Skew products of topological graphs and noncommutative duality

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work in progress

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Coverings

- can be done with small categories [Bridson-Haefliger]

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- in particular: k-graphs [Pask-Q-Raeburn]
- more particular: path categories of directed graphs [Deicke-Pask-Raeburn]

Definition

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$$\begin{array}{ccccc} v \in \mathcal{D}^0 = p^{-1}(x) & \qquad & v\mathcal{D} = r^{-1}(v) & & \mathcal{D}v \\ \big\downarrow & & {\scriptstyle p}\big\downarrow & \text{bijective} & \big\downarrow{\scriptstyle p} \\ x & & x\mathcal{C} & & \mathcal{C}x \end{array}$$

Induced homomorphisms

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fundamental group

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More precisely, coverings $(\mathcal{D}, p)$ and $(\mathcal{E}, q)$ of $\mathcal{C}$ are isomorphic $\iff$ $p_*\pi(\mathcal{D}, v)$ and $q_*\pi(\mathcal{E}, u)$ are conjugate subgroups of $\pi(\mathcal{C}, x)$.

Regular coverings

Definition

Simplifying assumption (for this talk): $p : \mathcal{D} \to \mathcal{C}$ *regular covering*, i.e., $p_*\pi(\mathcal{D}, v)$ normal in $\pi(\mathcal{C}, x)$

Definition

deck transformations: $G := \pi(\mathcal{C}, x)/p_*\pi(\mathcal{D}, v)$ acts freely on $(\mathcal{D}, p)$, transitively on fibres

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skew product covering:

$$\begin{array}{ccc} \mathcal{C} \times_c G & \qquad & (x, c(\lambda)g) \xleftarrow{(\lambda,g)} (y, g) \\ \big\downarrow & & \\ \mathcal{C} & & x \xleftarrow[\lambda]{\qquad\qquad} y \end{array}$$

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Theorem

- G *acts freely on* $\mathcal{C} \times_c G$ *via* $(\lambda, g) \cdot h = (\lambda, gh)$
- $(\mathcal{C} \times_c G)/G \cong \mathcal{C}$

Regular coverings = Skew products

Theorem (Gross-Tucker — well. . .)

If a group G acts freely on a small category $\mathcal{D}$, then:

- *$\mathcal{D}/G$ is a category*
- *the quotient map $\mathcal{D} \to \mathcal{D}/G$ is a covering*
- *there is a cocycle $c : \mathcal{D}/G \to G$ such that*

$$\mathcal{D} \xrightarrow{\text{quotient map}} \mathcal{D}/G \xleftarrow{\text{skew product covering}} (\mathcal{D}/G) \times_c G \qquad (\cong)$$

Special case

Specialize to: path categories of directed graphs (can also do for k-graphs)

$$E = (E^0, E^1, r, s)$$

$$E^1 \overset{r}{\underset{s}{\rightrightarrows}} E^0$$

edges vertices

edges go from sources to ranges:

$$r(e) \xleftarrow{e} s(e)$$

path category:

$$v_0 \xleftarrow{e_1} v_1 \xleftarrow{e_2} v_2 \quad \cdots \quad v_{n-1} \xleftarrow{e_n} v_n$$

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Theorem (Kaliszewski-Q-Raeburn)

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In fact, the coaction is easy to find:

$$\delta(s_e) = s(e) \otimes c(e) \qquad (e \in E^1)$$

where s_e is a typical generating partial isometry of $C^*(E)$.

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- $C_c(E^1)$ completes to C^*-correspondence $X(E)$ over $C_0(E^0)$
- $C^*(E) := \mathcal{O}_{X(E)}$ (Cuntz-Pimsner algebra)

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Two approaches:

- indirect: use *Landstad duality*
- direct: use *multiplier bimodules*

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Tentative Answer (Bridson-Haefliger)

$\pi(E, x) =$ *"free product" of fundamental group defined as in discrete case and "classical" homotopy fundamental group of* E^0

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But: how to compute that?

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Tentative Answer (Deaconu-Kumjian-Q)

geometric realization:

$$R(E) := \Big((E^1 \times [0,1]) \sqcup E^0 \Big) / \sim$$

$$(e,0) \sim s(e)$$
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Then define

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Hope to show these two approaches give isomorphic fundamental groups. . .

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Fact

This G bundle will be topologically trivial.

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The first part goes ok:

Theorem (Deaconu-Kumjian-Q)

If a locally compact group G acts freely and properly on a topological graph E, then the quotient E/G is a topological graph.

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Future work

What to do with nontrivial principal G-bundles...