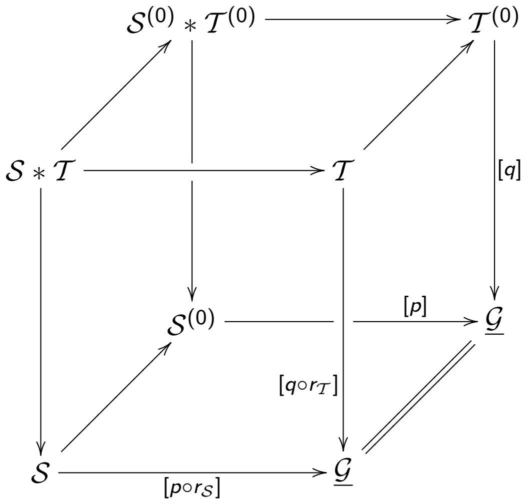

$\mathcal{S}^{(0)} * \mathcal{T}^{(0)}$
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$\mathcal{T}$
$[q]$
$\mathcal{S}^{(0)}$
$[p]$
$\underline{\mathcal{G}}$
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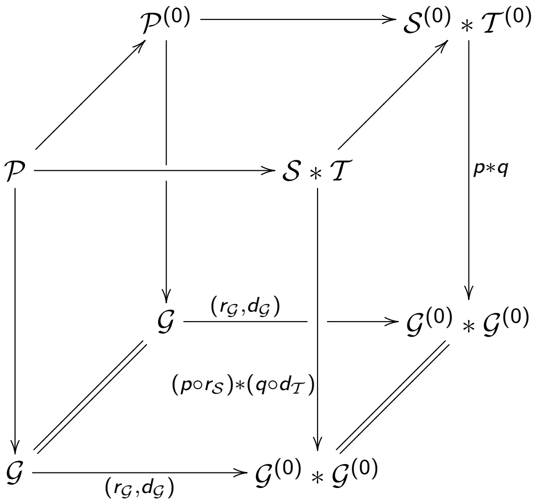

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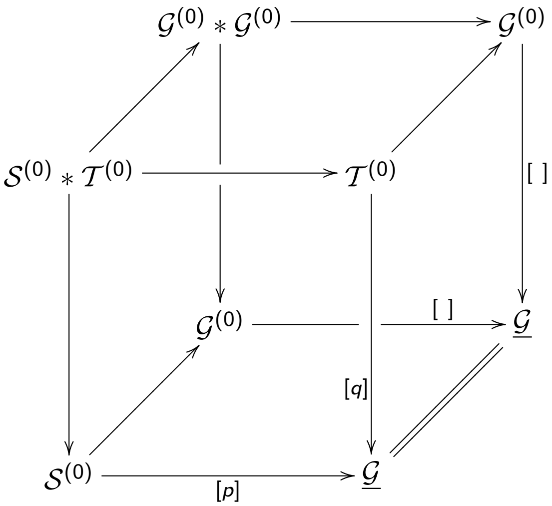
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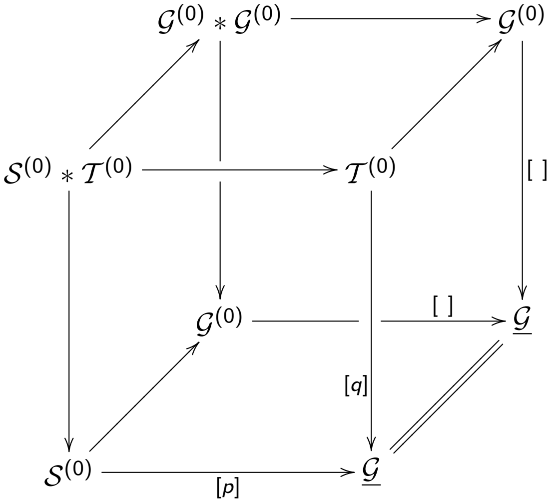
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And a measure on $P^{(0)}$ defined on $\Sigma \subseteq P^{(0)}$ by:

$$\mu_P^{(0)}(\Sigma) := \int_G \eta^x(\Sigma) d\mu_G(x)$$

where

$$\mu_G(E) = \int_{G^{(0)}} \lambda_G^u(E) d\mu_G^{(0)}(u)$$

and η^x is a system of measures corresponding to the projection $\pi_G : P^{(0)} \to G$, given by:

$$\eta^x = \gamma_p^{r(x)} \times \delta_x \times \gamma_q^{d(x)}$$

Thank you!