Towards Topological Groupoidification

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Counterrevolutionaries

Degroupoidification

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discrete groupoid		*vector space*

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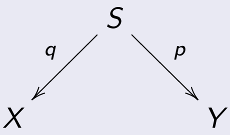

$\dashrightarrow$ $\widetilde{S} : \mathbb{R}^{\underline{Y}} \longrightarrow \mathbb{R}^{\underline{X}}$

span of groupoids — *linear operator*

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** *"Higher-Dimensional Algebra VII: Groupoidification"* by John Baez, Alex Hoffnung and Christopher Walker. arXiv:0908.4305v1

Goal

Extend degroupoidification from the realm of discrete groupoids to the topological and measure theoretic setting (with groupoidification of structures like operator algebras in mind).

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Suggests richness when considering topology: as a groupoid $\underline{G}$ is always cotrivial, but as a topological space it can be any $\mathcal{T}$ above.

Groupoid Cardinality

A key notion in the discrete setting.

Definition (Baez and Dolan, 2001)

$$||X|| = \sum_{[u]\in \underline{X}} \frac{1}{|Aut(u)|}$$

where $Aut(u) = X_u^u$ is the automorphism (or isotropy) group of a unit $u \in X^{(0)}$, which is assumed to be finite for every u.
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Example:
Let Γ be a finite group acting on a finite set S, and let $X = S \times \Gamma$ denote the corresponding transformation groupoid.

$$||X|| = \frac{|S|}{|\Gamma|}$$

Motivating example for Baez and Dolan.

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Remark:

Equivalent formula for groupoid cardinality:

$$||X|| = \sum_{u \in X^{(0)}} \frac{1}{|X^u|}$$

where $X^u = r^{-1}(u)$ is assumed to be finite for every u.

Groupoid Measure

Let G be a topological groupoid that admits a continuous left Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$, and let μ be a (Radon) measure on $G^{(0)}$. Assume that each λ^u is finite.

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* Please don't ask what is a stack.

Example

Let Γ be a locally compact group acting on a locally compact Hausdorff space X, and let $G = X \times \Gamma$ be the corresponding transformation groupoid. Note that $G^{(0)} \cong X$. Let μ be a (Radon) measure on X and let λ_Γ be the left Haar measure of Γ.

λ_Γ induces a left Haar system $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$ on G given by $\lambda^u = \lambda^{(x,e)} = \delta_x \times \lambda_\Gamma$, where δ_x is the point mass at $x \in X$.

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Proposition

If Γ is compact and μ is finite then:

$$\| G \|_{\lambda, \mu} = \frac{\mu(X)}{\lambda_\Gamma(\Gamma)}$$

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Let $\lambda = \{\lambda^u\}$ be the counting Haar system on the étale groupoid $\mathcal{G}$. Observe that for any $u \in \mathcal{G}^{(0)}$, the set $\mathcal{G}^u$ is nonempty and finite, and moreover

$$\left(\int_{\mathcal{G}^u} 1 d\lambda^u\right)^{-1} = \frac{1}{|\mathcal{G}^u|}.$$

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where $\widetilde{\Psi}([u]) := ||p^{-1}([u])||$

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What is $\tilde{p}$?

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Example:

For the open cover groupoid $\mathcal{G}$, we recover the original measure μ on $\underline{\mathcal{G}} \cong \mathcal{T}$ from the identity homomorphism.

$$id : \mathcal{G} \longrightarrow \mathcal{G} \quad \dashrightarrow \quad \widetilde{id}_{\mathcal{G}} = \mu$$

Spans

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But let me tell you a bit about our progress.

Recall that:

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X & & Y \end{array} \quad \dashrightarrow \quad \widetilde{S} : \mathbb{R}^{\underline{Y}} \longrightarrow \mathbb{R}^{\underline{X}}$$

span of groupoids → *linear operator*

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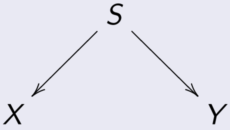

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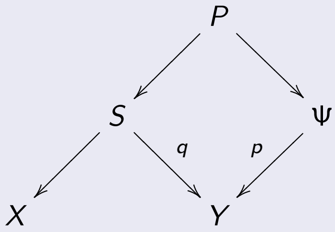
span of groupoids $\dashrightarrow$ *linear operator*

There is an explicit formula for $\widetilde{S}$.

In particular, the operator $\widetilde{S}$ is applied to the vector in $\mathbb{R}^{\underline{Y}}$ corresponding to $\Psi \to Y$ via a certain "weak pullback" construction, as in the following diagram:



Where $P = \{(s, y, \psi) \mid r(q(s)) = r(y) \;\; and \;\; r(p(\psi)) = d(y)\}$.

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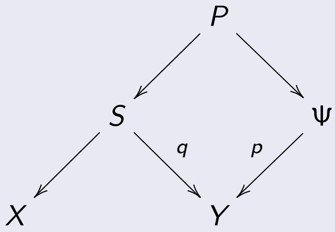

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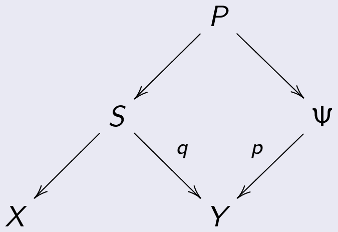

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Composition of spans is also achieved via weak pullbacks.

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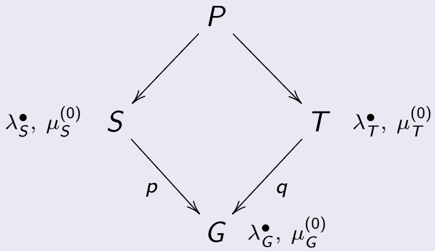

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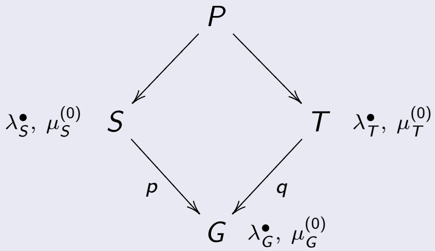

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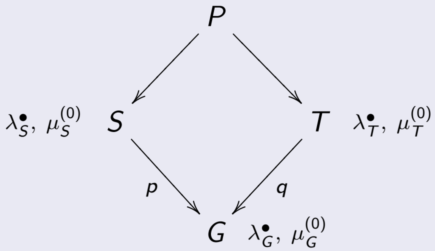

We need to define a Haar system $\lambda^\bullet_P$ for P, and a measure $\mu^{(0)}_P$ on $P^{(0)}$.

This we now (believe we) know how to do. It requires measure theoretic tools which we develop in arXiv:1004.3750v1.

BSMs and CSMs

We need to be able to carry out certain operations involving Borel and continuous systems of measures.

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Composition

$$X \xrightarrow[\alpha^\bullet]{p} Y \xrightarrow[\beta^\bullet]{q} Z$$

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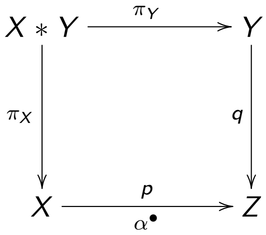
The composition $(\beta \circ \alpha)^\bullet$ is defined for any Borel set $E \subseteq X$ by

$$(\beta \circ \alpha)^z(E) = \int_Y \alpha^y(E)\ d\beta^z(y)$$

Lifting

Producing a system of measures $(q^*\alpha)^\bullet$ on π_Y in the following pull-back diagram:



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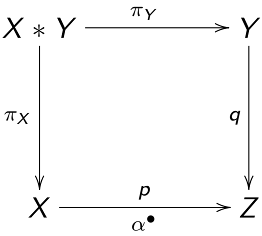

The lifting is given by

$$(q^*\alpha)^y = \alpha^{q(y)} \times \delta_y$$

Fibred Product

A system of measures $(\gamma_X * \gamma_Y)^\bullet$ on the map $f * g$ in the following diagram:

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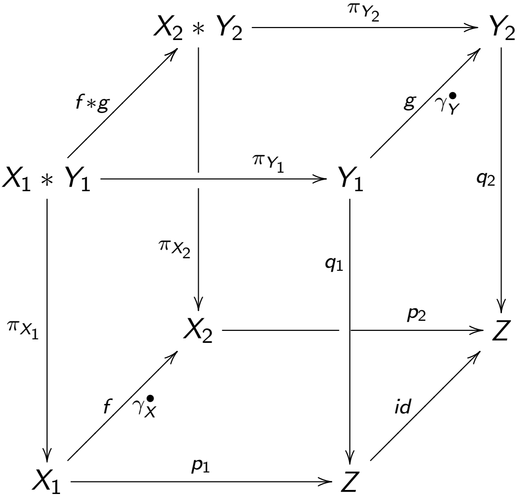

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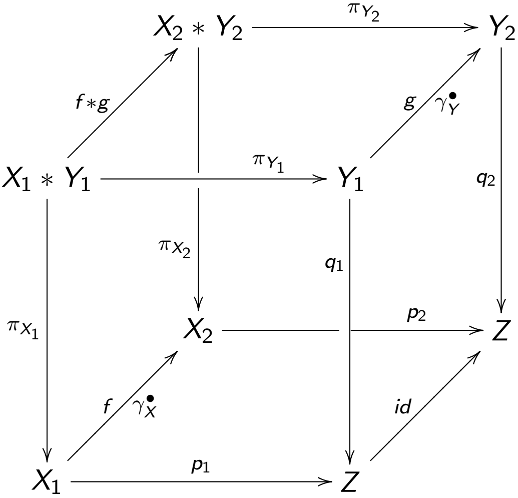

The fibred product is defined by $(\gamma_X * \gamma_Y)^{(x_2,y_2)} = \gamma_X^{x_2} \times \gamma_Y^{y_2}$.

Disintegration

If (X, μ) and (Y, ν) are measure spaces, and $f : X \to Y$ is a Borel map, then a system of measures $\gamma^\bullet$ on f is a disintegration of μ with respect to ν if $\mu(E) = \int_Y \gamma^y(E) d\nu(y)$ for every Borel set $E \subseteq X$.

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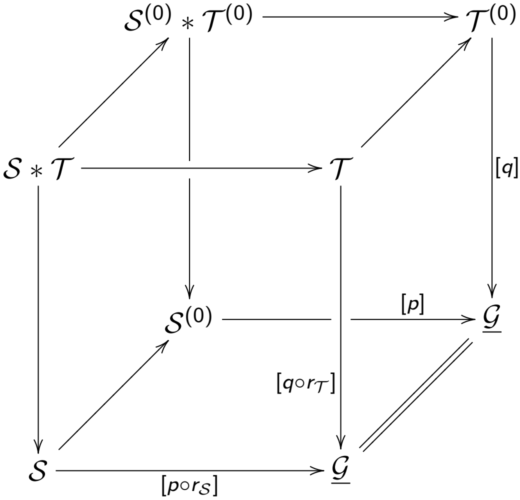
With these tools, and diagrams like:


$\mathcal{S}^{(0)} * \mathcal{T}^{(0)}$
$\mathcal{T}^{(0)}$
$\mathcal{S} * \mathcal{T}$
$\mathcal{T}$
$[q]$
$\mathcal{S}^{(0)}$
$[p]$
$\underline{\mathcal{G}}$
$[q \circ r_{\mathcal{T}}]$
$\mathcal{S}$
$\underline{\mathcal{G}}$
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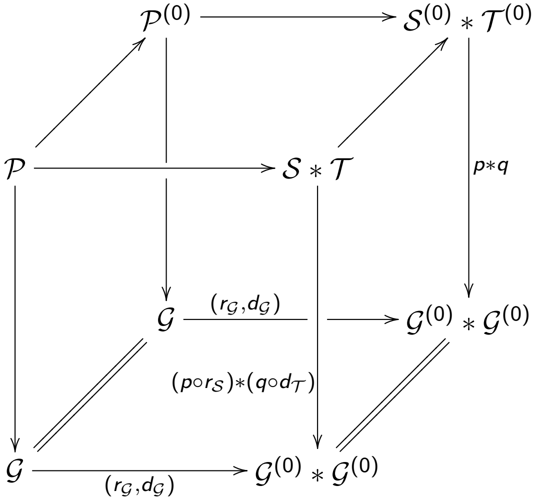
$\mathcal{P}^{(0)}$
$\mathcal{S}^{(0)} * \mathcal{T}^{(0)}$
$\mathcal{P}$
$\mathcal{S} * \mathcal{T}$
$p*q$
$\mathcal{G}$
$(r_{\mathcal{G}}, d_{\mathcal{G}})$
$\mathcal{G}^{(0)} * \mathcal{G}^{(0)}$
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$\mathcal{G}$
$\mathcal{G}^{(0)} * \mathcal{G}^{(0)}$
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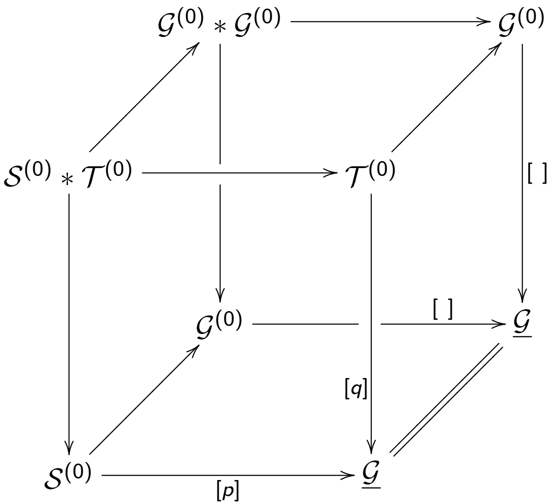
$\mathcal{G}^{(0)} * \mathcal{G}^{(0)}$
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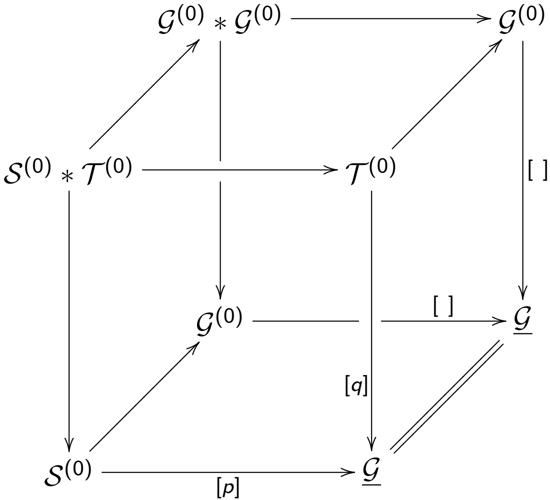
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We eventually construct the following Haar system on P:

$$u = (s, x, t) \in P^{(0)} \quad \Rightarrow \quad \lambda_P^u = \lambda_P^{(s,x,t)} = \lambda_S^s \times \delta_x \times \lambda_T^t$$

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And a measure on $P^{(0)}$ defined on $\Sigma \subseteq P^{(0)}$ by:

$$\mu_P^{(0)}(\Sigma) := \int_G \eta^x(\Sigma) d\mu_G(x)$$

where

$$\mu_G(E) = \int_{G^{(0)}} \lambda_G^u(E) d\mu_G^{(0)}(u)$$

and η^x is a system of measures corresponding to the projection $\pi_G : P^{(0)} \to G$, given by:

$$\eta^x = \gamma_p^{r(x)} \times \delta_x \times \gamma_q^{d(x)}$$

Thank you!