Haagerup's Property versus Relative Property (T)

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Haagerup's approximation property for groups

Γ-countable discrete group.

Definition

Γ is called Haagerup if one of the following equivalent properties is satisfied:

1. $\exists\ \phi_n \in c_o(\Gamma)$ positive definite functions such that $\lim_{n\to\infty} \phi_n(g) = 1,\ \forall g \in \Gamma$
2. $\exists\ \pi : \Gamma \to \mathcal{U}(\mathcal{H})$ unitary c_0-representation which contains almost invariant vectors

$$(\exists \xi_n \in \mathcal{H}_1 \text{ s. t. } \lim_{n\to\infty} \|\pi(\gamma)\xi_n - \xi_n\| = 0, \forall \gamma \in \Gamma (\Leftrightarrow 1_\Gamma \prec \pi))$$

3. There exists a proper conditionally negative definite function on Γ.
4. There exists a proper 1-cocycle on a c_0-representation of Γ.

Haagerup approximation property for groups

Examples:

- Amenable groups, free groups, $SL_2(\mathbb{Z})$, etc
- Γ_1, Γ_2 is Haagerup $\Rightarrow \Gamma_1 \times \Gamma_2$, $\Gamma_1 * \Gamma_2$ is Haagerup.
- Γ_1 is Haagerup and $\Gamma_1 \subset \Gamma_2$ coamenable $\Rightarrow \Gamma_2$ is Haagerup.
- Γ_1, Γ_2 are Haagerup $\Rightarrow \Gamma_1 \wr \Gamma_2$ is Haagerup (Cornulier-Stalder-Valette, '09)

Haagerup approximation property for von Neumann algebras

Definition

(M, τ) has Haagerup property if there exists a sequence $\phi_n : M \rightarrow M$ of subtracial, normal, completely positive maps such that $\phi_n \in \mathcal{K}(L^2(M))$ and $\lim_{n\rightarrow\infty} \|\phi_n(x) - x\|_2 = 0, \forall x \in M$.

Facts:

- $L\Gamma$ has Haagerup property $\Leftrightarrow$ Γ is Haagerup (Choda '83).
- The class of von Neumann algebras with Haagerup property is closed under free products, tensor products, coamenable extensions (Bannon-Fang '08).

Relative Property (T) of Kazhdan-Margulis for groups

Definition

An inclusion of groups ($\Gamma_0 \subset \Gamma$) is said to have relative property (T) of Kazhdan-Margulis if there exists finite set $F \in \Gamma$, $\delta > 0$ such that every unitary rep. $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ that satisfies $\|\pi(g)\xi - \xi\| < \delta, \forall g \in F$ for a unit vector $\xi \in \mathcal{H}$ must admit a Γ_0-invariant vector $\xi_0 \in \mathcal{H}$.

Remark

Equivalently (Jolissaint), ($\Gamma_0 \subset \Gamma$) have relative property (T) if for every $\varepsilon > 0$ there exists finite set $F \subset \Gamma, \delta > 0$ such that whenever ϕ is a p. d. function on Γ such that $|\phi(g) - 1| \leq \delta, \forall g \in F$ then $|\phi(g) - 1| \leq \varepsilon, \forall g \in \Gamma_0$

Examples:

1. $(SL_n(\mathbb{Z}) \subset SL_n(\mathbb{Z}))$, $n \geq 3$.
2. $(\mathbb{Z}^n \subset \mathbb{Z}^n \rtimes SL_n(\mathbb{Z}))$, $n \geq 2$
3. $(\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma)$ for every $\Gamma \subset SL_2(\mathbb{Z})$ nonamenable (Burger).

Relative Property (T) of Kazhdan-Margulis for von Neumann algebras

Definition (Popa)

(M, τ) von Neumann algebra $(B \subset M)$ has relative property (T) if for $\forall \varepsilon > 0$ $\exists F \subset M$ and $\delta > 0$ s.t. if $(\mathcal{H}, \xi)$ is a pointed Hilbert M-bimodule with ξ tracial vector and $\|x\xi - \xi x\| \leq \delta$ $\forall x \in F$ then there exists a B-central vector $\xi' \in \mathcal{H}$ s.t. $\|\xi - \xi'\| \leq \varepsilon$.

- $(\Gamma_0 \subset \Gamma)$ has relative property (T) iff $(L\Gamma_0 \subset L\Gamma)$ has rel. (T) (Connes-Jones, Popa)
- For every nonamenable subfactor $L(\mathbb{Z}^2) \subset N \subseteq L(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}))$ one has $(L(\mathbb{Z}^2) \subset N)$ has rel. (T) (Ioana, '08)

Question (Cherix-Cowling-Jolissaint-Julg-Valette, '01)

Is the presence of a infinite subgroup $\Gamma_0 \subset \Gamma$ with the relative property (T) the only obstruction for the Haagerup property of Γ?

Answer: No !- Y. de Cornulier, '05.
Counterexample:

- $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$, α not a cube.
- Γ_α doesn't have any infinite subgroup with rel. (T) but it has a infinite subset ($X_\alpha \subset \Gamma_\alpha$) with rel. (T). Specifically,

$$X_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \cap \mathcal{B},\ \mathcal{B} \text{ unit ball of } \mathbb{R}^3$$

von Neumann algebraic context

Observation

If M has Haagerup property then $\nexists$ diffuse $B \subset M$ rel. (T).

Proof.

B diffuse $\exists u_n \in \mathcal{U}(B)$ s.t. $u_n \to 0$ weakly. M Haagerup $\exists \phi_k$ compact maps on M

$$\|\phi_k(u_n) - u_n\|_2 \leq \varepsilon_k \to 0 \text{ when } k \to \infty, \ \forall n \in \mathbb{N}$$

$$\|\phi_k(u_n)\|_2 \to 0 \text{ when } n \to \infty, \ \forall k \in \mathbb{N}$$

Therefore $1 - \|\phi_k(u_n)\|_2 \leq \|\phi_k(u_n) - u_n\|_2 \leq \varepsilon_k$Contradiction! □

Question (von Neumann algebraic version of CCJJV)

Is the presence of a diffuse subalgebra $N \subset M$ with the relative property (T) the only obstruction for the Haagerup property of M?

De Cornulier's example in the von Neumann algebraic context

Theorem (C-Ioana, '09)

If $\Gamma_\alpha = \mathbb{Z}[\sqrt[3]{\alpha}]^3 \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$ for α not a cube, there exists a diffuse von Neumann subalgebra $B \subset L\mathbb{Z}[\sqrt[3]{\alpha}]^3 \subset L\Gamma_\alpha$ such that $B \subset L\Gamma_\alpha$ has rel (T) and $B' \cap L\Gamma_\alpha = L\mathbb{Z}[\sqrt[3]{\alpha}]^3$.

- For $(j,k) \in S = \{1,2,3\} \times \{0,1,2\}$ let $\mathbb{Z}[\sqrt[3]{\alpha}]^3 = \oplus \mathbb{Z} v_{j,k}$ for $v_{j,k} = (\alpha^{\frac{k}{3}} \delta_{i,j})_{1 \leq i \leq 3}$, $\mathbb{Z}[\sqrt[3]{\alpha}]^3 \cong \mathbb{Z}^9$.
- $\mathbb{R}^9/\mathbb{Z}^9 = \mathbb{T}^9 = \hat{\mathbb{Z}}^9 \cong \mathbb{Z}[\widehat{\sqrt[3]{\alpha}}]^3$
- $p : \mathbb{R}^3 \to \mathbb{R}^9$, $p(a) = (\langle a, v_{j,k} \rangle)_{(j,k) \in S}$, $\pi : \mathbb{R}^9 \to \mathbb{R}^9/p(\mathbb{R}^3)$
- $i : \mathbb{T}^9 \to [-\frac{1}{2}, \frac{1}{2})^9 \subset \mathbb{R}^9$, $i(x + \mathbb{Z}^9) = x + \mathbb{Z}^9 \cap [-\frac{1}{2}, \frac{1}{2})^9$
- $q = \pi \circ i : \mathbb{T}^9 \to \mathbb{R}^9/p(\mathbb{R}^3)$, $(Y, \nu) = (q(\mathbb{T}^9), q_* \lambda^9)$
- $L^\infty(Y, \nu) \subset L^\infty(\mathbb{T}^9, \lambda^9)$ via $L^\infty(Y, \nu) f \to f \circ q \in L^\infty(\mathbb{T}^9, \lambda^9)$.
- $L^\infty(Y, \nu) \subset L^\infty(\mathbb{T}^9, \lambda^9) \rtimes SO_3(\mathbb{Z}[\sqrt[3]{\alpha}])$ rel (T).

Generalized wreath product groups

Let A, Γ groups; X Γ-set.
Denote by $A^X = \oplus_X A$ and define $\rho : \Gamma \to Aut(A^X)$ action by automorphisms

$$\rho_g((a_x)_x) = (a_{g^{-1}x})_x, \quad g \in \Gamma$$

Then $A \wr_X \Gamma := A^X \rtimes_\rho \Gamma$.

Notation

$\Gamma_0 = \Gamma/\Lambda$ *then* $A \wr_{\Gamma_0} \Gamma = A \wr_X \Gamma$ *where* $X = \{g_\iota \Lambda\}_\iota$ *on which* Γ *acts by left multiplication.*

Theorem

- *A, Γ-(nontrivial) Haagerup groups.*
- *Γ_0-quotient of Γ which is not Haagerup.*

Then

- *$L(A \wr_{\Gamma_o} \Gamma)$ does not have Haagerup property (C-Ioana, '09)*
- *$L(A \wr_{\Gamma_o} \Gamma)$ does not admit any diffuse von Neumann subalgebra with relative property (T) (Popa '04, Ioana '05)*

Proof.

$$\text{If } B \subset L(A \wr_X \Gamma) =: M \text{ is diffuse with rel. (T)}$$

$$\Downarrow \text{ [Popa, Ioana]}$$

$$B \prec_M L\Gamma \text{ or } B \prec_M LA^F, F \subset X \text{ finite subset.}$$

Γ, A Haagerup $\Rightarrow$ Contradiction. □

Sketch of the proof

Theorem (C-Ioana, 2009)

- *A abelian, Γ, Γ_0 a quotient of Γ ($p : \Gamma \to \Gamma_0$ surj. hom.)*
- *$\rho : \Gamma_0 \to Aut(A)$ an action by automorphisms of A*

Define $\tilde{\rho} = \rho \circ p : \Gamma \to Aut(A)$. Then we have:

1. *If there exists $a \in A$ such that its stabilizer in Γ_0 is finite then*

$$\Gamma_0 \text{ not Haagerup } \Rightarrow A \rtimes_{\tilde{\rho}} \Gamma \text{ not Haagerup.}$$

2. *Suppose that $(X \subset \Gamma_0)$ has relative (T). If for every $a \in A$ one denote by $X_a = \{\rho_\gamma(a) | \gamma \in X\} \subset A$ then $(X_a \subset A \rtimes_{\tilde{\rho}} \Gamma)$ has relative property (T).*

Corollary

Let A, Γ, Γ_0 such that Γ_0 is a quotient of Γ. Then:

$$A \wr_{\Gamma_0} \Gamma \text{ has Haagerup} \Leftrightarrow A, \Gamma, \Gamma_0 \text{ have Haagerup.}$$

"$\Leftarrow$"- De Cornulier-Stalder-Valette '09.
"$\Rightarrow$"- immediate consequence of the previous theorem.

Remark

"$\Rightarrow$" disproves a conjecture of De Cornulier-Stalder-Valette which asserts that every generalized wreath product of two Haagerup groups is Haagerup.

Open Question

Find necessary and sufficient conditions on A, Γ and Γ-set X such that $A \wr_X \Gamma$ is Haagerup!

Important ingredient used in the proof:

Proposition (Peterson, '09)

Γ *is not Haagerup iff for* $\forall$ *sequence* $(\phi_n)_n$ *p. d. maps with* $\lim_{n\to\infty}\phi_n(g) = 1, \forall g \in \Gamma$ $\exists$ *infinite subset* $\mathcal{X} \subset \Gamma$ *and a infinite sequence* $(n_k)_k \in \mathbb{N}$ *such that* $lim_{k\to\infty} sup_{g\in\mathcal{X}} |\phi_{n_k}(g) - 1| = 0$

$A \rtimes_{\tilde{\rho}} \Gamma$ is Haagerup $\Rightarrow \pi : A \rtimes_{\tilde{\rho}} \Gamma \rightarrow \mathcal{H}$ unitary c_0-rep and $\xi_n \in \mathcal{H}$ s. t.

$$\lim_{n\rightarrow\infty} \|\pi(h)(\xi_n) - \xi_n\| = 0, \forall h \in A \rtimes_{\tilde{\rho}} \Gamma$$

$\exists \mu_n \in \mathcal{M}(\hat{A})$ such that $\langle \pi(a)\xi_n, \xi_n\rangle = \int_{\hat{A}} a d\mu_n, \ \forall a \in A.$
For every $\gamma \in \Gamma_0$ let $\tilde{\gamma} \in \Gamma$ s. t. $p(\tilde{\gamma}) = \gamma$

$$\|\gamma_*\mu_n - \mu_n\| = \|\tilde{\gamma}_*\mu_n - \mu_n\| \leq 2\|\pi(\tilde{\gamma})\xi_n - \xi_n\| \ (1)$$

Let $\{\gamma_i\} = \Gamma_0 \setminus \{e\}$ and define Γ_0-quasinvariant measure by:

$$\nu_n = (1 - \frac{1}{2^n})\mu_n + \sum_{i=1,\infty} \frac{1}{2^{i+n}}(\gamma_i)_*\mu_n$$

One has $\|\nu_n - \mu_n\| \leq \frac{1}{2^{n-1}}$ and therefore by (1)

$$\|\gamma_*\nu_n - \nu_n\| \leq \frac{1}{2^{n-2}} + 2\|\pi(\tilde{\gamma})\xi_n - \xi_n\|$$

Let $g_\gamma = (\frac{d\gamma_* \nu_n}{d\nu_n})^{\frac{1}{2}} \in L^2(\hat{A}, \nu_n)$ and define $\sigma_n : \Gamma_0 \to \mathcal{U}(L^2(\hat{A}, \nu_n))$ a unitary rep. by the formula

$$\sigma_n(\gamma)(f) = g_\gamma(f \circ \gamma^{-1})$$

If $\eta_n = 1_{\hat{A}} \in L^2(\hat{A}, \nu_n)$ a calculation shows that

$$\frac{1}{2}\|\gamma_*\nu_n - \nu_n\| \leq \|\sigma_n(\gamma)(\eta_n) - \eta_n\| \leq \|\gamma_*\nu_n - \nu_n\|^{\frac{1}{2}} \forall \gamma \in \Gamma_0$$

Assume by contradiction Γ_0 is not Haagerup. By Peterson's result $\exists$ an increasing sequence $(n_k)_k$ and $\mathcal{F} \subset \Gamma$ infinite subset s. t.

$$\lim_{k\to\infty} \sup_{\gamma \in \mathcal{F}} \|\sigma_{n_k}(\gamma)(\eta_{n_k}) - \eta_{n_k}\| = 0$$

Calculations similar to the above imply:

$$\|\gamma_*\mu_{n_k} - \mu_{n_k}\| \leq \frac{1}{2^{n_k - 1}} + 2\|\sigma_{n_k}(\gamma)(\eta_{n_k}) - \eta_{n_k}\|.$$

Hence $|\langle\pi(\rho_\gamma(a))\xi_{n_k},\xi_{n_k}\rangle - \langle\pi(a)\xi_{n_k},\xi_{n_k}\rangle| =$
$|\int \rho_\gamma(a)d\mu_{n_k} - \int a d\mu_{n_k}| \leq \|\gamma_*\mu_{n_k} - \mu_{n_k}\| \to 0$ when $k \to \infty$
Therefore

$$\|\pi(\rho_\gamma(a))\xi_{n_k} - \xi_{n_k}\|^2 = \|\pi(a)\xi_{n_k} - \xi_{n_k}\|^2 + 2(\frac{1}{2^{n_k-1}} + 2\varepsilon_{n_k}),\ \forall\gamma\in\mathcal{F}$$

and by the above we get

$$\lim_{k\to\infty}\sup_{\gamma\in\mathcal{F}}\|\pi(\rho_\gamma(a))\xi_{n_k} - \xi_{n_k}\| = 0,\ \forall\gamma\in\mathcal{F}.$$

Since $Stab_{\Gamma_0}(a)$ is finite and π is c_0-rep $\Rightarrow$
$\lim_{\gamma\to\infty}\langle\pi(\rho_\gamma(a))\xi_{n_k},\xi_{n_k}\rangle = 0$ which is a contradiction!

Summary

$$1.\exists H \leq G,\ (H, G)\text{rel. (T)}$$

$$\Downarrow$$

$$2.\exists B \subset LG,\ (B, LG)\text{rel. (T)}$$

$$\Downarrow$$

$$3.G \text{is not Haagerup}$$

1. $\not\Leftarrow$ 2.-De Cornulier's example.
2. $\not\Leftarrow$ 3.-$\mathbb{Z} \wr_{\Gamma_0} \mathbb{F}_n$, Γ_0 property (T) quotient.

Thank you for listening !